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Retracted:

Digit staircases and a general identity for multiplication by $(b-1)$

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Abstract. Multiples of nine in base 10 form a staircase pattern: the tens digit increases, the ones digit decreases, and their sum is always nine. We establish a general identity showing that in any base b, multiplication by $(b-1)$ produces the same phenomenon. Examples across numeral systems illustrate its link to digital roots and modular arithmetic, while its pedagogical and historical significance highlights how an elementary curiosity fits within number theory and mathematics education.

Keywords: digital root; divisibility by nine; modular arithmetic; number patterns; base-b representation
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1 Introduction

Among the multiplication tables, the row for nines in base 10 has long been noticed for its unusual regularity [2]. Each product displays a striking symmetry: the tens digit increases step by step, the ones digit decreases in the same fashion, and the two digits together always sum to nine. For instance,

$$\begin{aligned} 1 \times 9 = 09 \quad &\to \quad 0 + 9 = 9, \\ 2 \times 9 = 18 \quad &\to \quad 1 + 8 = 9, \\ 3 \times 9 = 27 \quad &\to \quad 2 + 7 = 9, \\ &\vdots \\ 8 \times 9 = 72 \quad &\to \quad 7 + 2 = 9, \\ 9 \times 9 = 81 \quad &\to \quad 8 + 1 = 9, \\ 10 \times 9 = 90 \quad &\to \quad 9 + 0 = 9. \end{aligned}$$

This pattern continues well beyond the basic table. For example,

$$\begin{aligned} 11 \times 9 = 99 \quad &\to \quad 9 + 9 = 18,\ 1 + 8 = 9, \\ 12 \times 9 = 108 \quad &\to \quad 1 + 0 + 8 = 9. \end{aligned}$$

At first sight, this may appear as a numerical curiosity, but it reflects deeper principles. The persistence of digit sums equal to nine is explained by the idea of *digital roots* and by congruences modulo 9 [3]. More generally, the increasing–decreasing digit structure arises from a compact identity that extends to all numeral bases.

The purpose of this article is to establish and explore this identity. We show that in any base b, multiplication by $(b-1)$ produces an analogous “staircase” structure. We examine its formulation, provide checks in different bases, and connect the result to divisibility rules, modular arithmetic, and elementary error-detection schemes. In addition, we reflect briefly on its educational and historical significance, situating a familiar classroom observation within the broader context of number theory.

2 The digital root and divisibility by 9

The process we have been using—adding digits until only a single digit remains—is known as finding the *digital root*. This notion appears in number theory texts such as [5, 4]. For example:

$$\begin{aligned} \mathrm{dr}(108) &= 1 + 0 + 8 = 9, \\ \mathrm{dr}(99) &= 9 + 9 = 18 \ \to \ 1 + 8 = 9. \end{aligned}$$

The digital root is always between 1 and 9 (or 0 if we start with 0).

We can make this idea precise for *any* numeral system, not just base 10:

Proposition 1 [Digital root in any base]. *Let* $b \geqslant 2$. *For any integer* $n \geqslant 0$,

$$\mathrm{dr}_b(n) = \begin{cases} 0, & n = 0, \\ 1 + ((n-1) \bmod (b-1)), & n > 0. \end{cases}$$

In particular, $\mathrm{dr}_b(n) \equiv n \pmod{b-1}$ *and takes values in* $\{1, \dots, b-1\}$ *(or* 0 *only when* $n = 0$*).*

Proof. In base b, writing $n = \sum d_i b^i$ gives $b \equiv 1 \pmod{b-1}$, hence

$$\sum d_i b^i \equiv \sum d_i \pmod{b-1}.$$

Repeated digit-summing does not change the residue modulo $(b-1)$, so the unique representative in $\{1, \ldots, b-1\}$ matching $n \pmod{b-1}$ is $1 + ((n-1) \bmod (b-1))$ (and 0 only when $n = 0$). □

An important principle follows: *a number is divisible by* 9 *if and only if its digit sum is divisible by* 9. This explains why every multiple of 9 eventually collapses to a digital root of 9.

Why does this work? Consider the number 234. In expanded form,

$$234 = 2 \times 100 + 3 \times 10 + 4.$$

Since $100 = 99 + 1$ and $10 = 9 + 1$, each power of ten is "one more than a multiple of 9". Thus, when we divide by 9, each power of ten behaves like 1. So 234 has the same remainder as the sum of its digits:

$$234 \equiv 2 + 3 + 4 \equiv 9 \pmod{9}.$$

That is, 234 and $2 + 3 + 4$ both leave the same remainder upon division by 9. Because 9 itself is divisible by 9, the original number 234 must be divisible as well. Here the phrase "modulo 9", or simply "mod 9", means that we are comparing only the remainders upon division by 9, and the symbol $\equiv$ (triple bar) denotes *congruence.*

This argument works for every integer: a number and the sum of its digits are congruent modulo 9. That is why digital roots give a quick and reliable divisibility test. Historically, this method has been known for centuries as *casting out nines*, a traditional way to check arithmetic by comparing the digital roots on both sides of a computation [9, 7].

Remark 1 [Beyond casting out nines]. In base 10, a stronger checksum works modulo 81. If $n = \sum_{i=0}^{k} d_i 10^i$ with digits $d_i \in \{0, \ldots, 9\}$, then

$$10^i \equiv (1+9)^i \equiv 1 + 9i \pmod{81},$$

so

$$n \equiv \sum_{i=0}^{k} d_i(1 + 9i) \pmod{81}.$$

Example. Numbers 5281 and 4210 both have digit sum 16 (so both $\equiv 7 \pmod{9}$), but modulo 81:

$$5281 \equiv 16, \qquad 4210 \equiv 79.$$

Thus the refined check distinguishes them.

Quick checks:

$$\begin{aligned} 99 &\to 9 + 9 = 18 \ \to \ 1 + 8 = 9, && \text{divisible by 9.} \\ 108 &\to 1 + 0 + 8 = 9, && \text{divisible by 9.} \\ 123456 &\to 1 + 2 + 3 + 4 + 5 + 6 = 21, \ \ 2 + 1 = 3 && \text{remainder 3 when divided by 9.} \end{aligned}$$

3 The "staircase" identity

Another way to see the behavior of the 9 times table is to track its digits systematically. From 1×9 up through 10×9, the tens digits increase step by step from 0 to 9, while the ones digits decrease from 9 down to 0. Placed in two rows, they form two ladders moving in opposite directions:

$$\begin{array}{cccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \rightarrow \\ \leftarrow & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & \end{array}$$

Each product can be written in the compact form

$$9n = 10(n-1) + (10-n), \quad n = 1, 2, \ldots, 10,$$

where the first term supplies the tens digit ($n-1$ climbing upward), and the second term supplies the ones digit ($10-n$ stepping downward). Together they always sum to 9. For example, with $n = 7$ we have $9n = 10(n-1) + (10-n) = 10 \cdot 6 + 3 = 63$.

The staircase is not confined to the first ten multiples. From 11×9 through 20×9, the same structure reappears in the last two digits, now accompanied by an extra hundreds digit:

$$\begin{array}{cccccccccccc} & 11(9) & 12(9) & 13(9) & 14(9) & 15(9) & 16(9) & 17(9) & 18(9) & 19(9) & 20(9) & \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\ & \underline{9}9 & \underline{10}8 & \underline{11}7 & \underline{12}6 & \underline{13}5 & \underline{14}4 & \underline{15}3 & \underline{16}2 & \underline{17}1 & \underline{18}0 & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ & \underline{09} & \underline{10} & \underline{11} & \underline{12} & \underline{13} & \underline{14} & \underline{15} & \underline{16} & \underline{17} & \underline{18} & \rightarrow \\ \leftarrow & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & \end{array}$$

Proposition 2 [The staircase with explicit carry]. *In base b and for $1 \leqslant n \leqslant b$,*

$$(b-1)n = b(n-1) + (b-n).$$

Thus the last two digits are $(n-1, b-n)$, which always sum to $(b-1)$.

For general n, write $n = qb + m$ with $q \geqslant 0$ and $m \in \{1, \ldots, b\}$ (taking $m = b$ when n is a multiple of b). Then

$$(b-1)n = (b-1)qb + \big[b(m-1) + (b-m)\big],$$

so each block of b consecutive multiples advances the higher-order digits by $(b-1)qb$ and repeats the same two-digit staircase in the lower digits.

Proof. The two-digit case is straightforward:

$$b(n-1) + (b-n) = bn - b + b - n = (b-1)n.$$

For general $n = qb + m$, expand

$$(b-1)n = (b-1)qb + (b-1)m,$$

and apply the two-digit identity to $(b-1)m$. This yields the stated decomposition into a carried block and a repeating staircase tail. □

This repetition can be expressed in general form. Writing $n = 10k + m$ with $1 \leqslant m \leqslant 10$, we obtain

$$9n = 90k + \big[\,10(m-1) + (10-m)\,\big].$$

The term $90k$ advances the higher-order digits, while the bracketed expression encodes the staircase that repeats every decade. In fact, shifting by ten steps gives

$$9(n+10) = 9n + 90,$$

so the staircase continues with the last two digits cycling and the leading digits increasing.

Examples.

- For $n = 13$, we have $k = 1, m = 3$:
$$9n = 90(1) + [20 + 7] = 117,$$
matching $9 \times 13 = 117$.
- For $n = 134$, we have $k = 13, m = 4$:
$$9n = 90(13) + [30 + 6] = 1206,$$
matching $9 \times 134 = 1206$.
- For $n = 1347$, we have $k = 134, m = 7$:
$$9n = 90(134) + [60 + 3] = 12123,$$
matching $9 \times 1347 = 12123$.

Remark (the case $m = 10$). When $m = 10$, the formula simplifies to

$$n = 10k + 10 = 10(k+1), \qquad 9n = 90(k+1).$$

Here the staircase "resets" into the next block of ten multiples. For instance, $n = 20$ gives $9 \cdot 20 = 180 = 90 \cdot 2$, while $n = 1350$ yields $9 \cdot 1350 = 12150 = 90 \cdot 135$. In both cases the boundary case fits smoothly into the continuing pattern.

Lemma 1 [Uniqueness of the staircase]. *Fix a base $b \geqslant 2$. Among multipliers $m \in \{1, \ldots, b-1\}$, the only one for which*

$$n \mapsto mn \quad (1 \leqslant n \leqslant b)$$

produces two digits that (i) *have constant sum and* (ii) *vary monotonically in opposite directions is $m = b - 1$.*

Proof [Sketch of proof]. If the two digits of mn always sum to a constant S, then $mn \equiv S \pmod{b-1}$ for all n. As n varies, this is possible only when $m \equiv 0 \pmod{b-1}$, i.e., $m = b - 1$. For this multiplier the constant sum is $(b-1)$, and the monotone up–down pattern follows directly from the staircase identity. □

4 Beyond base 10: staircases in other number systems

The phenomenon we have described for the number 9 in base 10 is not unique to the decimal system. In fact, it arises in every positional numeral system whenever we multiply by $(b-1)$ in base b. This can be captured in a single general identity:

$$(b-1)n = b(n-1) + (b-n), \quad n \geqslant 1.$$

The first term $b(n-1)$ raises the higher digit by one step, while the second term $(b-n)$ lowers the trailing digit. Together, they always sum to $(b-1)$, explaining why multiplication by $(b-1)$ produces a staircase pattern in any base.

A note on bases. In base b, each position represents a power of b: ones, bs, b^2s, and so on; our everyday decimal system is the case $b = 10$. In base 8, for instance, the numeral 43_8 means $4\times 8+3 = 35$ in decimal. In base 12, the digits are $0, 1, \ldots, 9, A, B$, where $A = 10$ and $B = 11$; thus 92_{12} corresponds to $9 \times 12 + 2 = 110$ in decimal.

Checks. The identity holds uniformly across bases:

- Base 10, multiplier 9, $n = 4$:

$$9 \cdot 4 = 10(3) + (10-4) = 30 + 6 = 36.$$

- Base 8, multiplier 7, $n = 5$:

$$7 \cdot 5 = 8(4) + (8-5) = 32 + 3 = 35 = 43_8.$$

- Base 12, multiplier B (eleven), $n = A$ (ten):

$$B \cdot A = 12(9) + (12-10) = 108 + 2 = 110 = 92_{12}.$$

Visualizing the staircases. In each base, the same up–down structure appears:

Base 10 (*multiplier* 9).

$$\begin{array}{ccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \rightarrow \\ \leftarrow & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & \end{array}$$

Base 8 (*multiplier* 7).

$$\begin{array}{cccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \rightarrow \\ \leftarrow & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & \end{array}$$

Base 12 (*multiplier* $B = 11$).

$$\begin{array}{cccccccccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & A & B & \rightarrow \\ \leftarrow & B & A & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & \end{array}$$

In every case, the top row climbs steadily from 0 up to $(b-1)$, while the bottom row descends from $(b-1)$ back to 0. Each vertical pair sums to $(b-1)$, illustrating the universal staircase identity.

Thus, the staircase pattern is not peculiar to our familiar base 10: it is a built-in feature of positional notation itself. Whenever one multiplies by $(b-1)$ in base b, the digits form this rising-and-falling staircase, their paired rungs always totaling $(b-1)$.

5 Limitations of the staircase trick

The staircase pattern is striking, but it is also quite special. It arises only when multiplying by $(b-1)$ in base b, and it depends on the positional nature of the numeral system. In other settings, the regularity breaks down.

Not every multiplier. In base 10, 8 is close to 9, but multiplying by 8 does not yield a staircase. For instance:

$$1 \times 8 = 08, \quad 2 \times 8 = 16, \quad 3 \times 8 = 24, \quad 4 \times 8 = 32.$$

The tens digits increase, but the ones digits do not decrease in a neat sequence—they vary irregularly. Thus, the staircase is unique to the multiplier 9 in base 10 (or more generally, to $(b-1)$ in base b).

Not in non-positional systems. The effect also depends on positional notation. For example, in Roman numerals

$$\text{IX, XVIII, XXVII, XXXVI,}$$

the products of 9 by $1, 2, 3, 4$ do not reveal any staircase, because Roman numerals lack place value. The pattern is therefore a feature of positional systems such as base 10, base 8, or base 12, not of numeral systems in general [6].

Even with these restrictions, the staircase idea has found surprising uses—from mental arithmetic checks to modern error-detection schemes—applications we turn to next.

6 Applications and connections

While the staircase of nines may appear at first as a numerical curiosity, it is closely tied to deeper mathematical ideas and even to practical applications.

Error checking. For centuries, the method of "casting out nines" has been used to verify arithmetic. Suppose a student computes 23×47 and mistakenly writes 1089. The digital root of the left side is

$$\mathrm{dr}(23) \times \mathrm{dr}(47) = 5 \times 2 = 10 \to 1,$$

while the digital root of 1089 is $1 + 0 + 8 + 9 = 18 \to 9$. Since $1 \neq 9$, the product must be incorrect. The correct product, 1081, has digital root $1+0+8+1 = 10 \to 1$, agreeing with the check.

Divisibility tests. The staircase reflects the divisibility rule for 9: a number is divisible by 9 precisely when the sum of its digits is divisible by 9. For instance, 234 has digit sum $2+3+4 = 9$, so it is divisible by 9. By contrast, 235 has digit sum 10, which is not a multiple of 9, so it is not divisible by 9.

Gateway to modular arithmetic. Such patterns provide a natural entry point for teaching modular arithmetic. Since $10 \equiv 1 \pmod 9$, the number 234 is congruent to $2 + 3 + 4 = 9$ modulo 9. This observation generalizes to any base b, where $b \equiv 1 \pmod{b-1}$, explaining why the staircase identity holds in every base.

Error detection in practice. Digit sums also play a role in modern error-detection systems. For example, ISBN-10 codes for books use a check digit derived from a weighted sum of the digits modulo 11 [1]. Similarly, barcodes employ check digits based on modulo 10 arithmetic [8]. These mechanisms extend the same logic as casting out nines, showing how a classroom trick connects directly to real-world technology.

In this way, the staircase of nines links playful observation with number theory and real applications. From classroom checks to modern coding systems, a simple digit pattern reveals unexpected depth.

7 Concluding remarks

We have shown that the familiar "row of nines" hides a simple mechanism: in any base b, multiplying by $(b-1)$ decomposes as $b(n-1)+(b-n)$, forcing a two–digit staircase whose rungs sum to $(b-1)$. This explains the repeating tail across decades, isolates why no other multiplier produces the same symmetry, and links the classroom trick directly to congruences and digit–sum checks. Beyond its charm, the identity offers a compact way to teach modular arithmetic and to connect elementary numeracy with modern checksum ideas.

Several extensions suggest themselves—for example, weighted staircases modulo $(b-1)^2$ (such as our mod-81 check in base 10) and visualizations of staircase tails across different bases—which we leave for future exploration.

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Conflict of interest

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REZIUMĖ

Skaitmenų laiptai ir bendra tapatybė dauginant iš $(b-1)$

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Straipsnyje nagrinėjamas devynetų „laiptų“ reiškinys: dešimtainėje sistemoje dešimtys didėja, vienetai mažėja, o jų suma visada lygi 9. Pateikiame bendrą tapatybę, rodančią, kad bet kurioje bazėje b dauginant iš $(b-1)$ gaunamas tas pats reiškinys. Aptariame pavyzdžius įvairiose skaičiavimo sistemose, ryšius su skaitmenų šaknimis ir modulio aritmetika, bei išryškiname pedagoginę ir istorinę šios tapatybės reikšmę.

Raktiniai žodžiai: skaitmeninė šaknis; dalumas iš devynių; modulinė aritmetika; skaičių dėsningumai; b-nė skaičiavimo sistema