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Transformation of criteria with a-priori chosen optimal values

Askoldas Podviezko[1], **Valentinas Podvezko**[2]

[1] *Institute of Economics, Mykolas Romeris University*
Ateities st. 20, LT-08303 Vilnius, Lithuania
[2] *Faculty of Fundamental Sciences, Vilnius Gediminas Technical University*
Saulėtekio al. 11, LT-10223 Vilnius, Lithuania
E-mail: valentinas.podvezko@vgtu.lt, askoldas@gmail.com

Abstract. Aim of multiple criteria decision-aid (MCDA) methods is to find the best alternative among the ones that are available or to rank alternatives in the order of preference. There are the following core pillars of the methods: the set of criteria and matrix with values of criteria that characterise the evaluated alternatives (decision matrix); and vector of weights that reflect relative importance of criteria. Usually, two types of criteria are used by researchers. Maximising criteria (e.g. profits) reflect a better situation whenever the larger value has been attained. While in case a criterion is minimising (e.g. costs), the better situation is reflected when its value is smaller. Such situations, when the best value of a criterion has a certain value, which differs from the maximal or the minimal, are usually not considered. This paper aims to fill this gap. Such criteria will be named as criteria with a-priori chosen optimal values. The aim of the paper is to propose appropriate types of transformation for criteria with a-priori chosen optimal values. Such transformations appear to be general and can be used with all three types of criteria.

Keywords: multiple criteria decision aid methods, transformation of data, normalisation, criteria with a-priori chosen optimal values.

Introduction

Aim of multiple criteria decision-aid (MCDA) methods is to find the best alternative among the ones that are available or to rank alternatives in the order of preference. There are the following core pillars of the methods: the set of criteria; matrix with values of criteria that characterise the evaluated alternatives (decision matrix); and vector of weights that reflect relative importance of criteria. In research usually two types of criteria are used. Maximising criteria (e.g. profits) reflect a better situation whenever the larger value has been attained. While in case a criterion is minimising (e.g. costs), the better situation is reflected when its value is smaller. Such situations, when the best value of a criterion has a certain value, which differs from the maximal or the minimal, are usually not considered. Nevertheless, situations when there is a desired value of a criterion rather than maximal or minimal value can often be encountered as is demonstrated by the following examples. The age criterion for a candidate for a job could be considered a minimising criterion, but it is not as a young candidate would lack experience, while an old candidate could lack energy or motivation. The size of a car to buy could be considered a maximising criterion, but it is not as many buyers would look for a certain size of a car. Height, weight, and age of a candidate to form a potential couple usually can be neither maximising, nor minimising criteria. The size of a house for many potential buyers cannot be a maximising criterion, because a large house requires more cleaning efforts, and is awkward for communication between family members, etc. Desired temperature in a room or a car, again, obviously is neither a maximising, nor a minimising criterion. We propose to call such criteria as criteria with a-priori chosen optimal values.

Available normalisations, even if attention should be paid to distortions they introduce [1], are designed for minimising or maximising criteria. For criteria with a-priori chosen optimal values two problems must be solved. First, appropriate normalisations should be proposed. Such normalisations should transform the desired optimal value into the largest value. Second, it should be possible to comprise all transformed values into a single cumulative criterion of a MCDA method, for all types of criteria: maximising, minimising, and criteria with a-priori chosen optimal values.

1 Transformation of criteria with a-priori chosen optimal values

There is a classic transformation which reflects preferences of a decision-maker and embeds his/her a-priori chosen optimal value. It is Gauss transformation [4, 2].

The authors believe that in the case of criteria with a-priori chosen optimal values standard deviation is not the right parameter for Gauss transformation. It reflects how the data is scattered around the mean, but not around the chosen optimal value. Consequently, the variance around the chosen optimal value r_i^0 is proposed in this paper as more appropriate parameter for the criteria of the named type. We denote it as $\overline{\sigma}_i$, where $\overline{\sigma}_i = \sqrt{E[(r_{ij} - r_i^0)^2]}$. Consequently, we propose the following transformation (1):

$$\hat{r}_{ij} = \exp\big(-\big|r_{ij} - r_i^0\big|^\alpha / \big(2\overline{\sigma}_i^2\big)\big), \tag{1}$$

where r_{ij} is value of the i-th criterion for the j-th alternative; r_i^0 the a-priori chosen optimal value of the criterion, and $\overline{\sigma}_i$ is standard deviation of values of the i-th criterion around r_i^0, $i = 1, \ldots, m$; $j = 1, \ldots, n$; m − the number of criteria, n − the number of evaluated alternatives. We note that $\overline{\sigma}_i$ could be also set by a decision-maker in accordance to his/her preferences of how values of criteria should be mapped into the interval $[0, 1]$. There could be more variations of such transformation. The idea is that it has to be dependent on the distance $|r_{ij} - r_i^0|$.

We added another parameter α to allow more flexibility. The parameter is for a decision-maker to choose. The more flexible is decision-maker for having value of r_{ij} near r_i^0, the smaller should be α; $\alpha < 2$. Contrary, if decision-maker is less flexible, α should be chosen $\alpha > 2$.

We can easily observe that the maximal value of the transformation (1) $\max \hat{r}_{ij} = 1$ is attained at the point of optimal value $r_{ij} = r_i^0$ as is shown on Fig. 1.

The formula (2) is a special case of former more generalised formula (1), when $\alpha = 2$. Consequently, we have the classic case of Gaussian transformation by slight altering formula and taking $\overline{\sigma}_i$ from (1) instead of σ_i (around the mean):

$$\hat{r}_{ij} = \exp\big(-\big(r_{ij} - r_i^0\big)^2 / \big(2\overline{\sigma}_i{}^2\big)\big). \tag{2}$$

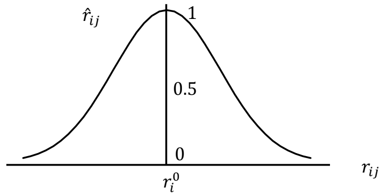

Fig. 1. Transformation for a criterion with a-priori chosen optimal value r_i^0.

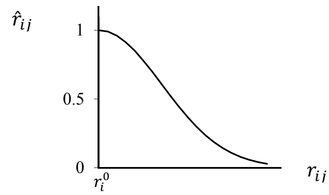

Fig. 2. Transformation a minimising criterion.

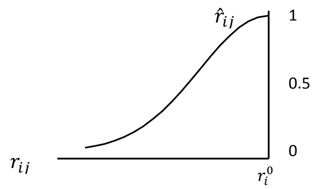

Fig. 3. Transformation a maximising criterion.

In addition to the following known transformation for criteria with a-priori chosen optimal values (3) we propose the linear transformation (4):

$$\hat{r}_{ij} = \begin{cases} \dfrac{r_{ij} - \min_j r_{ij}}{\max_j r_{ij} - \min_j r_{ij}}, & \text{if } i \text{ is a maximising criterion,} \\ \dfrac{\max_j r_{ij} - r_{ij}}{\max_j r_{ij} - \min_j r_{ij}}, & \text{if } i \text{ is a minimising criterion.} \end{cases} \tag{3}$$

Similarly to the transformation (1) it maps values of criteria r_{ij} into the interval $[0, 1]$:

$$\hat{r}_{ij} = 1 - \frac{|r_{ij} - r_i^0|}{d_i}, \quad \text{where } d_i = \max\left(r_i^0 - \min_j r_{ij}; \max_j r_{ij} - r_i^0\right). \tag{4}$$

Maximal value $\hat{r}_{ij} = 1$ is again attained at the optimal point $\hat{r}_{ij} = r_i^0$, and minimal value $\hat{r}_{ij} = 0$ is attained at the worst value of the i-th criterion.

Both transformations (1) and (4) have the same feature: the further is the value of a criterion from the optimal point r_i^0, the smaller is its transformed value. The formulae of proposed transformations are generalisations of known ones, because they can be used also for maximising and minimising criteria as special cases. We will use the transformation (1) in the following case study.

In fact, suppose the criterion is minimising, which makes the value $r_i^0 = \min_j r_{ij}$ desired. Then the mapping function would be of the shape as is shown in Fig. 2.

It will map the point $r_i^0 = \min_j r_{ij}$ to the maximal point of 1, and whenever $r_{ij} > r_i^0$, the function will map r_{ij} to gradually smaller values.

Contrary, in case a criterion is maximising, the desired value would be $r_i^0 = \max_j r_{ij}$. Then the mapping function would be of the shape as is shown in Fig. 3. It will map the point $r_i^0 = \max_j r_{ij}$ to the maximal point of 1, and whenever $r_{ij} < r_i^0$, the function will map r_{ij} to gradually smaller values.

All proposed transformations map values of criteria into the interval $[0, 1]$, and result in maximising objectives in terms of transformed values.

2 MCDA methods: criteria with a-priori chosen optimal values

As long as criteria with a-priori chosen optimal values are transformed (along with maximising and minimising ones) in accordance with proposed transformation formulae, we can use any MCDA method without paying attention, if we have to transform minimising criteria or if there are negative values of criteria [2, 3]. We will use the SAW and the TOPSIS methods in this paper to demonstrate how the proposed types of normalisation can be applied in practice. We note that the COPRAS method will yield identical results as the SAW method as all transformed values become values of maximising objectives.

The formula of the SAW method [5] is ready-to-use for already transformed values $\tilde{r}_{ij} = \hat{r}_{ij}$: $S_j = \sum_{i=1}^{m} \omega_i \tilde{r}_{ij}$, where ω_i are weights of criteria, which could be elicited from experts.

Usage of the TOPSIS method [5] is also slightly simplified as transformed values are representing only maximising objectives.

The TOPSIS vector normalisation has to be used for already transformed values: $\tilde{r}_{ij} = \frac{\hat{r}_{ij}}{\sqrt{\sum_{j=1}^{n} r_{ij}^2}}$ $(i = 1, \ldots, n;\ j = 1, \ldots m)$, where $\tilde{r}_{ij}$ is the normalised value of j-th criterion for i-th alternative.

For finding the best alternative V^* and the worst alternative V^- the following formulae are used: $V^* = \{V_1^*, V_2^*, \ldots, V_m^*\} = \{\max_j \omega_i \tilde{r}_{ij}\}$, $V^- = \{V_1^-, V_2^-, \ldots, V_m^-\} = \{\min_j \omega_i \tilde{r}_{ij}\}$.

The distance D_j^* of every considered alternative to the ideal (best) solutions and its distance D_j^- to the worst solutions is calculated as follows:

$$D_j^* = \sqrt{\sum_{i=1}^{m} \left(\omega_i \tilde{r}_{ij} - V_i^*\right)^2}, \qquad D_j^- = \sqrt{\sum_{i=1}^{m} \left(\omega_i \tilde{r}_{ij} - V_i^-\right)^2}.$$

The criterion C_j^* of the method TOPSIS was calculated by the following formula: $C_j^* = \frac{D_j^-}{D_j^* + D_j^-}$ $(j = 1, \ldots, n)$.

3 A case-study

As was mentioned in the introduction many examples can be found where criteria with a-priori chosen optimal values are the most appropriate to be used. As a special case, evaluation of alternative houses obviously cannot be properly done without choosing proprietary type of criteria along with maximising and minimising ones. We choose as such (proprietary) criteria the following three: the size of the house; the size of the yard; distance to city centre. It would be misleading to treat the size of the house as a maximising criterion, because too large houses require more time for movement and cleaning, and make communication between family members more difficult. Similarly, too large yard requires more time for taking care of, for mowing the lawn, etc. Some inhabitants may neither prefer to live in the centre of a city, nor too far from the city centre. Consequently, we believe that such criteria have optimal values for a decision-maker.

Values of criteria for 4 alternative houses to acquire are presented in Table 1.

Table 1. Values of criteria describing 4 alternative houses.

No.	Criteria	Type	Optimal	A	B	C	D	Weights
1.	Size, sq. m.	proprietary	90	100	55	150	230	20%
2.	Size of the yard, ares	proprietary	8	10	5	6	100	14%
3.	Distance to city centre, km	proprietary	7	12	3	10	20	6%
4.	Distance to public transport, km	min	–	5	0.5	1	10	14%
5.	Distance to job, km	min	–	8	3	9	22	10%
6.	Distance to nature or a park, km	min	–	0	1	3	0	7%
7.	Number of grocery stores within 5 km distance	max	–	3	21	1	0	5%
8.	Maximal persistent noise, dB	min	–	20	45	30	10	24%

Table 2. Transformed values of criteria of alternative houses.

No.	Criteria	Type	Alternatives			
			A	B	C	D
1.	Size, sq. m.	proprietary	0.992	0.905	0.746	0.202
2.	Size of the yard, ares	proprietary	0.999	0.998	0.999	0.136
3.	Distance to city centre, km	proprietary	0.796	0.864	0.921	0.214
4.	Distance to public transport, km	min	0.694	1.000	0.995	0.196
5.	Distance to job, km	min	0.888	1.000	0.843	0.181
6.	Distance to nature or a park, km	min	1.000	0.819	0.165	1.000
7.	Number of grocery stores within 5 km distance	max	0.573	1.000	0.503	0.469
8.	Maximal persistent noise, dB	min	0.500	0.088	0.250	1.000

It is known that perception of difference of sound level by 10 dB is perceived as twice as different [1], therefore more refined normalisation (power of 2 instead of power of e along with the adjustment to the argument of divided by 10) should be used in this case ($i = 8$):

$$\hat{r}_{8j} = \frac{2^{-r_{8j}/10}}{\min_j(2^{-r_{8j}/10})}.$$

It maps the most desirable level of the smallest noise to the maximal point 1, and gradually diminishes corresponding normalised value in accordance to the elicited perception of noise. Transformed values of all types of involved criteria are shown in Table 2. A good resemblance of preferences of a decision-maker in observed mappings looks convincing: the closer is value of a criterion to its desired (maximal, minimal or a-priori chosen value), the closer is its transformed value to 1.

Results of evaluation by two methods are shown in Table 3.

Correspondence of the results is not ideal, as it is the usual case, but the results fulfil the major task of finding the best alternative. Often appearing discrepancies with the TOPSIS method are not yet explored. Most probably, one of the major causes is

Table 3. Results of MCDA evaluation.

Method	Alternatives			
	A	B	C	D
SAW or COPRAS	0.791	0.741	0.665	0.451
TOPSIS	0.606	0.441	0.451	0.555
	Ranking			
SAW or COPRAS	1	2	3	4
TOPSIS	1	4	3	2

that the TOPSIS method does not fully, linearly, reflect preference of decision-maker in terms of weights. Also, hypothetic reference alternatives should be chosen more carefully as it is suggested in [3].

4 Conclusions

In the paper special attention is paid to the criteria with the best value for a decision-maker differing from maximum or minimum. Three types of linear and power normalisations are proposed for such criteria. One power normalisation is designed for transformation of sound in accordance to the believed perception of disturbance caused by noise. For all other cases the classic power normalisation is taken, which is derived from the theory of statistics. An essential adjustment is proposed in the paper: variance is measured not around the mean, but around more important a-priori chosen optimal value. Such a choice for calculating variance allows discerning alternatives based on the difference of the value of criterion from the desirable value. We note that the bell-shaped normalisation is suitable whenever stochasticity or slack in data is involved as small deviations from the desired value are accounted as insignificant. Deviations among very large values are also accounted as insignificant as is usually perceived by human beings.

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REZIUMĖ

Kriterijų transformavimas su iš anksto pasirinktu optimumu

A. Podviezko, V. Podvezko

Daugiakriterių metodų (angl. Multiple Criteria Decision Aid – MCDA) tikslas yra nustatyti geriausią iš vertinamų alternatyvų arba ranguoti alternatyvas nagrinėjamo tikslo atžvilgiu. MCDA metodų pagrindą sudaro charakterizuojančių nagrinėjamo proceso kriterijų reikšmių matrica (sprendimo matrica) ir kriterijų svorių vektorius. Paprastai tyrėjai naudoja dviejų tipų kriterijus: maksimizuojamus, kai didžiausia reikšmė yra geriausia (pvz., pelnas), arba minimizuojamus, kai mažiausia reikšmė yra geriausia (pvz., kaina). Tačiau neatsižvelgiama į atvejus, kai tam tikros kriterijų reikšmės yra geriausios. Šiame straipsnyje bandoma užpildyti tokią spragą. Straipsnyje siūlomi keli kriterijų su iš anksto pasirinktų optimumu transformacijos būdai. Pasiūlyti būdai yra universalūs ir gali būti taikomi tiek minimizuojantiems ar maksimizuojantiems, tiek kriterijams su iš anksto pasirinktu optimumu.

Raktiniai žodžiai: daugiakriteriai MCDA metodai, duomenų transformavimas, kriterijai su iš anksto pasirinktų optimumu.