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Loop-check free sequent calculi for unary fragment of temporal logic

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Abstract. This paper explores the construction of an efficient sequent calculus for a selected fragment of porpositional linear temporal logic (PLTL), extending the ideas of classical calculi discussed in [1], and builds upon previous investigations into the issue of loops in PLTL. Unary fragment of $PLTL$ is identified in which formulas can contain at most one outermost $\Box$ ("always") operator. Fragments are typically analyzed with the aim of defining more efficient calculi for formulas belonging to the fragment, especially when such an approach is not feasible for the full logic (such a strategy is employed in [3, 4]). New or-type rule $(\vdash \Box_L *)$ is introduced by the authors. Authors propose newly developed sequent calculus $PLTL - F_1$ for this fragment, which eliminates the need for loop axioms while improving derivation efficiency. Furthermore, the authors also introduce a totally loop-check free sequent calculus $PLTL - F_3$. Elimination of loops was achieved by proving restrictions on the applications of the $(\vdash \Box)$ and $(\circ)$ rules.

Keywords: loops; sequent calculi; fragments; PLTL

AMS Subject Classification: 03B44

Introduction

Sequent calculi for temporal logic can describe the reasoning about how the truth values of assertions change over time. There are quite a few sequent calculi created for such logic (see the works of [2] and [5]), however they are not suitable for automatic reasoning at all, or they have an efficiency problems due to the loop check invloved.

Since loop check elimination for the full PLTL is still a big problem, in this paper the authors focused on solving the problem of a meaningful PLTL fragment, here called fragment α. A new sequent calculus $PLTL - F_3$, which fully eliminates loop check from the derivation, for the PLTL fragment α is presented.

Definition 1. PLTL fragment α is a minimal fragment satisfying the following:

1. if F is non-modular formula, F and $\Box F \in \alpha$
2. if $F, G \in \alpha$, then $\neg F, F \wedge G, F \vee G \in \alpha$.

For every PLTL sequent S_0, which contains only formulas from the fragment α, by applying only invertible logical rules of the sequent calculus, one can obtain sequent $S = \Pi, \Box G_1, \ldots, \Box G_m \vdash \Sigma, \Box F_1, \ldots, \Box F_n$ (here Π, Σ contains only non-modular formulas, $G_1, \ldots, G_m, F_1, \ldots F_n$ are also non-modular formulas), which is derivable if and only if S_0 is derivable in PLTL. Therefore, without loss of generality, in this article we will consider sequents of the shape S as sequents from the fragment α.

1 Sequent calculus for PLTL

1.1 Syntax and semantics

Propositional linear temporal logic uses standard propositional logic operators and modal operators for the time – $\Box$ ("always") and $\circ$ ("next"). The modal operator $\Diamond$ is not used, since it can be expressed as $\Diamond p = \neg\Box\neg p$.

- $\circ p$ is understood as "p is true in the next time moment".
- $\Box p$ is understood as "p is true now and at all time moments in the future".

In the paper [5], a sequent calculus $G_L PLTL$ for logic PLTL is described, in which both positive (loop-axioms) and negative (non-derivable) loops can be constructed.

Definition 2. $S \to S'$ is a loop in a sequent derivation tree if S and S' are sequents presented in the same branch of the tree (S' is above S), S' and S contains the same formulas on their antecedents and succedents, and S can be obtained from S' by using the structural rule of weakening in a bottom-up direction.

Definition 3. S' is a loop axiom in a sequent calculus $G_L PLTL$ if there is a loop $S \to S'$ and there exists such a rule ($\vdash \Box$) application with the principal formula $\Box F$ between S and S', that S' is on the right branch of this application, and there is no such a rule ($\vdash \Box$) application with the same principal formula $\Box F$ between S and S', that S' is on the left branch of this application.

The rule ($\vdash \Box$) is defined in the next definition. $\Box A$ presneted in the rule ($\vdash \Box$) definition is its principal formula.

Definition 4. Sequent calculus $G_L PLTL$ for logic PLTL is a calculus with 2 axioms – traditional $A, \Gamma \vdash A, \Delta$ and a loop-axiom, classical rules for propositional logic and specific temporal rules:

$$\frac{\Gamma_1 \vdash \Gamma_2}{\Sigma_1, \circ\Gamma_1 \vdash \Sigma_2, \circ\Gamma_2}(\circ) \qquad \frac{A, \circ\Box A, \Gamma \vdash \Delta}{\Box A, \Gamma \vdash \Delta}(\Box\vdash) \qquad \frac{\Gamma \vdash \Delta, A \quad \Gamma \vdash \Delta, \circ\Box A}{\Gamma \vdash \Delta, \Box A}(\vdash\Box)$$

here, A is any formula. $\Gamma, \Gamma_1, \Gamma_2, \Delta, \Sigma_1, \Sigma_2$ are the multisets of formulas.

According [1], calculus $G_L PLTL$ is sound and complete for PLTL. Note, that structural rule of weakening ($Weak$) is admissible in the calculus $G_L PLTL$.

1.2 Features of calculus G_LPLTL

In this paper, we will denote a set of non-modular formulas with Π or Σ. In these proofs we exploit the known fact, that rules $(\vee\ \vdash), (\vdash\ \vee), (\wedge\ \vdash), (\vdash\ \wedge), (\neg\ \vdash), (\vdash \neg), (\Box \vdash), (\vdash \Box)$ are invertible in the sequent calculus G_LPLTL.

Lemma 1. *If the sequent $S = \Pi, \Box\Gamma \vdash \Sigma, \Box F_1, \ldots, \Box F_n$ from the fragment α is derivable in the G_LPLTL calculus (F_i is a non-modular formula for every $i = 1, \ldots, n$), then*:

1. $\Pi, \Box\Gamma \vdash \Sigma$ *is derivable, or*
2. $\Box\Gamma \vdash \Box F_1, \ldots, \Box F_n$ *is derivable.*

Proof. We apply all possible $(\vdash \Box)$ rules to the sequent S and obtain 2^n sequents, all of them are derivable, since only invertible rules were applied.

One of these derivable sequents is a sequent $S_1 = \Pi, \Box\Gamma \vdash \Sigma, \circ\Box F_1, \ldots, \circ\Box F_n$ (obtained by always taking only the right branch of the rule $(\vdash \Box)$ application). Sequent $S_2 = \Pi, \Gamma, \circ\Box\Gamma \vdash \Sigma, \circ\Box F_1, \ldots, \circ\Box F_n$ is obtained by applying all possible $(\Box \vdash)$ rules for the sequent S_1. Here Π, Γ, Σ are non-modular sets of formulas, because the sequent S is from the fragment α, and there will be no formulas bounded by the $\circ$ operator in these sets. All applied rules are invertible and S_1, S_2 are derivable if S is derivable. Since S_2 is derivable, formulas in sets Π, Γ, Σ are non-modular formulas, and during rule $(\circ)$ application all formulas without outermost operator $\circ$ should be deleted, then:

a) $\Pi, \Gamma \vdash \Sigma$ is derivable, or

b) $\circ\Box\Gamma \vdash \circ\Box F_1, \ldots, \circ\Box F_n$ is derivable.

In case (a), sequent $\Pi, \Box\Gamma \vdash \Sigma$ will also be derivable, since by applying all $(\Box \vdash)$ rules to it, we obtain sequent $\Pi, \Gamma, \circ\Box\Gamma \vdash \Sigma$, which is derivable because $\Pi, \Gamma \vdash \Sigma$ is derivable.

In case (b), sequent $\Box\Gamma \vdash \Box F_1, \ldots, \Box F_n$ will also be derivable, because sequent $\circ\Box\Gamma \vdash \circ\Box F_1, \ldots, \circ\Box F_n$ is derivable and only rule $(\circ)$ can be applied to it. $\Box$

Lemma 2. *If the sequent $S = \Box\Gamma \vdash \Box F_1, \ldots, \Box F_n$ from the fragment α is derivable in calculus G_LPLTL and $\Box F_i$ is any modal formula from the succedent of S, then there are two possible cases*:

1. *The sequent* $\Box\Gamma \vdash \Box F_1, \ldots, \Box F_{i-1}, \Box F_{i+1}, \ldots, \Box F_n$ *is derivable, or*
2. *The sequent* $\Gamma \vdash F_i$ *is derivable.*

Here, F_i is a non-modular formula for every $i = 1, \ldots, n$.

Proof. We apply all possible $(\Box \vdash)$ rules to the initial sequent S and thus obtain the sequent $S_2 = \Gamma, \circ\Box\Gamma \vdash \Box F_1, \ldots, \Box F_n$. S_2 is derivable because S is derivable and only invertible rules were applied.

We apply all possible $(\vdash \Box)$ rules to the sequent S_2 and obtain 2^n sequents, all of which are derivable because only invertible rules were applied. One of these sequents is sequent $S_3 = \Gamma, \circ\Box\Gamma \vdash \circ\Box F_1, \ldots, \circ\Box F_{i-1}, F_i, \circ\Box F_{i+1} \ldots, \circ\Box F_n$, in which all formulas F are bounded by $\circ\Box$ operators, except for F_i.

Now we apply all logical rules $(\vee, \wedge, \neg)$ to the formulas from Γ to the sequent S_3 and obtain m new sequents, in whose antecedents and succedents additional atomic formulas Γ_i, Γ_i' may have appeared. If we obtain an axiom during the derivation process, we continue to apply logical rules to this branch as long as it is possible to do so. In all these sequents, the formulas bounded by $\circ$ are the same, since Γ consists only of non-modular formulas, because the original sequent is from the fragment α.

Let $S_4 = \Gamma_z, \circ\Box\Gamma \vdash \Gamma_z', \circ\Box F_1, \ldots, \circ\Box F_{i-1}, F_i, \circ\Box F_{i+1} \ldots, \circ\Box F_n$ be one of these sequents. This sequent is derivable because it is obtained from S_3 by applying only invertible rules. Also, Γ_z and Γ_z' consist only of atomic formulas and do not contain modal operators.

We have two possible cases:

1. Sequent S_4 is derivable by applying rule $(\circ)$,
2. Sequent S_4 is derivable without applying rule $(\circ)$.

Sequent S_4 is derivable because of the only logical rules applications for the formula F_i (case 2), or it is derivable because of the rule $(\circ)$ application (case 1). Therefore:

- $S_5 = \Gamma_z \vdash \Gamma_z', F_i$ is derivabel, or
- $S_6 = \circ\Box\Gamma \vdash \circ\Box F_{i-1}, \circ\Box F_{i+1} \ldots, \circ\Box F_n$ is derivable.

It follows that:

- S_5 is derivable (it is either an axiom itself, or we obtain axioms in the leaves of the derivation tree of this sequent by applying the logical rules F_i), or
- The sequent S_6 is derivable, to which the rule $(\circ)$ applies.

Finally, we obtain one of the two cases:

1. $\exists z \in \{1, 2, \ldots, m\}$ such that S_6 is derivable.
   Then $S_1 = \Box\Gamma \vdash \Box F_1, \ldots, \Box F_{i-1}, \Box F_{i+1}, \ldots, \Box F_n$ is also derivable.
2. $\forall z \in \{1, 2, \ldots, m\}$ $\Gamma_z \vdash \Gamma_z', F_i$ is derivable. Then $\Gamma \vdash F_i$ is also derivable. $\Box$

Lemma 3. *If the sequent $S = \Box\Gamma \vdash \Box F_1, \ldots, \Box F_n$ is derivable from the fragment α in the calculus $G_L PLTL$, then there exists such an i that the sequent $S_1' = \Box\Gamma \vdash F_i$ or $S_1'' = \Box\Gamma \vdash$ is also derivable.*

Proof. From Lemma 2, we know that if S is derivable, then

1. The sequent $\Box\Gamma \vdash \Box F_1, \ldots, \Box F_{i-1}, \Box F_{i+1}, \ldots, \Box F_n$ is derivable, or
2. The sequent $\Gamma \vdash F_i$ is derivable

Let us assume that there exists such an i that $S_2 = \Gamma \vdash F_i$ is derivable (case 2 of Lemma 2). Then we will show that S_1' is also derivable.

Apply the rule $(\Box \vdash)$ to all the formulas from $\Box\Gamma$ and $(Weak)$ to the sequent S_1':

$$\cfrac{(\Box\vdash)*l \quad \cfrac{\vdots}{\;}}{S_1' = \Box\Gamma \vdash F_i}$$

$$(Weak)\;\cfrac{\cfrac{\ldots}{\Gamma \vdash F_i}}{\circ\Box\Gamma, \Gamma \vdash F_i} \qquad \vdots \qquad (\Box\vdash)*l\;\cfrac{\vdots}{S_1' = \Box\Gamma \vdash F_i}$$

We obtain $\Gamma \vdash F_i$ (which is derivable), which means that S_1' is also derivable.

If such an i does not exist, we will show that S_1'' is also derivable. In this case, case 1 of Lemma 2 applies. For each i, $\Box F_i$ is unnecessary and can be removed from the derivation tree using the weakening rule. After removing all $\Box F_i$, the derivable sequent S_1'' remains. $\Box$

2 Loop-check free sequent calculi for the fragment

Definition 5. Rule $(\vdash \Box_L *)$.

$$(\vdash \Box_L *) \qquad \frac{\Box\Gamma \vdash B_1 \quad || \qquad \dots \qquad || \quad \Box\Gamma \vdash B_m \quad || \qquad \Pi, \Box\Gamma \vdash \Sigma}{\Pi, \Box\Gamma \vdash \Sigma, \Box B_1, \dots, \Box B_m},$$

here $n, m \in \mathbb{N}$, $\Box\Gamma = \Box A_1, \dots, \Box A_n$. Π, Σ – sets of atomic, non-modal formulas.

Note: $(\vdash \Box_L *)$ is or-type rule, which means that sequent is derivable if at least one (not all) of the or-type rule premise is derivable for every or-type rule application.

Definition 6. $PLTL - F_1$ is a sequent calculus for PLTL with axiom $\Gamma, F \vdash \Delta, F$ and rules $(\vee \vdash), (\vdash \vee), (\wedge \vdash), (\vdash \wedge), (\neg \vdash), (\vdash \neg), (\Box \vdash), (\vdash \Box_{L^*}), (\circ)$.

Lemma 4. *Rule $(Weak)$ is admissible in the sequent calculus $PLTL - F_1$.*

The proof using mathematical induction goes straightforward from the fact, that every $(Weak)$ rule application may be lifted by one level in the derivation tree without loosing derivability.

Theorem 1. *A sequent S from the fragment α is derivable in calculus $PLTL - F_1$ if and only if it is derivable in calculus $G_L PLTL$.*

Let us denote $\Box\Gamma = \Box G_1, \Box G_2, \dots, \Box G_k$. Π, Σ are sets of atomic formulas only.

Proof. **(=>)**. Suppose that the sequent S is derivable in the calculus $PLTL - F_1$, then we will demonstrate that S is also derivable in the calculus $G_L PLTL$. If M_1 is sequent S derivation tree in the $PLTL - F_1$, we construct the tree M_2 as follows:

1. If we encounter a rule from the set
   $T = \{(\vee \vdash), (\vdash \vee), (\wedge \vdash), (\vdash \wedge), (\circ), (\neg \vdash), (\vdash \neg), (\Box \vdash)\}$
   we leave this derivation step.
2. Suppose $S' = \Pi, \Box\Gamma \vdash \Sigma, \Box F_1, \dots, \Box F_n$ is the conclusion of the rule $(\vdash \Box_{L*})$ application. Since the sequent is derivable in the calculus $PLTL - F_1$, one of the OR branches must be derivable.
   (a) Suppose that the branch $\Box\Gamma \vdash F_i$ is derivable. If we encounter the rule $(\vdash \Box_{L*})$ applictation in the derivation tree M_1, then the branch B :

$$\frac{\overset{\dots}{\Box\Gamma \vdash F_1} \quad || \qquad \dots \quad || \qquad \overset{\dots}{\Box\Gamma \vdash F_i} \quad || \qquad \dots \quad || \qquad \overset{\dots}{\Pi, \Box\Gamma \vdash \Sigma}}{S' = \Pi, \Box\Gamma \vdash \Sigma, \Box F_1, \dots, \Box F_n},$$

is changed by the branch B' :

$$
(Weak)\,\dfrac{(Weak)\,\dfrac{(\vdash\Box)\,\dfrac{\dfrac{\text{(derivable)}}{S_1^* = \Box\Gamma \vdash F_i} \qquad k*(\Box\vdash)\,\dfrac{\begin{array}{c}(\circ)\,\dfrac{\dfrac{\oplus - \text{loop axiom}}{S_2^* = \Box\Gamma \vdash \Box F_i}}{\Gamma, \circ\Box\Gamma \vdash \circ\Box F_i} \\ \vdots\end{array}}{\Box\Gamma \vdash \circ\Box F_i}}{\Box\Gamma \vdash \Box F_i}}{S_2 = \Pi, \Box\Gamma \vdash \Sigma, \Box F_i}}{S' = \Pi, \Box\Gamma \vdash \Sigma, \Box F_1, \dots, \Box F_n}
$$

(b) If there is no such branch $\Box\Gamma \vdash F_i$ that should be proven, then $\Pi, \Box\Gamma \vdash \Sigma$ can be proven. In this case, we replace branch B of the derivation tree M_1 with branch B'' :

$$
(Weak)\,\dfrac{\Pi, \Box\Gamma \vdash \Sigma}{S' = \Pi, \Box\Gamma \vdash \Sigma, \Box F_1, \dots, \Box F_n}
$$

Here, $k*(\Box\vdash)$ means that rule T was applied k times. After replacing every rule $(\vdash \Box_{L^*})$ application, we will get derivation tree M_2. All leaves of the derivation tree M_2 contains axioms or loop axioms (see case S_2^*). M_2 contains only calculus $G_L PLTL$ rules and $(Weak)$ which is admissible, therefore, sequent S is derivable in $G_L PLTL$.

(<=) Suppose that the sequent S is derivable in the calculus $G_L PLTL$, then we will show that S is also derivable in calculus $PLTL - F_1$. If M_1 is sequent S derivation tree in the $G_L PLTL$, we construct the tree M_2 as follows:

1. If we encounter a rule from the set in the derivation tree M_1 $T = \{(\vee\vdash), (\vdash\vee), (\wedge\vdash), (\vdash\wedge), (\circ), (\neg\vdash), (\vdash\neg), (\Box\vdash), (Weak)\}$ we leave this derivation step.
2. Let $S' = \Pi, \Box\Gamma \vdash \Sigma, \Box F_1, \dots, \Box F_n$ is a conclusion of the rule $(\vdash\Box)$ application. We know that sequent S' is derivable in the $G_L PLTL$ calculus.
3. According to Lemma 1, if sequent S' is derivable in the $G_L PLTL$ calculus, then:
   a) $\Pi, \Box\Gamma \vdash \Sigma$ is derivable, or
   b) $\Box\Gamma \vdash \Box F_1, \dots, \Box F_n$ is derivable.
4. Suppose case 3a). If the sequent $\Pi, \Box\Gamma \vdash \Sigma$ is derivable in calculus $G_L PLTL$, then the existing derivation branch is replaced with the derivation branch:

$$
(Weak)\,\dfrac{\begin{array}{c}\dots \\ \Pi, \Box\Gamma \vdash \Sigma\end{array}}{S' = \Pi, \Box\Gamma \vdash \Sigma, \Box F_1, \dots, \Box F_n}
$$

   In the calculus $PLTL - F_1$, the derivation of this sequent is the same, because there are no formulas bounded by the $\Box$ operator in the succedent, and rule $(\vdash\Box)$ cannot be applied.
5. Suppose case 3b). If the sequent $\Box\Gamma \vdash \Box F_1, \dots, \Box F_n$ is derivable, we apply Lemma 3 and obtain two possible cases:
   a) $S_1'' = \Box\Gamma \vdash F_i$ is derivable, or
   b) $S_2'' = \Box\Gamma \vdash$ is derivable.
6. Suppose case 5a). If S_1'' is derivable, the branch of the derivation tree B :

$$(\vdash \Box)\ \frac{\overset{\dots}{\Pi, \Box\Gamma \vdash \Sigma, \Box F_1, \dots, F_i, \dots, \Box F_n} \qquad \overset{\dots}{\Pi, \Box\Gamma \vdash \Sigma, \Box F_1, \dots, \circ\Box F_i, \dots, \Box F_n}}{S' = \Pi, \Box\Gamma \vdash \Sigma, \Box F_1, \dots, \Box F_n}$$

is changed by the branch B' :

$$(\vdash \Box_{L*})\ \frac{\dots \quad || \quad \dfrac{(\text{derivable in } PLTL - F_1)}{S_1'' = \Box\Gamma \vdash F_i} \quad || \quad \dots}{S' = \Pi, \Box\Gamma \vdash \Sigma, \Box F_1, \dots, \Box F_n}$$

7. Suppose case 5b). If S_2'' is derivable, S' is derivable in $PLTL - F_1$, since we can apply $(Weak)$ to obtain derivable S_2'', which is derivable without $(\vdash \Box)$.

   Derivation tree M_2 contains only calculus $PLTL - F_1$ rules and $(Weak)$ which is admissible in $PLTL - F_1$, therefore, sequent S is derivable in $PLTL - F_1$.

□

Definition 7. Sequent S is called **primary** if it is of the shape $\Sigma_1, \circ\Gamma_1 \vdash \Sigma_2, \circ\Gamma_2$, where multisets $\Sigma_1, \circ\Gamma_1, \Sigma_2, \circ\Gamma_2$ may be empty. Σ_1, Σ_2 consists of only atomic formulas, $\Sigma_1 \cap \Sigma_2 = \emptyset$. $\circ\Gamma_1, \circ\Gamma_2$ consists of only formulas of shape $\circ F, \circ\Gamma_1 \cap \circ\Gamma_2 = \emptyset$.

Definition 8. Rule $(\circ_p)$ is a rule

$$(\circ_p)\ \frac{\Gamma \vdash \Delta}{\Sigma, \circ\Gamma \vdash \circ\Delta, \Pi}$$

here Σ, Π contain only atomic formulas, i.e. $\Sigma, \circ\Gamma \vdash \circ\Delta, \Pi$ is a primary sequent [2].

Definition 9. $PLTL - F_2$ is a sequent calculus for PLTL with axiom $\Gamma, F \vdash \Delta, F$ and rules $(\vee\vdash), (\vdash\vee), (\wedge\vdash), (\vdash\wedge), (\neg\vdash), (\vdash\neg), (\Box\vdash), (\vdash\Box_{L^*}), (\circ_p)$.

Lemma 5. *Rule $(Weak)$ is admissible in the sequent calculus $PLTL - F_2$.*

The proof using mathematical induction goes straightforward from the fact, that every $(Weak)$ rule application may be lifted by one level in the derivation tree without loosing derivability.

Theorem 2. *A sequent S from fragment α is derivable in calculus $PLTL - F_1$ if and only if it is derivable in calculus $PLTL - F_2$.*

Proof. Π, Δ denote finite sets of formulas, which may be empty.

(=>) Suppose that the sequent S is derivable in the calculus $PLTL - F_1$, then we will show that S is also derivable in the calculus $PLTL - F_2$. If M_1 is sequent S derivation tree in the $PLTL - F_1$, we construct the tree M_2 as follows:

1. If we encounter a rule $(\circ)$ application in the tree that was not applied to the primary sequent, then we change the branch B :

$$(\circ)\ \frac{\overset{\dots}{S_1 = \Delta_1 \vdash \Delta_2}}{S' = \Pi, \circ\Delta_1 \vdash \Sigma, \circ\Delta_2} \quad \text{with the branch } B' : \quad (Weak)\ \frac{(\circ_p)\ \dfrac{\overset{\dots}{S_1 = \Delta_1 \vdash \Delta_2}}{S'' = \circ\Delta_1 \vdash \circ\Delta_2}}{S' = \Pi, \circ\Delta_1 \vdash \Sigma, \circ\Delta_2}$$

2. If we encounter a rule $(\circ)$ application in the tree that was applied to the primary sequent – rule $(\circ)$ application is replaced by the $(\circ_p)$ without any modifications.
3. If we encounter a rule from the set
$T = \{(\vee \vdash), (\vdash \vee), (\wedge \vdash), (\vdash \wedge), (\neg \vdash), (\vdash \neg), (\Box \vdash), (\vdash \Box_{L^*})\}$
we leave this derivation step.

After replacing every rule $(\circ)$ application by $(\circ_p)$, we will get derivation tree M_2. Derivation tree M_2 contains only calculus $PLTL - F_2$ rules and $(Weak)$, which is admissible in $PLTL - F_2$, therefore, sequent S is derivable in $PLTL - F_2$.

(<=) Suppose sequent S is derivable in the calculus $PLTL - F_2$. Sequent S derivation tree in $PLTL - F_1$ is obtained from its derivation tree in $PLTL - F_2$ by replacing every rule $(\circ_p)$ application by $(\circ)$ without any other modification. $\Box$

Lemma 6. *If sequent S from fragment α is derivable in $PLTL - F_2$ calculus, then there exists a derivation tree in which there are no rule $(\circ_p)$ applications.*

Proof. Suppose $S = \Pi, \Box A_1, \ldots, \Box A_n \vdash \Sigma, \Box F_1, \ldots, \Box F_n$ is derived in $PLTL - F_2$ and the rule $(\circ_p)$ was used in its derivation, then we have the fragment:

$$
(T) * k \cfrac{\vdots \qquad (\circ_p) \cfrac{\cfrac{\cdots}{S_1 = \Box A_1, \ldots, \Box A_n \vdash}}{\cfrac{S_p = \Pi', \circ\Box A_1, \ldots, \circ\Box A_n \vdash \Sigma'}{\vdots}} \qquad \vdots}{S = \Pi, \Box A_1, \ldots, \Box A_n \vdash \Sigma, \Box F_1, \ldots, \Box F_n}
$$

S_p is derivable since only invertable rules were applied between S and S_p (rule $(\circ_p)$ was not applied). Where is no forumalas bounded by $\Box$ in the succedent of S_p, since they were removed by the rule $(\vdash \Box_{L^*})$ applications. If $S_1 = \Box A_1, \ldots, \Box A_n \vdash$ is derivable, then $S_3 = A_1, \ldots, A_n, \circ\Box A_1, \ldots, \circ\Box A_n \vdash$ is also derivable (since it is obtained from S_1 by applying only invertible $(\Box \vdash)$ rules). $A_1, \ldots, A_n$ are non-modular formulas, so either $S_2 = A_1, \ldots, A_n \vdash$ is derivable, or $\circ\Box A_1, \ldots, \circ\Box A_n \vdash$ is derivable. In the second case, applying $(\circ_p)$ would result in a loop, which lead S_1 is non derivable (contradiction). Thus, if S_1 is derivable, then S_2 is also derivable (which is derivable without the $(\circ_p)$ rule, since it contains only non-modular formulas).

Then we replace the fragment of the derivation tree of sequence S with a fragment without applying the rule $(\circ_p)$:

$$
(Weak) \cfrac{(\Box \vdash) * n \cfrac{(Weak) \cfrac{\cfrac{\cdots}{S_2 = A_1, \ldots, A_n \vdash}}{\cfrac{A_1, \ldots, A_n, \circ\Box A_1, \ldots, \circ\Box A_n \vdash}{\vdots}}}{\Box A_1, \ldots, \Box A_n \vdash}}{S = \Pi, \Box A_1, \ldots, \Box A_n \vdash \Sigma, \Box F_1, \ldots, \Box F_n}
$$

Constructed derivation tree contains only calculus $PLTL - F_2$ rules and $(Weak)$ rule, which is admissible, but omitting $(\circ_p)$ applications. Therefore, we may construct sequent S derivation tree in $PLTL - F_2$ in which the rule $(\circ_p)$ is not used at all. $\Box$

Definition 10. $PLTL - F_3$ is a sequent calculus for PLTL with axiom $\Gamma, F \vdash \Delta, F$ and rules $(\vee \vdash), (\vdash \vee), (\wedge \vdash), (\vdash \wedge), (\neg \vdash), (\vdash \neg), (\Box \vdash), (\vdash \Box_{L^*})$.

Note: Sequent calculus $PLTL - F_3$ does not contain any rule for the "next" operator – neither $(\circ)$, nor $(\circ_p)$.

Premises of the rule $(\vdash \Box_{L^*})$ is simplier than rule conclusion. Premise of the rule $(\Box \vdash)$ contains extended formulas (of the sahpe $\circ\Box G$) from the rule conclusion, but for these formulas there is no rule to be applied (due to $(\circ_p)$ elimination). Summarizing this, it is impossible to construct any loop (positive loop-axioms or negative non-derivable loops) in a derivation tree constructed according $PLTL - F_3$.

Theorem 3. *A sequent S from fragment α is derivable in calculus $PLTL - F_2$ if and only if it is derivable in calculus $PLTL - F_3$.*

The poof goes straightforward from the Lemma 6.

Theorem 4. *Calculi $G_L PLTL$ and $PLTL - F_3$ are equivalent in fragment α.*

Proof. Since we know that $PLTL - F_3 \iff PLTL - F_2$, $PLTL - F_2 \iff PLTL - F_1$ and $PLTL - F_1 \iff G_L PLTL$ in fragment α, then $G_L PLTL \iff PLTL - F_3$ in fragment α. $\Box$

Conclusions

$PLTL - F_3$ is a totally loop-check free sequent calculus for the PLTL fragment α, since it is impossible to construct a loop (neither loop axiom, neither non-derivable loop) without rule $(\circ)$, $(\circ_p)$ applications.

$PLTL - F_3$ is more efficient than $G_L PLTL$ for sequents from the fragment α, since derivation trees in $PLTL - F_3$ are smaller, and there is no need to apply loop check, which usually takes most of the inference time.

References

[1] R. Alonderis, R. Pliuškevičius. Sequent systems for PLTL. In *Lithuanian Mathematical Collection*, pp. 1–5. Vilnius University Press, Vilnius, Lithuania, 2013.

[2] A. Birštunas. Restrictions for loop-check in sequent calculus for temporal logic. *Lith. Math. J.*, **48**:269–274, 2008.

[3] I. Kokkinis, T. Studer. Cyclic proofs for linear temporal logic. In *Concepts of Proof in Mathematics, Philosophy, and Computer Science*, pp. 171–192. De Gruyter, Berlin, Germany, 2016.

[4] R. Pliuškevičius. Loop-free verification of termination of derivation for a fragment of dynamic logic. In *Lithuanian Mathematical Collection*, pp. 283–287. Vilnius University Press, Vilnius, Lithuania, 2008.

[5] R. Pliuškevičius. Method of marks for propositional linear temporal logic. *Lith. Math. J.*, **55**:46–50, 2014.

REZIUMĖ

Becikliai sekvenciniai skaičiavimai unariniam laiko logikos fragmentui

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Šiame darbe nagrinėjamas efektyvaus sekvencinio skaičiavimo konstravimas pasirinktam tiesinės teiginių laiko logikos (PLTL) fragmentui, praplečiant klasikinio skaičiavimo idėjas, aptartas [1], ir remiantis ankstesniais tyrimais apie ciklus PLTL logikoje. Apibrėžtas unarinis tiesinės laiko logikos fragmentas, kuriame formulėse gali būti daugiausiai vienas išorinis $\Box$ („visada") operatorius. Fragmentai paprastai analizuojami siekiant apibrėžti efektyvesnius skaičiavimus formulėms, priklausančioms fragmentui, ypač kai tai nėra įmanoma visai logikai (tokia strategija naudojama [3, 4]). Pristatoma nauja or-tipo sukcedente taiyklė ($\vdash \Box_L *$). Autoriai siūlo naują sekvencinį skaičiavimą $PLTL - F_1$ fragmentui α, kuris pašalina ciklines aksiomas ir pagerina išvedimo efektyvumą. Be to, autoriai pristato ir pilnai beciklį skaičiavimą $PLTL - F_3$. Ciklų pašalinimas buvo pasiektas įrodžius ($\vdash \Box$) ir ($\circ$) taisyklių taikymo apribojimus.

Raktiniai žodžiai: ciklai; sekvenciniai skaičiavimai; fragmentai; PLTL