ESTIMATION OF DEFAULT PROBABILITY FOR LOW DEFAULT PORTFOLIOS

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Abstract. *This article presents several approaches to estimating the probabilities of default for low default portfolios, their advantages and disadvantages, and provides exemplary calculations using data of one external credit register of Lithuania. The results show that three approaches seem to be most appropriate: those of K. Pluto and D. Tasche (2005) without correlation, and those of N. M. Kiefer (2006) and A. Forrest (2005) without correlation. The first one could be easily implemented by banks; however, if the ordinal ranking of obligors is incorrect, then the monotony of probabilities of default is not ensured. The same problem exists with the second approach. The A. Forrest (2005) approach without correlation ensures the monotony of default probabilities and allows estimating conservative PDs; however, it requires programming skills, otherwise iterative recalculation will be very time-consuming.*

Key words: *low-default portfolios, probability of default*

Introduction

According to the New Capital Adequacy Directive, banks applying the internal-rating-based approach have to estimate their own probabilities of default (thereinafter PDs) for their obligors. However, in practice a substantial part of bank assets often consists of low default portfolios. This impedes not only the development of a statistical scoring model, but also the estimation of PDs and other credit risk parameters, as well as the validation process. The key concern for regulators is that credit risk might be underestimated because of data scarcity. Supervisory requirements (Basel II, New Capital Adequacy Directive and local supervisory regulations) provide no excuse or relief for low default portfolios (thereinafter LDP). To avoid excluding LDPs from the internal ratings based approach, it is recommended to use some data-enhancing tools. Banks should put more emphasis on alternative data sources, apply alternative methods with more emphasis on qualitative tools. At the same time, the Basel Committee on Banking Supervision (BCBS, 2005b) has advised to use larger margins of conservatism if an uncertainty in PDs estimated for LDPs remains.

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Defining LDP is not a straightforward task. Different authors and supervisory institutions have used different definitions of LDP (see BBA, LIBA, ISDA 2004; 2005; FSA 2005; CEBS 2006; Bank of Lithuania 2006a). For example, the Bank of Lithuania defines LDP as a portfolio with only few actual defaults, or a portfolio free from any actual defaults. As all these definitions have the drawback of being judgmental and introduce the question of degree, the FSA (2006) proposed using a concrete number of defaults in order to define LDP without taking into account the total portfolio size. It was proposed to use 20 defaults on the rating level; this definition will be used further in this article.

Till now, the problem of LDPs has not been analysed by Lithuanian researchers. L. Dzidzevičiūtė (2010b) only mentioned the LDP problem in the context of statistical scoring model development.

Even though there is a range of statistical techniques available to choose from, there is no consensus on the best technique to estimate PDs for LDPs. Various authors have proposed the approaches related to rating transition matrices and bootstrapping, the distribution of numbers of defaults and simulation, the CAP and ROC curves, macroeconomic variables, etc. The purpose of this article is to analyze various approaches to PD estimation for LDPs, their advantages and disadvantages, to provide a comparative analysis and exemplary calculations. The LDP problem has an effect on the statistical scoring model development, the estimation of credit risk parameters and their validation. The Basel II determines three credit risk parameters needed to calculate risk-weighted assets and expected loss amount; these are the probability of default (PD), loss-given default (LGD) and exposure at default (EAD). The article focuses on the estimation of only one risk parameter – PD.

Part 1 of the article shows the spheres of the LDP problem, approaches to PD estimation for LDPs, and Part 2 presents exemplary calculations with data of one external credit register of Lithuania, defining LDP as a rating having no more than 20 actual defaults.

1. Comparative analysis of PD estimation approaches to LDPs

1.1. The spheres of LDP problem

Often, the insufficiency of defaults impedes the development of statistical scoring models. However, if obligors are assigned to ratings based on the result of expert scoring models, the LDP problem is not actual (see Fig. 1). When obligors are already assigned to ratings, banks applying the internal ratings based approach for capital adequacy calculation purposes have to estimate PD for each rating (Directive, 2006). To estimate the rating PD is recommended even for banks that not apply internal ratings based approach (Dzidzevičiūtė, 2010b). Rating PD may be estimated applying various methods (see Table 1).

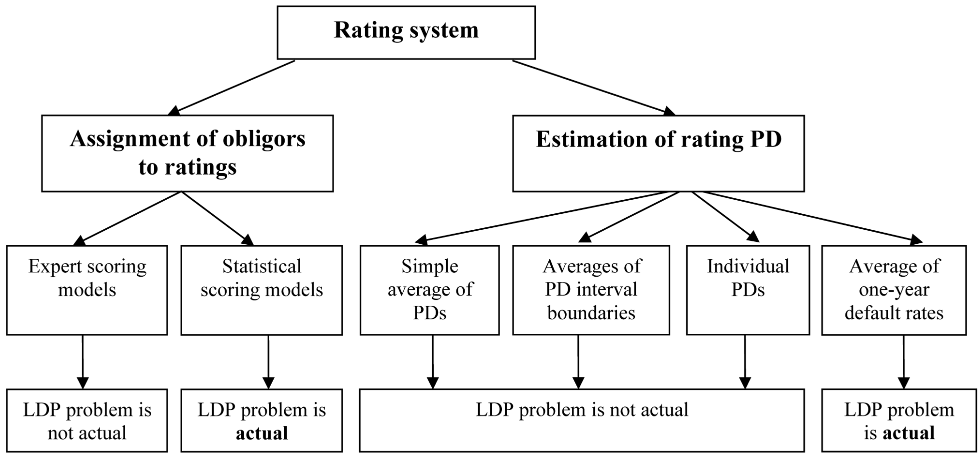

FIG. 1. **LDP problem**
Source: compiled by the author.

The LDP problem is actual only when estimating rating PD from long-run averages of one-year default rates (i. e. PD(4)). If bank assigns obligors to ratings based on the score of the expert scoring model or on the score of the statistical scoring model not allowing to estimate individual PDs (e.g., discriminant analysis), PD(4) is the only possible method of PD estimation. As very often in better ratings the number of actual defaults is too low, banks have to find the way how to solve the LDP problem.

Statistical scoring models are not very popular among the banks of Lithuania. The survey related to commercial banks and branches of foreign banks operating in Lithuania has shown that statistical scoring models are applied only in four banks, and only one bank applies statistical scoring models allowing to estimate individual PD (logistic regression) (Dzidzevičiūtė, 2010c). Thus, for the majority of banks in Lithuania, the most appropriate method to estimate PDs for ratings is PD(4) (see Table 1).

Further in this article, the LDP problem is analyzed only as regards PD (4) estimation.

1.2. Approaches based on rating transition matrices and bootstrapping

PDs for ratings can be estimated from upgrades and downgrades to other ratings during a certain period of time. There are two ways to estimate migration matrices (Schuermann, Hanson, 2004): the cohort and the duration approaches. In simple terms, the cohort approach just takes observed proportions from the beginning of a year to the end (for the case of annual migration matrices) as estimates of migration probabilities; any movements within a year are not accounted for, i.e.: $P_{ij} = \frac{N_{ij}}{N_i}$, where P_{ij} is migration

TABLE 1. **Methods to estimate PDs for ratings**

<table>
<tr><th>Method</th><th>Formula</th><th>Comments</th></tr>
<tr><td>Simple average of PDs</td><td>$PD(1)_{rating} = \frac{\sum_{i=1}^{n} PD_i}{n}$
PD_i is individual PD of obligor i assigned to that rating;
n is number of obligors assigned to that rating</td><td rowspan="4">Methods may be used only applying statistical scoring model allowing to estimate individual *PD* (e.g., logistic, probit regression etc.)</td></tr>
<tr><td>Arithmetical average of PD interval boundaries</td><td>$PD(2)_{rating} = \frac{PD_{upper} + PD_{lower}}{2}$
PD_{upper} – upper PD boundary of individual PD interval defined for that rating;
PD_{lower} – lower PD boundary of individual PD interval defined for that rating</td></tr>
<tr><td>Geometrical average of PD interval boundaries</td><td>$PD(3)_{rating} = \sqrt{(PD_{upper} \cdot PD_{lower})}$</td></tr>
<tr><td>Individual PD</td><td></td></tr>
<tr><td>Average of one-year default rates</td><td>$PD(4)_{rating} = \frac{\sum_{i=1}^{n} PD_i}{n}$
$PD_i = \frac{number\_of\_defaults\_during\_the\_year}{number\_of\_obligors\_at\_the\_beginning\_of\_the\_year}$
PD_i – rating's default rate for year *i*;
n – number of years used to estimate PD</td><td>Method may be used applying statistical and / or expert scoring models</td></tr>
</table>

Source: Dzidzevičiūtė (2010b).

probability from rating *i* to rating *j* during a year, N_{ij} is the total number of transitions from rating *i* to rating *j* during a year, and N_i is the number of obligors at the beginning of a year.

The duration approaches, on the contrary, count all rating changes during a year. The probabilities to migrate to default status estimated applying duration approaches may be used as PDs for capital requirements calculation purposes. Lando and Skφdeberg (2002) propose two duration approaches: parametric, based on time-homogeneity, and nonparametric, based on time non-homogeneity. Applying one of duration approaches, it is possible to get migration probabilities even for ratings without actual defaults and use them as PDs, so both duration approaches are recommended for LDPs. The research of Y. Jafry and T. Schuermann (2004) has shown that even the second estimator imposes fewer assumptions on the data generating process by allowing for time non-homogeneity while fully accounting for all movements within a year; both approaches yield statistically indistinguishable transition matrices. However, computationally, the second, non-parametric, estimator is more intensive than the first one, so the authors recommend the first duration approach.

Meanwhile, applying the cohort approach we will not get PDs for zero defaults; in this approach, the probability to migrate to a default status is equal to the actual default rate. Schuermann and Hanson (2004), Christensen et al. (2004) propose to apply bootstrapping to obtain confidence sets for estimated migration probabilities. In such a case, it is possible to get PDs for ratings with no actual defaults even applying the cohort approach, using the upper boundary of a set. Confidence sets may also be calculated analytically, using the Wald interval; however, this is not recommended by the authors as PD bands are too wide (see Schuermann, Hanson, 2004).

In this research, the information about rating transitions during a year and the exact time of default was not received from the external credit register, so it was impossible to apply approaches based on rating transition matrices and bootstrapping in Part 2.

1.3. Approaches based on CAP and ROC curves

M. V. Burgt (2007) proposes an alternative way how to derive the CAP curve:

$$y(x) = \frac{1 - \exp^{-kx}}{1 - \exp^{-k}},$$

where x is the cumulative part of obligors, $y(x)$ is the proportion of defaults, in x, and k is the concavity parameter defining the slope of the CAP curve; when k converges to 0, the CAP curve converges to a diagonal line (more about CAP and ROC curves, see BCBS, 2005a).

PDs can be derived from the CAP curve, using the following equations, when AR is > 60% (or AUC > 80%):

$$PD_R = \frac{k \cdot D}{1 - \exp^{-k}} \cdot \exp(-kx_R);\ k \approx \frac{2}{1 - AR};\ k \approx \frac{1}{1 - AUC};$$

$$x_R = \frac{z_N + z_{N+1} + \ldots + z_{R-1} + (z_R/2)}{z},$$

where x_R is the cumulative percentage of obligors in the rating R, D is the average observed default rate for the whole portfolio in question; AR is the accuracy ratio, AUC is the area under curve measure, z is the total number of obligors, z_R is the number of obligors in rating R, z_N is the number of obligors in the worst rating, and z the is total number of obligors.

The formulas above imply that CAP curve approach needs at least some defaults, i.e. it cannot be applied when there are no defaults in the whole portfolio (but it is enough to have defaults in at least one rating).

D. Tasche (2009) proposes the two-parametric ROC curve approach described below. The ROC curve may be derived using the following equations:

$$R_{a,b}(F_N(s)) = \Phi(a + b \cdot \Phi^{-1}(F_N(s))); u \in (0,1); b = \frac{\sigma_N}{\sigma_D}; a = \frac{\mu_N - \mu_D}{\sigma_D}$$

$$F_N(s) = \frac{P[\{S <= s\} \cap N]}{1 - D},$$

where $R_{a,b}(F_N(s))$ is the cumulative proportion of defaults till rating s, D is an average observed default rate for the whole portfolio in question, $F_N(s)$ is a false alarm rate till rating s, i.e. the cumulative proportion of non-defaulters till rating s that were treated as defaulted. The numerator is calculated as the product of two probabilities, i.e. the probability that the rating is lower than or equal to s (if lower ratings indicate a higher risk) and the cumulative probability of non-default till rating s, $\mu_{N,}$ σ_N are the mean and standard deviations of non-defaulters' ratings, μ_D,σ_D are mean and standard deviation of defaulters' ratings, Φ() is a cumulative normal distribution function for a standard normal random variable; it is possible to calculate it with the MS Excel function = NORMSDIST(); Φ^{-1}() is the inverse cumulative distribution function for a standard normal random variable; it is possible to calculate it with MS Excel function = NORMSINV().

PD for ratings may be derived as presented below:

$$P[D|S = s] = \frac{D \cdot R'_{a,b}(F_N(s))}{D \cdot R'_{a,b}(F_N(s)) + 1 - D}; R'_{a,b}(F_N(s)) = b\frac{\varphi(a + b\Phi^{-1}(F_N(s)))}{\varphi(\Phi^{-1}(F_N(s)))},$$

where φ() is a standard normal density; it is possible to calculate it using MS Excel function = NORMDIST(x; 0; 1; false).

1.4. Approaches based on the distribution of default numbers and simulation

A. Forrest (2005) proposes two types of PD estimation approaches for LDPs: without correlation (see Table 2) and with correlation. The basic idea is that for each chosen confidence level the interval of PDs is derived (not one concrete PD value). The author recommends taking conservative PDs from this interval.

When there are no actual defaults in several ratings in succession, we are interested in conservative combinations of PDs on the dashed line (FIG. 2). As for several LDP ratings, even for each chosen confidence level, many conservative combinations of PDs are derived, the question is how to choose only one combination. The author recommends using the combination of PDs giving the maximum risk-weighted assets.

When there are several defaults in several ratings in succession, the minimum and maximum values of PDs are found separately in the same way as for a single LDP rating, adding all defaults and obligors up to that rating, for example, for rating A (see Fig. 3):

TABLE 2**. Approaches without correlation**

Description	Formulas	Comments
Single LDP rating, no actual defaults	$LR(PD)=\frac{L(PD)}{ML}$; $L(PD)=(1-PD)^N$ $ML=(1-DR)^N$ *L(PD)* – likelihood, i.e. probability of obtaining data actually observed on the subjects in the study as a function of the unknown parameters in the model. In the *LDP* context, the only parameter is *PD* *ML* – maximum likelihood, i.e. the largest value of likelihood among all relevant combinations of the model parameters. As in this case, the actual default rate (*DR*) = 0, *ML* = 1. *N* – the number of obligors in rating.	To get conservative *PD*, equation of likelihood ratio *LR(PD)* is solved iteratively for hypothetical *PD*, recalculating until the value reaches a 100%-confidence level. For example, if we choose the 95% confidence level, we have to find the *PD* giving *LR(PD)* equal to 0.05. *PD* may be also calculated using MS Excel formula = BETAINV(confidence level;1;*N*)
Single LDP rating, several actual defaults	$LR(PD)=\frac{L(PD)}{ML}$; $L(PD)=PD^D\cdot(1-PD)^{N-D}$ $ML=DR^D\cdot(1-DR)^{N-D}$ $MIN\_PD<DR<MAX\_PD$ *D* – number of actual defaults in rating *DR* – actual default rate of rating	*LR(PD)* is rescaled as a positive quantity expressed as -2ln *LR(PD)*. As the value of -2ln LR(PD) is expected to be chi-squared distributed, the conservative *PD* is the higher of two PDs for which -2ln*(LR(PD))* equals to the inverse of the one-tailed probability of the chi-square distribution that may be calculated with MS Excel function = CHIINV using the the 100% confidence level and 1 degree of freedom as there' is only one LDP rating
Several LDP ratings, no actual defaults	$L(PD)=(1-PD_A)^{NA+NB}$ (for rating A) $L(PD)=(1-PD_B)^{NB}$ (for rating B) *NA*, *NB* – numbers of obligors in ratings A and B, respectively.	Maximum values of PD_A and PD_B are found iteratively where respective L(PD) equals to (100%-confidence level). Conservative combinations of PDs are on the dashed line (see Figure 2).
Several LDP ratings, several actual defaults	$LR(PD)=\frac{L(PD)}{ML}$ $L(PD)=PD_A^{DA}(1-PD_A)^{NA-DA}\cdot PD_B^{DB}(1-PD_B)^{NB-DB}$ $ML=DR_A^{DA}(1-DR_A)^{NA-DA}\cdot DR_B^{DB}(1-DR_B)^{NB-DB}$	Conservative combination of PDs has to comply with three conditions: a) $PD_A<PD_B$; b) -2ln (*LR(PD)*) = CHIINV((100% confidence level);2); c) combination of *PDs* has to be on the most distant line of the graph (see Fig. 3) The number of degrees of freedom has to be equaled to the number of *LDP* ratings in succession.

Source: compiled by the author in accordance with A. Forrest (2005).

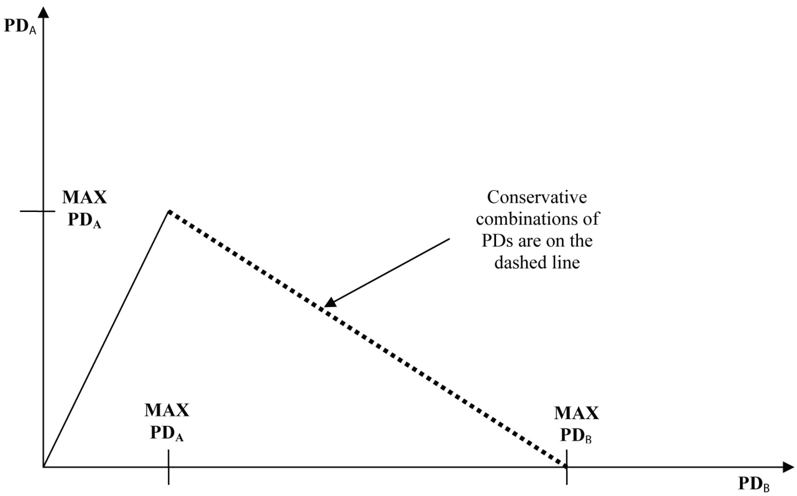

FIG. 2. **Estimation of PDs for several LDP ratings with no actual defaults**
Source: A. Forrest (2005).

$$LR(PD) = \frac{L(PD)}{ML};\ L(PD) = PD_A^{\ DA+DB}(1-PD_A)^{NA+NB-DA-DB}$$
$$ML = DR_{PORTFOLIO}^{\ DA+DB}(1-DR_{PORTFOLIO})^{NA+NB-DA-DB}$$

For rating B:

$$LR(PD) = \frac{L(PD)}{ML};\ L(PD) = PD_B^{\ DB}(1-PD_B)^{NB-DB}$$
$$ML = DR_B^{\ DB}(1-DR_B)^{NB-DB}$$

The number of degrees of freedom iteratively searching for the minimum and maximum PDs for both ratings will be 2 in this example because we have two LDP ratings in succession. However, choosing the maximum PDs for both ratings would be over-conservative (see Fig. 3, the point where the lines of MAX PD_A and MAX PD_B intersect).

The dark lines restrict the conservative region of PD, within which -2ln (LR(PD)) <=CHIINV((100% confidence level);2). From all conservative combinations on the most distant line, only one giving maximum risk-weighted assets should be chosen.

If A. Forrest's (2005) approach is modified introducing correlations, the conservative regions of PDs are *ceteris paribus* wider than without correlations; the values of conservative PDs are bigger. So, further in Part 2 only the approach without correlations will be applied.

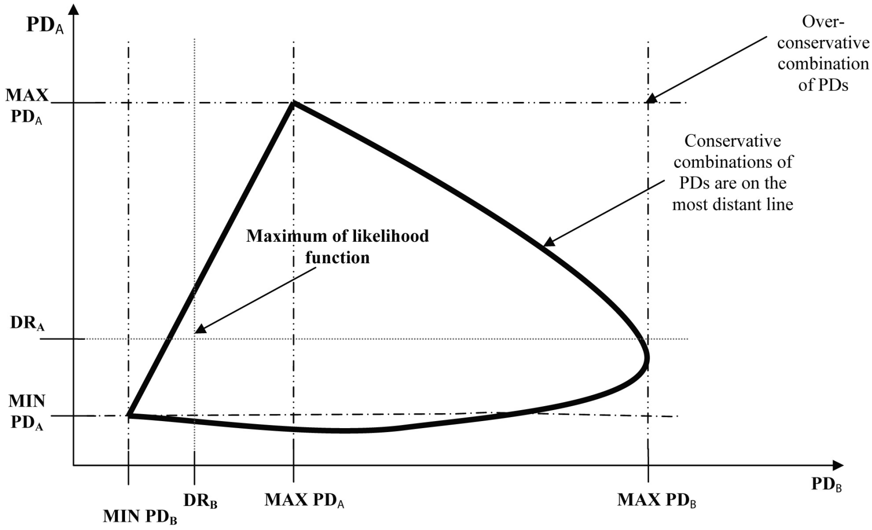

FIG. 3. **Estimation of PDs for several LDP ratings with several actual defaults**

Source: A. Forrest (2005).

Pluto and Tasche (2005) have proposed three ways to get most prudent estimates of PDs for LDPs: 1) without correlation, 2) with correlation, and 3) a multi-period case. If there are no actual defaults in single LDP rating, the approach without correlation is identical to A. Forrest's (2005) approach without correlation. However, if there are several LDP ratings in succession with no actual defaults, Pluto and Tasche (2005) propose using the extreme values of PDs (see MAX PD_A and MAX PD_B in Fig. 2). If there are actual defaults in LDP ratings, the authors assume that the number of defaults in the portfolio is binomially distributed as long as the default events are independent. For example, for three LDP ratings (A, B and C), the most prudent PD estimates are calculated using the expressions below. The right-hand side of the equations shows the probability of observing not more than DA + DB + DC defaults, not more than DB + DC defaults and not more than DC defaults, respectively:

$$1-\gamma = \sum_{i=0}^{DA+DB+DC} \left[\binom{NA+NB+NC}{i} \cdot PD_A{}^{i} \cdot (1-PD_A)^{NA+NB+NC\text{-}i} \right];$$

$$1-\gamma = \sum_{i=0}^{DB+DC} \left[\binom{NB+NC}{i} \cdot PD_B{}^{i} \cdot (1-PD_B)^{NB+NC-i} \right];$$

$$1-\gamma = \sum_{i=0}^{DC} \left[\binom{NC}{i} \cdot PD_C{}^{i} \cdot (1-PD_C)^{NC-i} \right],$$

where: $= \binom{n}{k} = C_n^k = \dfrac{n!}{k!\,(n-k)!}$, i. e. the number of possible k combinations from the total number of n observations; γ is chosen at the confidence level.

The tail of a binomial distribution can be expressed in terms of an appropriate beta distribution function. PDs may be calculated using MS Excel formula = BETAINV (confidence level; $D + 1$; $N–D$).

If a correlation is introduced, analogically as in A. Forrest's (2005) approach, most prudent estimates of PDs are *ceteris paribus* higher than without correlations. In the multi-period case, authors introduce an additional correlation measure, i.e. an inter-temporal correlation. An unrealistic assumption is made that only the number of obligors N_1 in the first year is known and the portfolio is closed for new obligors, so that $N_t = N_1$. Besides, PDs seem to be too low if compared to the approach without correlation. Most prudent estimates of annual PDs are derived for the whole period. Taking into account that in this article the LDP problem is discussed only as regards the estimation of PDs for rating from long run averages of one-year default rates (i.e. PD(4)), the multi-period case will not be further analyzed in this article. In Part 2, only Pluto and Tasche's (2005) approach without correlations will be applied.

N. M. Kiefer (2006) uses the Bayes rule to estimate PDs for LDPs. PD is estimated as the posterior expectation $\overline{\theta} = E(\theta|r, e)$. The posterior distribution $p(\theta | r, e)$, describing the uncertainty about θ given observation of r, actual defaults in rating with n obligors and having expert informatikon, e is expressed:

$$p(\theta|r,e) = \frac{p(r|\theta,e)\cdot p(\theta|e)}{p(r|e)},$$

were $p(r | \theta, e)$ is the distribution of r defaults given that *PD* (i. e. the probability of success on each trial) is θ and expert information *e is availalle*. Using the Bernoulli scheme, the right-hand side of the equation below shows the probability of observing r defaults in rating with n obligors:

$$p(r|\theta,e) = \binom{n}{r}\theta^r(1-\theta)^{n-r}.$$

This distribution may be calculated as the values of probability mass function applying MS Excel function = BINOMDIST().

$p(\theta | e)$ is the prior distribution of θ. The challenging step is to represent the expert's assessments with a statistical distribution. As the usual approach is to fit a parametric form, the author proposes using the beta distribution. The probability density function of the two-parameter beta distribution for the random variable $\theta \in [0.1]$ is

$$p(\theta|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1},$$

where $\Gamma(n)$ is gamma function (if n is a positive integer, then $\Gamma(n) = (n-1)!$); parameters α, β may be estimated by the method of moments to fit the parametric probability statements:

$$\hat{\alpha} = \bar{\theta}\left(\frac{\bar{\theta}(1-\bar{\theta})}{v} - 1\right); \ \hat{\beta} = (1-\bar{\theta})\left(\frac{\bar{\theta}(1-\bar{\theta})}{v} - 1\right),$$

where $\bar{\theta}$ is the sample mean and v is the sample variance.

The $p(r|e)$ is the unconditional distribution of the number of defaults. For the two-prameter beta family, the exact functional form can be calculated:

$$p(r|e) = \frac{\Gamma(r+\alpha)\Gamma(n-r+\beta)\Gamma(\alpha+\beta)\Gamma(n+1)}{\Gamma(r+1)\Gamma(n-r+1)\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)}.$$

PD is derived searching for the maximum value of the posterior distribution $p(\theta|r; e)$. N. M. Kiefer suggests using the four-parameter beta distribution that allows flexibility within the PD range $[a, b]$, but in some situations it may be too restrictive. Also, the seven-parameter distribution is discussed. However, the approach becomes more complicated, it is difficult to derive an unconditional distribution of the number of defaults $p(r|e)$.

This author has also proposed further modifications of his approach (see Kiefer 2007; 2008).

1.5. Other approaches

Wilde and Jackson (2006) proposed to estimate PDs analytically by calibrating CreditRisk+ to the Merton model of default behaviour. The approach is most advantageous where there are data of five or more years; it is possible to get PDs even when there are no defaults in the whole portfolio. However, PDs seem to be too big, even bigger than applying the Pluto and Tasche (2005) approach with a correlation.

G. Sabato (2006) proposed to relate the estimation of PDs with unemployment rates in a particular age or education category. This approach is appropriate only for the estimation of PDs for physical persons. Of course, it is possible to modify the approach making it appropriate for companies, for example, to use common variables of different economic sectors etc.; however, this wouldn't allow deriving reasonable PDs because companies in the same sector may represent different levels of risk. Besides, the approach is appropriate only to derive PDs for specific sub-groups of age, education, etc., but not for ratings.

Besides, the problem occurs not only when choosing the most appropriate methodology to estimate PDs for LDPs. If banks choose the methodology themselves, in different banks PDs derived for LDP ratings having the same number of obligors and the same number of defaults may be very different, i.e. banks may choose not only different methodologies, but also different parameters of the same methodology (confidence levels, values of correlation, etc.). Thus, the supervisors would face the problem of fair comparability. Supervisors could use the approach proposed by the Financial Services Authority (FSA, 2006). In their approach, banks compare their PDs with the so-called “look-up PDs” in the supervisory table. If the weighted average PD is less than the look-up PD, the bank adjusts it upwards until the weighted average PD is equal to or above the look-up PD. Look-up PDs are derived by the supervisor using one of the approaches discussed above, for example, the Pluto and Tasche (2005) approach without correlations. In such a way, PDs for LDP ratings with a similar risk in different banks would be comparable.

2. Estimation of PD for LDPs using data of one external credit register of Lithuania

For the purpose of this chapter, following the FSA definition, LDP shall be treated as a rating with the total number of defaults not more than 20.

10404 “company-years” at three scoring dates were assigned to nine ratings according to individual PDs estimated by the statistical scoring model of Lithuanian companies, developed by L. Dzidzevičiūtė (2006a). To develop this model, data on the Lithuanian companies from all economic sectors for 2005–2008 were obtained from the external loan register JSC Creditinfo Lietuva. It is possible to say that the data sample used to develop the model represents all the companies of Lithuania. An additional validation sample consisting of 10404 “company-years” was used to test the suitibility of LDP approaches.

The first rating indicates the lowest risk of companies and the 9^{th} the highest risk. Rating PDs were estimated for the point of 31 December 2007 (see PD(4) in Table 3). Data about defaults in 2008 were used for validation purposes.

PD(4) was calculated as a simple average of annual default rates in 2006 and 2007, respectively. One could notice that in ratings 1–3, both in 2006 and 2007, there are no more than 20 defaults. In 2007, also rating 7 should be treated as an LDP rating (as there are only 9 defaults). An especially severe problem is the rating 1 as there are no defaults either in 2006 or in 2007. Therefore, PD(4) for ratings 1–3 and 7 should be recalculated using one of the proposed approaches (see Table 4):

- M. Burgt’s (2007) CAP curve approach;
- D. Tasche’s (2009) ROC curve approach;
- A. Forrest’s (2005) approach without correlation;
- K. Pluto and D. Tasche’s (2005) approach without correlation;
- N. M. Kiefer (2006) Bayes’ approach.

M. Burgt (2007) CAP curve and D. Tasche (2009) ROC curve approaches. Even though both approaches ensure the monotony of PDs, they seem to be too low (see marked PDs in Tables 5 and 6).

TABLE 3. **Assignment to ratings and determination of rating PDs**

Rating	Lower PD boundary, %	Higher PD boundary,%	31 December 2005			31 December 2006			31 December 2007		PD(4), %
			All	Defaulted till 31 12 2006	Default rate, %	All	Defaulted till 31 12 2007	Default rate, %	All	Defaulted till 31 12 2008	
A	B	C	D	E	F=E/D	G	H	I=H/G	J	K	L=(F+I)/2
1	0.01	1.00	99	**0**	**0.00**	222	**0**	**0.00**	369	2	**0.00**
2	1.01	2.20	292	**3**	**1.03**	554	**7**	**1.26**	706	10	**1.15**
3	2.21	3.70	344	**0**	**0.00**	259	**8**	**3.09**	361	11	**1.55**
4	3.71	8.00	732	31	4.23	660	33	5.00	889	49	4.62
5	8.01	16.00	726	44	6.06	278	36	12.95	464	52	9.51
6	16.01	28.00	568	78	13.73	254	47	18.50	326	68	16.12
7	28.01	40.50	333	90	27.03	38	**9**	**23.68**	50	17	**25.36**
8	40.51	61.00	275	64	23.27	412	259	62.86	482	272	43.07
9	61.01	99.99	151	70	46.36	184	144	78.26	376	279	62.31
Total			3520	380	10.80	2861	543	18.98	4023	760	

Source: calculated by the author.

TABLE 4. **Comparison of PDs for LDPs***

	M. Burgt (2007) CAP curve approach			D. Tasche (2009) ROC curve approach			A. Forrest (2005) approach without correlation			K. Pluto, D. Tasche (2005) approach without correlation			N. M. Kiefer (2006) Bayes' approach		
Rating	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006} %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %
1	**0.28**	**0.09**	***0.19***	**0.06**	**0.002**	***0.03***	**0.85**	**1.49**	***1.17***	**0.35**	**1.07**	***0.71***	**0.60**	**1.51**	***1.06***
2	**0.37**	**0.26**	***0.32***	**0.18**	**0.05**	***0.12***	**0.879**	**1.78**	***1.33***	**0.40**	**1.36**	***0.88***	**0.68**	**1.89**	***1.29***
3	**0.61**	**0.78**	***0.70***	**0.80**	**0.73**	***0.77***	**0.88**	**3.63**	***2.26***	**0.43**	**2.91**	***1.67***	**0.44**	**2.91**	***1.68***
4	4.23	5.00	*4.62*	4.23	5.00	*4.62*	4.23	5.00	*4.62*	4.23	5.00	*4.62*	4.23	5.00	*4,62*
5	6.06	12.95	*9.51*	6.06	12.95	*9.51*	6.06	12.95	*9.51*	6.06	12.95	*9.51*	6.06	12.95	*9,51*
6	13.73	18.50	*16.12*	13.73	18.50	*16.12*	13.73	18.50	*16.12*	13.73	18.50	*16.12*	13.73	18,50	*16,12*
7	27.03	**28.06**	***27.54***	27.03	**34.63**	***30.83***	27.03	**38.64**	***32.83***	27.03	**17.31**	***22.17***	27.03	**24.57**	***25.80***
8	23.27	62.86	*43.07*	23.27	62.86	*43.07*	23.27	62.86	*43.07*	23.27	62.86	*43.07*	23.27	62,86	*43,07*
9	46.36	78.26	*62.31*	46.36	78.26	*62.31*	46.36	78.26	*62.31*	46.36	78.26	*62.31*	46.36	78,26	*62,31*

Source: calculations of the author.

Note: * PD_{2006} and PD_{2007} for the ratings not complying with LDP definition were calculated in an ordinary way, i.e. the number of defaults was divided by the number of companies.

Recalculated PD_{2006} and PD_{2007} for LDP ratings are marked. PD(4) for all ratings is estimated using the formula: $PD(4)_{rating} = \frac{PD_{2006} + PD_{2007}}{2}$

TABLE 5. **PDs for ratings applying CAP curve and ROC curve approaches for 2006**

Rating	Number of defaults	All	Actual default rate, %	x_{R}, %	CAP curve PDs, %	$F_N(s)$, %	$R_{a,b}'(F_N(s))$	$R_{a,b}(F_N(s))$, %	ROC curve PDs, %
1	0	99	0.00	98.59	**0.28**	100.00	0.0051	100.00	**0.06**
2	3	292	1.03	93.04	**0.37**	96.85	0.0153	99.98	**0.18**
3	0	344	0.00	84.01	**0.61**	87.64	0.0668	99.61	**0.80**
4	31	732	4.23	68.72	1.41	76.69	0.1501	98.44	1.78
5	44	726	6.06	48.01	4.33	54.36	0.4268	92.33	4.91
6	78	568	13.73	29.63	11.77	32.64	0.9672	77.94	10.48
7	90	333	27.03	16.83	23.61	17.04	1.8218	57.10	18.06
8	64	275	23.27	8.20	37.75	9.30	2.7285	39.89	24.82
9	70	151	46.36	2.14	52.46	2.58	4.7902	16.11	36.70
Total	**380**	**3520**	**AR, %**	**63.21**		**a**	**1.2990**		
			k	**5.44**		**b**	**1.1759**		
			D, %	**10.80**					

Source: calculations of the author.

TABLE 6. **PDs for ratings applying CAP curve and ROC curve approaches for 2007**

Rating	Number of defaults	All	Actual default rate, %	x_{R}, %	CAP curve PDs, %	$F_N(s)$, %	$R_{a,b}'(F_N(s))$	$R_{a,b}(F_N(s))$, %	ROC curve PDs, %
1	0	222	0.00	96.12	**0.09**	100.00	0.0001	100.00	**0.002**
2	7	554	1.26	82.56	**0.26**	90.42	0.0023	99.99	**0.05**
3	8	259	3.09	68.35	**0.78**	66.82	0.0315	99.68	**0.73**
4	33	660	5.00	52.29	2.66	56.00	0.0691	99.16	1.59
5	36	278	12.95	35.90	9.33	28.95	0.4188	93.94	8.93
6	47	254	18.50	26.60	18.99	18.51	0.9281	87.34	17.86
7	9	38	23.68	21.50	**28.06**	9.58	2.2611	74.33	**34.63**
8	259	412	62.86	13.63	51.20	8.33	2.6541	71.27	38.34
9	144	184	78.26	3.22	113.56	1.73	10.6226	37.81	71.33
	543	2861	**AR, %**	**73.85**		**a**	**2.2119**		
			k	**7.65**		**b**	**1.1932**		
			D, %	**18.98**					

Source: calculations of the author.

One could notice that for better ratings PDs are significantly lower than the actual default rates, especially in D. Tasche's (2009) ROC curve approach. In both years this approach gives too low PDs for ratings 1 and 2, even if compared with M. Burgt's (2007) CAP curve approach. For example, for rating 2, the actual default rate in 2006 is 1.03 per cent (see Table 3), M. Burgt's (2007) CAP curve approach gives 0.37 per cent, meanwhile D. Tasche's (2009) ROC curve approach gives 0.18 per cent. So, the values seem to be too low if compared with other approaches (see Table 4).

For worse ratings PDs are not too low (see rating 7 in Table 6), but usually in practice a low number of defaults is an issue for better ratings. Besides, both approaches are very sensitive to the discriminatory power of the scoring model. As in 2007 the model discriminates better (the accuracy ratio is 73.85 per cent and in 2006 only 63.21 per cent), PDs for better ratings in 2007 are comparatively lower. The other three approaches (see Table 4) give higher PDs for ratings 1, 2 and 3 in 2007 than in 2006, and this seems to be reasonable because ratings 1, 2 and 3 are riskier in 2007 than in 2006. M. Burgt's (2007) CAP curve and D. Tasche's (2009) ROC curve approaches, on the contrary, give lower PDs for ratings 1, 2 and 3 in 2007 than in 2006; thus, these PDs don't fully reflect the riskiness of ratings.

The Pluto and Tasche (2005) approach without correlation. Table A.1 in Appendix provides the PDs for LDP ratings applying this approach with various confidence levels. PDs for ratings 1, 2 and 3 are derived on the cumulative basis adding all defaults and all obligors up to this rating, i.e. in 2006 for rating 1 the number of defaults will be 3 and the number of obligors 735; for rating 2, the number of defaults will be 3 and the number of obligors 636; for rating 3, the number of defaults will be 0 and the number of obligors 344. However, the PD for rating 7 in 2007 was derived on a single basis as rating 7 does not follow the other LDP ratings. One could notice that PDs in 2006 don't comply with the monotony requirement as almost allways the PD for rating 3 is lower than for rating 2 (except only the 99.99% confidence level; however, then PDs are too high). Scaled PDs were also estimated as proposed by the authors (see Table A.1 in Appendix), i.e.:

$$SCALED\_PD_i = K \cdot PD_i$$

$$K = \frac{PD_{PORTFOLIO}}{\frac{PD_1 \cdot N_1 + PD_2 \cdot N_2 + PD_3 \cdot N_3}{N_1 + N_2 + N_3}}$$

where PD_i is the estimated PD for rating *i*; *K* is the scaling factor, and N_i is the number of obligors in rating *i*.

As LDP ratings were excluded from ordinary ratings, the $PD_{portfolio}$ was treated as an average PD of the portfolio consisting of only the first three ratings in 2006 (for 2007,

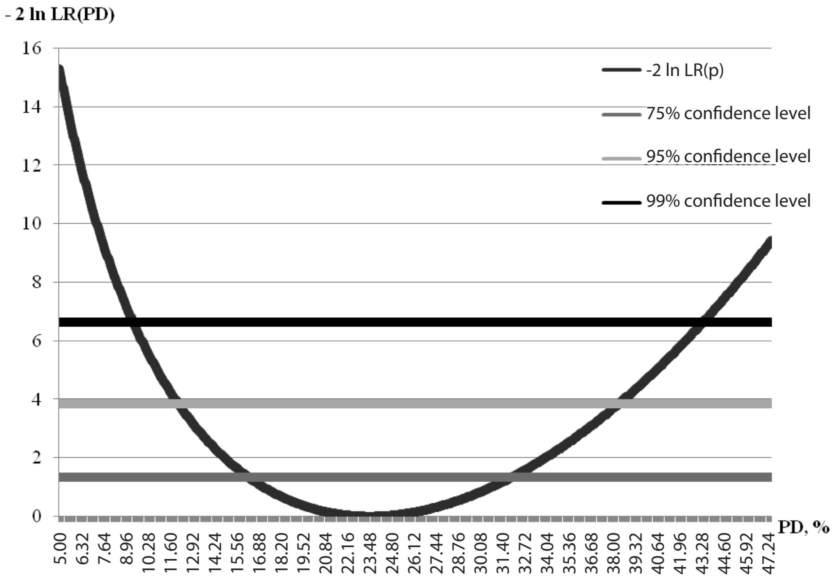

FIG. 4. **Estimation of PD_{2007} for rating 7**

Source: calculations of the author.

also rating 7 was added). In 2006, the $PD_{PORTFOLIO}$ is 0.41 per cent (i. e. 3/735), and in 2007 it is 2.24 per cent (i.e. 24/1073). For the final purposes of anglysis, it was decided to use scaled PDs with the 99.99% confidence level. The estimates comply with the monotony requirement and are not too high.

A. Forrest (2005) approach without correlation. To estimate the $PD_{2007,}$ rating 7 was treated as a single LDP rating. Graphically (see Fig. 4), the conservative PD_{2007} may be determined where the line of the rescaled likelihood ratio (i.e. -2ln LR(PD)) intersects the cut line of the chosen confidence level on the right side of the graph. A. Forrest argues that classically the 95% confidence level is chosen. If we choose the confidence level reccomended by this author, the PD_{2007} lies between the minimum PD of 12.16 per cent and the maximum PD of 38.64 per cent. The maximum likelihood is found at 23.68 per cent, i.e. at the actual default rate.

As we are interested in getting a conservative value, we will choose 38.64 per cent. The cut lines were derived using MS Excel function =CHIINV(100%-chosen confidence level;1) (see Table 2). The conservative PD_{2007} for rating 7, derived using the 95 per cent confidence level, seems to be most reasonable and will be used further.

As both in 2006 and 2007, ratings 1, 2 and 3 are LDP ratings and they are in succesion, PDs for them will be derived together (see Fig. 3). Table 7 provides the minimum and maximum values of PDs for these ratings.

TABLE 7. Minimum and maximum values of PDs for ratings (percentages)

	2006		2007	
Rating	Minimum PD	Maximum PD	Minimum PD	Maximum PD
1	0.05	1.45	0.64	2.74
2	0.05	1.68	0.82	3.49
3	0.00	1.13	0.96	7.07

Source: calculations of the author.

As one could see in Fig. 3, combinations of PDs can break through the line of the minimum PD, so the iterative checking of PDs was started from 0.01 per cent for rating 1, from 0.02 per cent for rating 2 and from 0.03 per cent for rating 3 up to the maximum PD of a respective rating. From all the conservative combinations of PDs complying with these three conditions, i.e.

- $PD_1 < PD_2 < PD_3$,
- -2ln(LR(PD)) =CHIINV((100% – 95%);3),
- combination of PDs has to be on the most distant line of the graph,

only one combination was chosen, giving maximum risk-weighted assets. For 2006, this is a combination of 0.85%/0.879%/0.88% and for 2007 it is 1.49%/1.78%/3.63% (see Table 4). To compare risk-weighted assets, the formulas applicable for retail exposures were used:

$$Risk\_weighted\_assets\ (RWA) = RW * EAD;$$

$$Risk\_weight\ \ (RW) = (\,LGD \cdot \Phi\left(\frac{\Phi^{-1}(\,PD\,)}{\sqrt{1-R}} + \sqrt{\frac{R}{1-R}} \cdot \Phi^{-1}(\,0.999\,)\right) - PD \cdot LGD\,) \cdot 12.5 \cdot 1.06;$$

$$Correlation\ (R) = 0.03 \cdot \frac{1-e^{-35 \cdot PD}}{1-e^{-35}} + 0.16 \cdot \left[1 - \frac{1-e^{-35 \cdot PD}}{1-e^{-35}}\right],$$

where *LDG* is a loss given default; for the sake of comparability, always the value of 45% was used; *EAD* is exposure at default; for the sake of comparability, always the value of 100 LTL was used.

It should be noted that the application of this approach starting from three LDP ratings in succession requires programming skills, otherwise the iterative checking of various combinations of PDs will be very time-consuming. However, the derived combinations of PDs comply with the monotony requirement and seem to be very reasonable for the calculation of capital adequacy.

N. M. Kiefer (2006) Bayes' approach. The first step is to decide upon the representation of the prior distribution $p(\theta \mid e)$. As N. M. Kiefer (2006) says that the four-parameter beta distribution in some situations may be too restrictive, in this article we use the two-parameter beta distribution. For ratings 1, 2 and 3, in both years hypothetical PDs from 0.01 per cent to 7.00 per cent were used with the step equal to 0.01 per cent. Thus, parameters α and β are 3 and 79, respectively. However, parameters for rating 7 in 2007 have to be different as the PD for this rating is expected to be significantly higher than in other three LDP ratings, so hypothetical PDs from 12.00 per cent to 45.00 per cent were used with the step equal to 0.01 per cent. Thus, parameters α and β are 6 and 15, respectively.

Similarly as in the Pluto and Tasche (2005) approach, PDs for ratings 1, 2 and 3 are derived on the cumulative basis adding all defaults and all obligors up to that rating. For rating 7, posterior distribution was derived on a single basis, as this rating is not in succesion with other LDP ratings.

Figures 5 and 6 show the posterior distributions $p(\theta | r,e)$ of PDs. The PD for a respective rating is derived searching for the maximum value of this posterior distribution.

One could notice that the posterior distribution of PD for rating 3 in 2006 is shifted to the left as compared with PD distributions for ratings 1 and 2. Thus, PDs in 2006 don't comply with the monotony requirement as the PD for rating 3 is lower than for rating 2 and even than for rating 1.

The estimated PDs need to be validated in order to check their suitability. According to the regulation of the Bank of Lithuania, banks applying the internal ratings based approach shall carry out a regular (at least annual) validation of the PD quantification

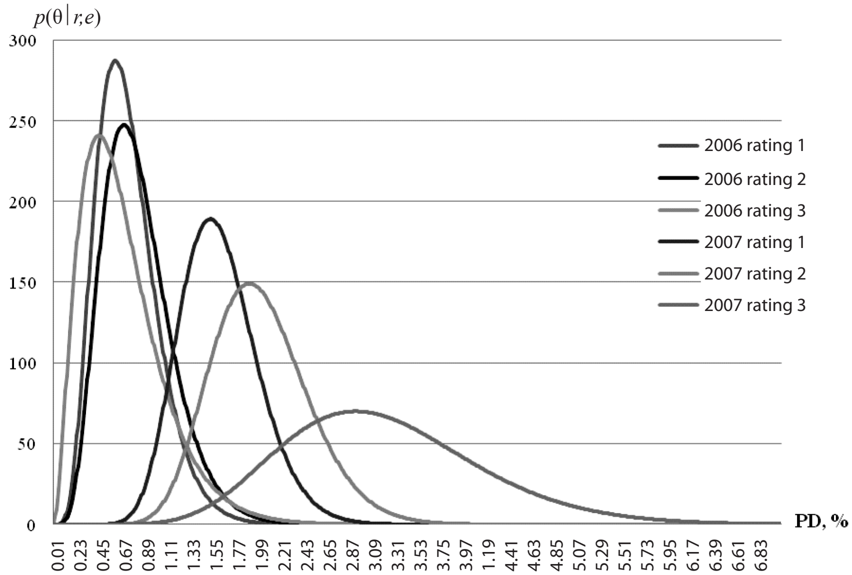

FIG. 5. Posterior distributions of PDs for ratings 1, 2 and 3 in 2006 and 2007

Source: calculations of the author.

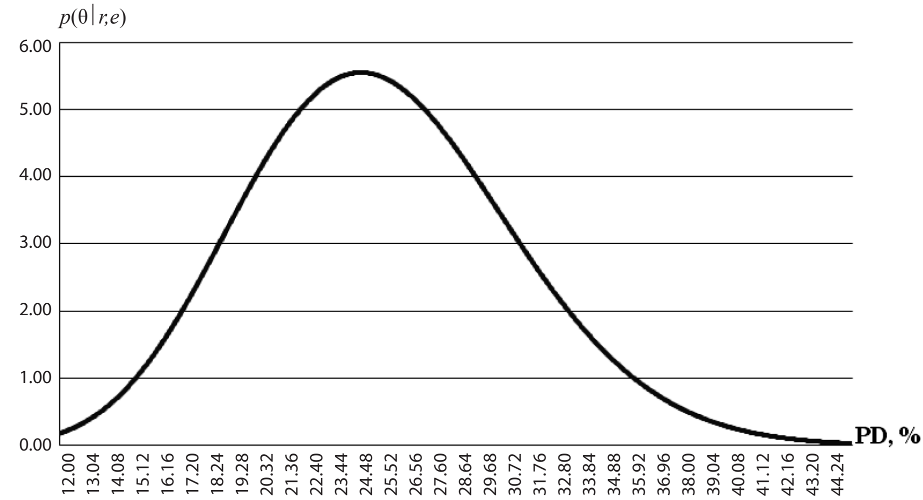

FIG. 6. Posterior distribution of PD for rating 7 in 2007

Source: calculations of the author.

process (Bank of Lithuania, 2006b). Even banks not applying the internal ratings based approach should *mutatis mutandis* comply with the regulation on validation (Bank of Lithuania, 2008). One of the recommended validation methods is the binomial test (BCBS, 2005a; Bank of Lithuania, 2006b; Tasche, 2006; Burgt, 2007; SAS, 2009). This method tests whether the estimated PD(4) presented in Table 4 falls within a 95% confidence level around the PD_{real} (i.e. the actual default rate in 2008). The $PD_{estimated}$ should lie in the interval as presented below:

$$\left[PD_{real} - \Phi^{-1}\left(\frac{1+\alpha}{2}\right)\cdot\sqrt{\frac{PD_{real}\cdot(1-PD_{real})}{N}};\right.$$

$$\left.PD_{real} + \Phi^{-1}\left(\frac{1+\alpha}{2}\right)\cdot\sqrt{\frac{PD_{real}\cdot(1-PD_{real})}{N}}\right],$$

where α is the confidence level which is chosen as 95%, and Φ^{-1} is the inverse of the cumulative standard normal distribution.

Results of the binomial test have shown that only in three approaches the $PD_{estimated}$ allways falls into the interval between the lower and the upper boundaries of PD_{real} (see Table 8). So, it is reasonable to reject the other two approaches where this requirement is not fulfilled. PDs in the Pluto and Tasche (2005) approach without correlation using scaled PDs with 99.99 per cent confidence level seem to be quite reasonable, they allways fall into the defined interval. In Forrest's (2005) approach without correlation and Kiefer (2006) Bayes' approach, PDs also fall into the defined interval; besides, they are more

TABLE 8. **Validation of PDs for LDPs**

	1 rating	2 rating	3 rating	7 rating
Number of obligors by 2007.12.31	369	706	361	50
Defaulted till 2008.12.31	2	10	11	17
PD_{real}, %	0.54	1.42	3.05	34.00
Lower boundary of $PD_{estimated}$, %	0.00	0.54	1.27	20.87
Higher boundary of $PD_{estimated}$, %	1.29	2.29	4.82	47.13
M. BURGT (2007) CAP CURVE APPROACH				
$PD_{estimated,}$%	0.19	0.32	0.70	27.54
Binomial test	TRUE	FALSE	FALSE	TRUE
D. TASCHE (2009) ROC CURVE APPROACH				
$PD_{estimated}$, %	0.03	0.12	0.77	30.83
Binomial test	TRUE	FALSE	FALSE	TRUE
A.Forrest (2005) approach without correlation				
$PD_{estimated}$, %	1.17	1.33	2.26	32.83
Binomial test	TRUE	TRUE	TRUE	TRUE
K. PLUTO, D. TASCHE (2005) APPROACH WITHOUT CORRELATION				
$PD_{estimated}$, %	0.71	0.88	1.67	22.17
Binomial test	TRUE	TRUE	TRUE	TRUE
N. M. KIEFER (2006) BAYES' APPROACH				
$PD_{estimated}$, %	1.06	1.29	1.68	25.80
Binomial test	TRUE	TRUE	TRUE	TRUE

Source: calculation of the author.

conservative than PDs in the approach discussed above. In the Burgt (2007) CAP curve and the Tasche (2009) ROC curve approaches, the $PD_{estimated}$ for ratings 2 and 3 is less than the lower boundary. For rating 7, all PDs fall into the defined interval; however, the Pluto and Tasche (2005) approach without correlation here gives quite a low value, very chose to the lower boundary.

Conclusions

The author of this article recommends to apply LDP approaches on the rating (and not on the portfolio) level, using a concrete number of defaults in order to define LDP without accounting for the total size of rating or portfolio. For ratings not complying with LDP definition (having more than 20 defaults), PDs should be calculated in an ordinary way. If a concrete rating in one year is treated as an LDP and in another doesn't comply with the LDP definition, LDP approaches should be applied only for the first year.

The Pluto and Tasche (2005) approach without correlation could be easily implemented in banks. However, if the ordinal ranking of obligors is incorrect, then this approach doesn't ensure the monotony of PDs in LDP ratings. The same problem exists in Kiefer's (2006) approach. Forrest's (2005) approach without correlation ensures the monotony and conservatism of PDs; however, it requires programming skills, otherwise the iterative recalculation of PDs will be very time-consuming. PDs estimated in these three approaches passed the binomial test.

A numerical example has shown that PDs estimated in Burgt's (2007) CAP curve and Tasche's (2009) ROC curve approaches are too low for better ratings; PDs didn't pass the binomial test.

If it is impossible to extract the information about rating transitions during a year and the exact time of defaut, it makes no sence to apply the approaches based on rating transition matrices; in any case, they are quite time-consuming. However, some supervisors (e.g., the Bank of Lithuania) require banks to estimate rating transition matrices; so, at the same time the LDP problem is solved.

Applying Forrest's (2005) and the Pluto and Tasche (2005) approaches with a correlation, the conservative values of PDs may be too high, thus the calculated capital adequacy requirements to cover credit risk may not satisfy banks and their supervisors, taking into account that the internal ratings based approach in Basel II should ensure not an over-conservative but an accurate calculation of capital requirements. Multi-period approaches, proposed by Pluto, Tasche (2005) and Wilde, Jackson (2006), give either too high or too low PDs; in some cases, assumptions are unrealistic and cannot be fulfilled in practice. The approach based on unemployment rates proposed by G. Sabato (2006), is appropriate only to estimated PDs for a physical person. Modifications of the approach to estimate PDs for companies wouldn't allow deriving reasonable PDs. Besides, the approach is appropriate only to derive PDs for specific sub-groups of age, education, etc., but not for ratings.

As the rating system used in this article was developed using a large sample of Lithuanian companies' data, the conclusions are most actual to banks of Lithuania and their ratings systems to Lithuanian companies. Besides, it is recommended to supervisors to prepare a common methodology applicable in all their jurisdiction, or to prepare look-up tables of PDs for banks.

REFERENCES

Bank of Lithuania (2006a). General Regulations for the Calculation of Capital Adequacy approved by Bank of Lithuania Board Resolution No. 138 of 9 November 2006, "Valstybes zinios" (Official Gazette), 2006, No. 142 (5442).

Bank of Lithuania (2006b). Regulations on Validation and its Assessment approved by Bank of Lithuania Board Resolution No. 140 of 9 November 2006, "Valstybes zinios" (Official Gazette), 2006, No. 142 (5444).

Bank of Lithuania (2008). Regulations on Organization of Internal Control and Risk Assessment (Management) approved by Bank of Lithuania Board Resolution No. 149 of 25 September 2008, "Valstybes zinios" (Official Gazette), 2008, No. 127 (4888).

Basel Committee on Banking Supervision (BCBS) (2005a). Studies on the Validation of Internal Rating Systems.

Basel Committee on Banking Supervision (BCBS) (2005b): Validation of Low-default Portfolios in the Basel II Framework.

Basel Committee on Banking Supervision (BCBS) (2006). International Convergence of Capital Measurement and Capital Standards: A Revised Framework Comprehensive Version.

British Bankers Association, London Investment Banking Association, International Swaps and Derivatives Association (BBA, LIBA, ISDA) (2004). Introductory Paper on Low Default Portfolios.

British Bankers Association, London Investment Banking Association, International Swaps and Derivatives Association (BBA, LIBA, ISDA) (2005). Low Default Portfolios.

Burgt, M. (2007). Calibrating low-default portfolios, using the Cumulative Accuracy Profile.

Committee of European Banking Supervisors (CEBS) (2006): Guidelines on the implementation, validation and assessment of advanced measurement (AMA) and internal ratings based (IRB) approaches.

Christensen, J. H. E.; Hansen, E.; Lando, D. (2004). Confidence sets for continuous-time rating transitions probabilities. – Journal of Banking & Finance, Vol. 28 (2004), p. 2575–2602.

Directive 2006/48/EC of the European Parliament and of the Council Relating to the Taking up and Pursuit of the Business of Credit Institutions.

Dzidzevičiūtė, L. (2010a). Statistical scoring model for Lithuanian companies. Ekonomika (2010) Vol. 89, issue 4, 96–115.

Dzidzevičiūtė, L. (2010b). Statistinių vertinimo balais modelių kūrimo ir taikymo ypatumai. Pinigų studijos No 1, p. 35–54.

Dzidzevičiūtė, L. (2010c). Statistinių vertinimo balais modelių taikymas Lietuvos bankuose Pinigų studijos, No 2, p. 70–86.

Financial Services Authority (FSA). (2005). Expert Group article on low default portfolios.

Financial Services Authority (FSA). (2006). Low default portfolios: a proposal for conservative estimation of low default probabilities.

Forrest, A. (2005). Likelihood approaches to low default portfolios. Version 1.2 14/9/05.

Jafry, Y., Schuermann, T. (2004). Measurement, estimation and comparison of credit migration matrices – Journal of Banking & Finance, (2004), Vol. 28, p. 2603–2639.

Kiefer, N. M. (2006). Default Estimation for Low-Default Portfolios. CAE Working Paper 06–08.

Kiefer, N. M. (2007). Bayesian Methods for Default Estimation in Low-Default Portfolios.

Kiefer, N. M. (2008). Default Estimation, Correlated Defaults, and Expert Information.

Lando, D., Skφdeberg, T. M. (2002). Analyzing rating transitions and rating drift with continuous observations. Journal of Banking & Finance, Vol. 26, p. 423–444.

Pluto, K., Tasche, D. (2005). Estimating probabilities for low default portfolios.

Sabato, G. (2006). Managing credit risk for retail low-default portfolios.

SAS (2009). Credit Risk Modeling Using SAS ^{R}. Course Notes.

Schuermann, T.; Hanson, S. (2004). Estimating Probabilities of Default.

Tasche, D. (2006).Validation of internal rating systems and PD estimates.

Tasche, D. (2009). Estimating discriminatory power and PD curves when the number of defaults is small.

Wilde, T., Jackson, L. (2006). Low-default portfolios without simulation.

Appendix

TABLE A.1. **PDs for ratings in K. Pluto, D. Tasche (2005) approach without correlation**

	50% confidence level, no scalling			75% confidence level, no scalling			90% confidence level, no scalling			95% confidence level, no scalling			99% confidence level, no scalling			99.9% confidence level, no scalling			99.99% confidence level, no scalling		
Ra-ting	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %
1	0.50	1.51	1.01	0.69	1.78	1.24	0.91	2.05	1.48	1.05	2.22	1.64	1.36	2.57	1.96	1.77	3.00	2.38	2.15	3.38	2.76
2	0.58	1.93	1.25	0.80	2.27	1.54	1.05	2.61	1.83	1.21	2.83	2.02	1.57	3.27	2.42	2.04	3.80	2.92	2.48	4.29	3.38
3	0.20	3.34	1.77	0.40	4.15	2.28	0.67	4.97	2.82	0.87	5.50	3.19	1.33	6.60	3.96	1.99	7.96	4.98	2,64	9.19	5.92
4	4.23	5.00	4.62	4.23	5.00	4.62	4.23	5.00	4.62	4.23	5.00	4.62	4.23	5.00	4.62	4.23	5.00	4.62	4.23	5.00	4.62
5	6.06	12.95	9.51	6.06	12.95	9.51	6.06	12.95	9.51	6.06	12.95	9.51	6.06	12.95	9.51	6.06	12.95	9.51	6.06	12.95	9.51
6	13.73	18.50	16.12	13.73	18.50	16.12	13.73	18.50	16.12	13.73	18.50	16.12	13.73	18.50	16.12	13.73	18.50	16.12	13.73	18.50	16.12
7	27.03	25.22	26.13	27.03	30.12	28.58	27.03	34.80	30.91	27.03	37.69	32.36	27.03	43.24	35.14	27.03	49.54	38.29	27.03	54.70	40.86
8	23.27	62.86	43.07	23.27	62.86	43.07	23.27	62.86	43.07	23.27	62.86	43.07	23.27	62.86	43.07	23.27	62.86	43.07	23.27	62.86	43.07
9	46.36	78.26	62.31	46.36	78.26	62.31	46.36	78.26	62.31	46.36	78.26	62.31	46.36	78.26	62.31	46.36	78.26	62.31	46.36	78.26	62.31
	50% confidence level, scalling			75% confidence level, scalling			90% confidence level, scalling			95% confidence level, scalling			99% confidence level, scalling			99.9% confidence level, scalling			99.99% confidence level, scalling		
	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %	PD_{2006}, %	PD_{2007}, %	PD(4), %
1	0.52	1.13	0.82	0.47	1.11	0.79	0.44	1.09	0.76	0.42	1.08	0.75	0.39	1.08	0.73	0.36	1.07	0.72	0.35	1.07	0.71
2	0.60	1.43	1.02	0.55	1.41	0.98	0.50	1.39	0.95	0.48	1.38	0.93	0.45	1.37	0.91	0.42	1.36	0.89	0.40	1.36	0.88
3	0.21	2.49	1.35	0.27	2.57	1.42	0.32	2.64	1.48	0.34	2.69	1.51	0.38	2.76	1.57	0.41	2.84	1.63	0.43	2.91	1.67
4	4.23	5.00	4.62	4.23	5.00	4.62	4.23	5.00	4.62	4.23	5.00	4.62	4.23	5.00	4.62	4.23	5.00	4.62	4.23	5.00	4.62
5	6.06	12.95	9.51	6.06	12.95	9.51	6.06	12.95	9.51	6.06	12.95	9.51	6.06	12.95	9.51	6.06	12.95	9.51	6.06	12.95	9.51
6	13.73	18.50	16.12	13.73	18.50	16.12	13.73	18.50	16.12	13.73	18.50	16.12	13.73	18.50	16.12	13.73	18.50	16.12	13.73	18.50	16.12
7	27.03	18.76	22.89	27.03	18.67	22.85	27.03	18.52	22.77	27.03	18.40	22.71	27.03	18.11	22.57	27.03	17.70	22.37	27.03	17.31	22.17
8	23.27	62.86	43.07	23.27	62.86	43.07	23.27	62.86	43.07	23.27	62.86	43.07	23.27	62.86	43.07	23.27	62.86	43.07	23.27	62.86	43.07
9	46.36	78.26	62.31	46.36	78.26	62.31	46.36	78.26	62.31	46.36	78.26	62.31	46.36	78.26	62.31	46.36	78.26	62.31	46.36	78.26	62.31
Scal-inng fac-tor	1.04	0.74		0.68	0.62		0.48	0.53		0.40	0.49		0.29	0.42		0.21	0.36		0.16	0.32	