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Large deviations for endomorphisms of torus, II

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Let Ω_2 be a two-dimensional torus. Let $W = \|a_{ij}\|$, $i, j = 1, 2$, be a square matrix, a_{ij} be integers, $|\det W| = 1$. Let the endomorphism $T\colon \Omega_2 \to \Omega_2$ be defined by

$$T\vec{x} = \vec{x}\, W \pmod 1.$$

Let

$$\vec{\xi} = \vec{\xi}(t) = \bigl\{(\varphi(t), \psi(t)),\ a \leqslant t \leqslant b\bigr\} \tag{1}$$

be a smooth parametric curve on Ω_2, $\Phi(x)$ be a standard normal distribution function. Let $\vec{w}_i = (w_{i1}, w_{i2})$, $i = 1, 2$, be eigenvectors of W corresponding to the eigenvalues θ_1 and θ_2, $|\theta_1| > 1$, $|\theta_2| = |\theta_1|^{-1}$.

We consider a real function $h(\vec{x})$, $\vec{x} \in \Omega_2$. The problem of large deviations for endomorphisms on torus is formulated in terms of the function h as follows.

Let

$$S_n(\vec{x}) = \sum_{k=0}^{n-1} h\bigl(\vec{x}\, W^k\bigr), \qquad Z_n(\vec{x}) = \frac{1}{\sigma\sqrt{n}} S_n(\vec{x}),$$

where

$$\sigma^2 = \lim_{n\to\infty} \int_{\Omega_2} \Bigl(\frac{1}{\sqrt{n}} S_n(\vec{x})\Bigr)^2 \mathrm{d}\vec{x}, \qquad \sigma^2 > 0.$$

Then

$$F_{n,\xi}(x) = \frac{1}{b-a}\mu\bigl(t \in [a, b]\colon Z_n(\vec{\xi}\,) < x\bigr) \quad (\mu \text{ is the Lebesque measure})$$

is a distribution function for which the theorem of large deviations can be proved under some regularity conditions for the function h.

We assume that h satisfies the following conditions:

$$\max\bigl\{|h(\vec{x})|,\ \vec{x} \in \Omega_2\bigr\} \leqslant H, \tag{2}$$

$$\int_{\Omega_2} h(\vec{x})\, \mathrm{d}\vec{x} = 0, \tag{3}$$

$$\int_{\Omega_2} \big(h(\vec{x}) - h(\vec{x} + \vec{\delta})\big)^2 \,\mathrm{d}\vec{x} \leqslant H^2(\delta_1^2 + \delta_2^2), \tag{4}$$

where $\vec{\delta} = (\delta_1, \delta_2)$.

The main result is formulated in the following manner.

THEOREM. *Let the function $h(\vec{x})$, $\vec{x} \in \Omega_2$, and the curve $\vec{\xi}(t)$, $t \in [a, b]$, satisfy the above mentioned conditions. Then, in the interval*

$$0 < x < \frac{c\sqrt{n}}{\ln^2 n}, \quad c > 0,$$

the following relations for the large deviations are valid:

$$\frac{1 - F_{n,\xi}(x)}{1 - \Phi(x)} = \exp\big\{L(x)\big\}\bigg(1 + \mathrm{O}\Big(\frac{x \ln^2 n}{\sqrt{n}}\Big)\bigg),$$

$$\frac{F_{n,\xi}(-x)}{\Phi(-x)} = \exp\big\{L(-x)\big\}\bigg(1 + \mathrm{O}\Big(\frac{x \ln^2 n}{\sqrt{n}}\Big)\bigg),$$

where

$$L(x) = \sum_{k=3}^{\infty} \lambda_k x^k,$$

the coefficients λ_k are expressed in terms of cumulants of the sum Z_n.

In [2] the analogous result is proved for the functions h verifying the condition $|h(\vec{x}_1) - h(\vec{x}_2)| \leqslant H\varrho(\vec{x}_1, \vec{x}_2)$ where $\varrho(\cdot,\cdot)$ is the distance on torus. Therefore the theorem of large deviations is proved now for a wider class of functions h.

The following auxiliary propositions are needed for the proof of Theorem.

LEMMA 1. *Let the functions $f(\vec{x})$ and $g(\vec{x})$, $\vec{x} \in \Omega_2$, satisfy the condition* (2) *with the constants A and B respectively,*

$$\max_{\vec{x} \in \Omega_2} \big|f(\vec{x})\big| \leqslant A, \quad \max_{\vec{x} \in \Omega_2} \big|g(\vec{x})\big| \leqslant B.$$

Then

$$\int_a^b f(\vec{\xi}\,) g\big(\vec{\xi}\, W^m\big) \,\mathrm{d}t = \int_a^b f(\vec{\xi}\,) \,\mathrm{d}t \cdot \int_{\Omega_2} g(\vec{x}) \,\mathrm{d}\vec{x} + \mathrm{O}\Big(\frac{mAB}{\varepsilon^3 \theta_1^m}\Big), \tag{5}$$

$$\int_{\Omega_2} f(\vec{x}) g\big(\vec{x}\, W^m\big) \,\mathrm{d}\vec{x} = \int_{\Omega_2} f(\vec{x}) \,\mathrm{d}\vec{x} \cdot \int_{\Omega_2} g(\vec{x}) \,\mathrm{d}\vec{x} + \mathrm{O}\Big(\frac{mAB}{\theta_1^m}\Big), \tag{6}$$

where $\vec{\xi}(t)$ is a curve defined by (1)*, θ_1 is an eigenvalue, $|\theta_1| > |\theta_2|$,*

$$\varepsilon = \min_{t \in [a,b]} \big|w_{21}\varphi'(t) - w_{22}\psi'(t)\big| > 0.$$

Proof. See [1].

Let $\Gamma_k(S_n)$ denote the cumulant of k-th order of the sum $S_n(\vec{x})$.

LEMMA 2. *The cumulants $\Gamma_k(S_n)$ are estimated as follows:*

$$\left|\Gamma_k(S_n)\right| \leqslant H_0 H^k k! (\ln^2 n)^{k-2} n, \quad H_0 > 0.$$

Proof. Denote

$$S_{k,l} = S_{k,l}(\vec{x}) = \sum_{k \leqslant i \leqslant l} h\bigl(\vec{x} W^i\bigr).$$

For some natural numbers n_1 and n_2, let

$$p = \Bigl[\frac{n}{n_1 + n_2}\Bigr].$$

The numbers n_1 and n_2 will be chosen later. Denote

$$\begin{aligned}
\eta_k(\vec{x}) &= S_{(k-1)(n_1+n_2)+1, kn_2+(k-1)n_1}, \quad 1 \leqslant k \leqslant p,\\
\eta_k^0(\vec{x}) &= S_{kn_2+(k-1)n_1+1, k(n_1+n_2)}, \quad 1 \leqslant k \leqslant p,\\
\eta_{p+1}^0(\vec{x}) &= S_{p(n_1+n_2)+1, n}.
\end{aligned}$$

We will assume that $\eta_{p+1}^0(\vec{x}) = 0$ if $n = p(n_1 + n_2)$. Then the sum $S_n(\vec{x})$ can be expressed by

$$S_n(\vec{x}) = \sum_{k=1}^{p} \eta_k(\vec{x}) + \sum_{k=1}^{p+1} \eta_k^0(\vec{x}).$$

Following [2] let us consider the function

$$f_n(t) = \int_{\Omega_2} \prod_{k=1}^{p} \exp\bigl(it\eta_k(\vec{x})\bigr)\,\mathrm{d}\vec{x}.$$

As the functions $\eta_k(\vec{x})$ satisfy the conditions of Lemma 1, we get from equation (6):

$$f_n(t) = \Bigl(\int_{\Omega_2} \exp\bigl(it\eta_1(\vec{x})\bigr)\,\mathrm{d}\vec{x}\Bigr)^p + \mathrm{O}\bigl(n^2 t H \theta_1^{-n_1}\bigr), \tag{7}$$

$$\int_{\Omega_2} h(\vec{x}) h\bigl(\vec{x} W^r\bigr)\,\mathrm{d}\vec{x} = \mathrm{O}\bigl(r\theta_1^{-r}\bigr), \tag{8}$$

$$\mathrm{E}\Bigl(\eta_k(\vec{x}) - \mathrm{E}\bigl(\eta_k(\vec{x})\bigr)\Bigr)^2 = n_2\sigma^2 + \mathrm{O}\Bigl(\frac{1}{(b-a)\epsilon^3}\Bigr). \tag{9}$$

Let a function $g(\alpha_1,\dots,\alpha_\nu)$ have the Taylor expansion on the vicinity of $0\in\mathbb{R}^\nu$:

$$g(\alpha_1,\dots,\alpha_\nu)=\sum_{k_1,\dots,k_\nu=0}^{\infty} g_{k_1\dots k_\nu}\cdot\alpha_1^{k_1}\cdots\alpha_\nu^{k_\nu}.$$

The function $g(\alpha_1,\dots,\alpha_\nu)$ is said to be of type M (see [1]) if $g_{k_1\dots k_\nu}\neq 0$ only if $\max\{k_1,\dots,k_\nu\}\geqslant 2$.

Functions of type M have the following properties:

(a) The linear combination and the product of any number of functions of type M is a function of type M.

(b) If the function $g_1(\alpha_1,\dots,\alpha_\nu)$ is of type M, the function $g_2(\alpha_1,\dots,\alpha_\nu)$ is analytic and

$$\inf_{|(\alpha_1,\dots,\alpha_\nu)|<\varepsilon} g_2(\alpha_1,\dots,\alpha_\nu)>0\quad\text{for some }\varepsilon>0$$

then g_1/g_2 is the function of type M.

The function

$$\varphi_0(\alpha_1,\dots,\alpha_\nu)=\mathrm{E}\exp(\alpha_1X_{t_1}+\cdots+\alpha_\nu X_{t_\nu}),\qquad X_n=h(\vec{x}W^n),\ n\geqslant 1,$$

has the above mentioned properties. We write down the Taylor expansion for this function, then apply Lemma 1 and (5), and estimate the coefficients of the expansion:

$$\varphi_0(\alpha_1,\dots,\alpha_\nu)=\sum_{k_1,\dots,k_\nu=0}^{\infty}\mathrm{E}X_{t_1}^{k_1}\cdots X_{t_\nu}^{k_\nu}\cdot\alpha_1^{k_1}\cdots\alpha_\nu^{k_\nu},\tag{10}$$

$$\mathrm{E}X_{t_1}^{k_1}\cdots X_{t_\nu}^{k_\nu}=\mathrm{E}\prod_{i=1}^{q'}h^{k_i}\big(\vec{x}W^{t_i}\big)\prod_{i=q'+1}^{\nu}h^{k_i}\big(\vec{x}W^{t_i}\big)+\mathrm{O}\big(d\theta_1^{-d}H^{k_1+\cdots+k_\nu}\big).\tag{11}$$

Here d depends on the partition of the set $\{t_1,\dots,t_k\}$ into blocks $I_1,\dots,I_\nu$ (see [3]).

Using (10) and (11) we can express the function $\varphi_0(\alpha_1,\dots,\alpha_\nu)$ as follows:

$$\varphi_0(\alpha_1,\dots,\alpha_\nu)=\varphi_1(\alpha_1,\dots,\alpha_{q'})\cdot\varphi_2(\alpha_{q'+1},\dots,\alpha_\nu)+d\theta_1^{-d}\psi_0(\alpha_1,\dots,\alpha_\nu)$$

where

$$\varphi_1(\alpha_1,\dots,\alpha_{q'})=\mathrm{E}\exp(\alpha_1X_{t_1}+\ldots+\alpha_{q'}X_{t_{q'}}),$$
$$\varphi_2(\alpha_{q'+1},\dots,\alpha_\nu)=\mathrm{E}\exp(\alpha_{q'+1}X_{t_{q'+1}}+\ldots+\alpha_\nu X_{t_\nu})$$

and the function $\psi_0(\alpha_1,\dots,\alpha_\nu)$ is analytic, $\psi_0(0,\dots,0)=0$.

For the evaluation of cumulants $\Gamma_k(S_n)$ we chose n_1 from the relation $0\leqslant n-p(n_1+n_2)\leqslant p$, $n_2=[\omega_2\ln n]$ and make use of the properties of functions of type M for evaluation of functions $f_n(t)$. These are only the main points of the proof of Lemma 2.

The proof of Theorem follows from Lemma 1, Lemma 2 and is analogous to Theorem 3 in [4].

References

1. D.A. Moskvin, Metric theory of automorphisms of two-dimensional torus, *Izv. Akad. Nauk*, Ser. mat., **45**, 69–100 (1981).
2. B. Kryžienė, G. Misevičius, Large deviations for endomorphisms of torus, *Liet. matem. rink.*, **44** (spec. issue), 72–76 (2004).
3. L. Saulis, V. Statulevičius, *Limit Theorems for Large Deviations*, Kluwer Academic Publishers, Dordrecht (1991).
4. G. Misevičius, On large deviations for a sum of type $\sum f(T^j t)$, *Acta et Commentationes Universitatis Tartuensis*, **893**, 3–10 (1990).

REZIUMĖ

B. Kryžienė, G. Misevičius. Toro endomorfizmų didieji nuokrypiai, II

Darbe suformuluota ir įrodyta teorema apie didžiuosius nuokrypius dydžiams $h(\vec{x}W^k)$, $k = 0, 1, 2, \ldots$, kur W yra toro Ω_2 endomorfizmas. Lyginant su ankstesniu autorių darbu teorema įrodoma platesnei funkcijų h klasei. Įrodymui naudojami D. Moskvino ir autorių ankstesni rezultatai bei V. Statulevičiaus centruotų momentų ir semiinvariantų įverčiai.