UNIVERSALITY THEOREMS IN PHYSICS

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Introduction

In recent times, somewhat unexpectedly, the number theory started being applied by physicists to solve physical problems and, perhaps even more unexpectedly, techniques developed by physicists are applied to problems in number theory. Physicists become acquainted with special functions early in their studies. D. Schumayer, D. A. W. Hutchinson [1] present, that for perennial model, the harmonic oscillator needs Hermite functions, or the Laguerre functions in quantum mechanics. Here physicists choose a particular number of theoretical functions, such as the Riemann zeta function. Physicists examine numerous models from different branches of physics, from classical mechanics to statistical physics where this function plays an integral role. Physicists also observe how this function relates to quantum chaos and how its pole-structure encodes when particles can undergo Bose-Einstein condensation at low temperatures [1].

The universality is a very interesting and useful property of zeta and *L*-functions. The property for the Riemann zeta-function was discovered by S. M. Voronin. Later, many mathematicians (S. M. Gonek, A. Reich, B. Bagchi, A. Laurinčikas, K. Matsumoto, R. Garunkštis, J. Steuding, W. Schwarz, H. Mishou, R. Macaitienė, R. Kačinskaitė, D. Šiaučiūnas, V. Garbaliauskienė and others) improved and generalized Voronin's theorem. Also the Linnik-Ibragimov conjecture exists stating that all functions in some half-plane given by Dirichlet series, analytically continuable to the left of the absolute convergence half-plane and satisfying some natural growth conditions that are universal in the Voronin sense. Hence, the universality of dzeta-functions is a new property of a very important mathematical object; on the other hand, it supports the Linnik-Ibragimov conjecture.

The aim of this note is to present the universality of dzeta-functions and its applications in physics.

Riemann-zeta function in Physics

D. Schumayer, D. A. W. Hutchinson present the applications of ***Riemann-zeta function in Quantum mechanics*** [1]. At the dawn of the 20th century, Bohr postulated a series of rules for describing the spectrum of the hydrogen atom well before the birth of Schrodinger's and Heisenberg's quantum mechanics. In these early days 'quantisation' meant to restrict the possible values of action variables of the classical system (Bohr-Sommerfeld, Wentzel Kramers-Brillouin, etc.) and the rules worked well, up to an additive constant. Classical mechanics works well for large systems; therefore, quantum mechanics must give the same predictions for a large system as classical mechanics (Bohr's correspondence principle). This unproven principle ties these two theories firmly together and the same principle inspired the use of the Riemann ζ function in investigating the relationship of classical to quantum mechanics.

Interpreting the $\zeta(s)$ zeros as energy levels, their distribution is breathtakingly similar to those of a quantum system's. This has inspired physicists to examine whether one could associate a dynamical system with the Riemann zeta function. The advantage of this approach would be that the huge number of $\zeta(s)$ zeros are known and quick numerical algorithms have also been developed to find further zeros, thus solving the Schrodinger equation for large energies would be unnecessary. The Riemann zeta function could play the same role in the examination of chaotic quantum systems as the harmonic oscillator does for integrable quantum systems.

Riemann-zeta function in Condensed matter physics [1]. One of the fundamental bases of modern condensed matter physics is the geometrical structure of solids; the lattice. The examination of this mathematical structure is necessary to understand even the basic properties of matter. The regular structure of a perfect lattice is suitable for immediate comparison with regularities among the natural numbers, and therefore it is not a surprise that many number-theoretical functions arise in crystallography. Moreover, not only the perfect regularity of a lattice, but also the lack of this regularity can be related to the Riemann zeta function, as Dyson indicated recently [2]: "A fourth joke of nature is a similarity in behaviour between quasi-crystals and the zeros of the Riemann Zeta function."

Riemann-zeta function in Statistical physics
[3]. Although statistical physics (the physics of systems with a large number of degrees of freedom) relied heavily upon combinatorics, well before the birth of quantum mechanics, presumably the first appearance of the Riemann zeta function in statistical physics occurred in Planck's momentous work on black body radiation, the dawn of the quantum era. From then on, the Riemann zeta function pops up in numerous different branches of statistical physics, from Brownian motion to lattice gas models. Since the topic of ultra-cold quantum gases has expanded rapidly in the past decade, we interpret the implications of the distribution of the Riemann zeta zeros in this area first.

Universality of some analytic functions

Now we will briefly discuss the universality property in analysis. The first result in this direction was achieved by M. Fekete in 1914, and is mentioned in [4]. He showed that there exists a real power series

$$\sum_{m=1}^{\infty} a_m x^n, \quad x \in [-1, 1],$$

which diverges at every point $x \neq 0$, and, moreover, for every continuous function $f(x)$, $x \in [-1,1]$, $f(0) = 0$, there exists a sequence $\{n_k, n_k \in \mathrm{N}\}$ such that

$$\sum_{m=1}^{\infty} a_m x^n \xrightarrow[k \to \infty]{} f(x)$$

uniformly in x.

J. Marcinkiewicz was the first who, in 1935, used the name of the universality. He proved [5] that if $\{h_n\}$ is a sequence of real numbers and $h_n \to \infty$ as $h \to \infty$, then there exists a continuous function $f(x) \in C_{[0,1]}$ such that for every measurable function $g(x)$, $x \in [0,1]$, we can find an increasing sequence $\{n_k, n_k \in \mathrm{N}\}$ satisfying

$$\frac{f(x + h_{n_k}) - f(x)}{h_{n_k}} \xrightarrow[k \to \infty]{} g(x)$$

almost everywhere on $[0, 1]$.

A result of G. D. Birkhoff [6] is related to the shifts of entire functions. He proved that there exists an entire function $f(s)$ such that for every entire function $g(s)$ there exists a sequence $\{a_n\} \in \mathbf{C}$ such that $f(s + a_n) \xrightarrow[k \to \infty]{} g(s)$ uniformly on compact subsets of the complex plane $\mathbf{C}$.

Now many universal objects are known. All above examples of universal objects are not explicitly given. Their existence obtained by using some non-effective theorems. Only in 1975 S. M. Voronin found an explicitly given universal object, and this object is the Riemann zeta-function $\zeta(s)$ defined, for $\sigma > 1$, by Dirichlet series

$$\varsigma(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

and by analytic continuation elsewhere. The universality of $\zeta(s)$ in some sense is related to the mentioned above Birkhoff result. Now we state the original Voronin theorem.

Theorem A [7]. *Suppose that* $0 < r < \frac{1}{4}$. *Let* $f(s)$ *be a continuous non-vanishing function on the disc* $|s| \leq r$ *which is analytic in the interior of this disc. Then, for every* $\varepsilon > 0$, *there exists a real number* $\tau = \tau(\varepsilon)$ *such that*

$$\max_{|s| \leq r} \left| \zeta \left(s + \frac{3}{4} + i\tau \right) - f(s) \right| < \varepsilon.$$

The Voronin theorem was improved and generalized. Denote by meas A the Lebesgue measure of a measurable set $A \subset \mathbf{R}$, and let, for $T > 0$,

$$\nu_T(\ldots) = \frac{1}{T} \mathrm{meas}\{\tau \in [0, T] : \ldots\},$$

where in place of dots a condition satisfied by τ is to be written. Then the modern version of the Voronin theorem is the following statement [8].

Theorem B. *Let K be a compact subset of the strip* $D = \left\{ s \in \mathbf{C} : \frac{1}{2} < \sigma < 1 \right\}$ *with connected complemenst. Let* $f(s)$ *be a continuous and non-vanishing on K function which is analytic in the interior of K. Then, for every* $\varepsilon > 0$,

$$\liminf_{T \to \infty} \nu_T \left(\sup_{s \in K} |\zeta(s + i\tau) - f(s)| < \varepsilon \right) > 0.$$

In Theorem A the approximation of a given function by a translation of $\zeta(s)$ is uniform only on a disc, while in Theorem B the disc is replaced by a more general compact set. On the other hand, by Theorem A there exists at least one number τ with an approximation property, and in Theorem B the set of such τ is sufficiently wide: it has a positive lower density. However, both Theorems A and B are non-effective, since it is impossible to indicate at least one value of τ.

In 1977, A. Reich obtained [9] the universality for Euler's product

$$\prod_{k=1}^{\infty}\left(1 - e^{2\pi i x_k}\left|p_k\right|^{-s}\right)^{-1},$$

where $0 \leq x_k < 1$, $k \in \mathbf{N}$, and p_k, $k \in \mathbf{N}$, are prime elements of some commutative semigroup with norm $|\cdot|$ and satisfying a certain axiom for the number of elements with norm $\leq x$.

A. Reich and S. M. Voronin for the proof of the universality used a theorem on the rearrangement of series in Hilbert spaces. The proof of Theorem B is based on a limit theorem in the sense of weak convergence of probability measures in the space of analytic functions.

The Hurwitz zeta-function $\varsigma(s, \alpha)$, $0 < \alpha \leq 1$, for $\sigma > 1$ is defined by

$$\varsigma(s, \alpha) = \sum_{m=0}^{\infty} \frac{1}{(m + \alpha)^s},$$

and by analytic continuation elsewhere. S. M. Gonek [10] and B. Bagchi [11] proved independently the universality of $\varsigma(s, \alpha)$ for rational and transcendental values of the parameter α. Since the function $\varsigma(s, \alpha)$ has no the Euler product over primes, the function $f(s)$ in Theorem B is not necessarily non-vanishing.

The Lerch zeta-function $L(\lambda, \alpha, s)$ is a generalization of the Hurwitz zeta-function, and is defined, for $\sigma > 1$ by

$$L(\lambda, \alpha, s) = \sum_{m=0}^{\infty} \frac{e^{2\pi i \lambda m}}{(m + \alpha)^s}.$$

If $\lambda \in \mathbf{Z}$ the Lerch zeta-function reduces to $\varsigma(s, \alpha)$. If $\lambda \notin \mathbf{Z}$ then the Lerch zeta-function is analytically continuable to an entire function. A. Laurinčikas proved [12] the universality of $L(\lambda, \alpha, s)$ with transcendental parameter α, and in [13], under some additional conditions, with rational α.

The works of H. Mishou and H. Bauer are devoted to universality of L-functions considered in algebraic number theory. H. Mishou in [14] and [15] proved the universality for L-functions with ideal class characters and for Hecke L-functions, respectively. H. Bauer obtained [16] the universality of Artin L-functions and applied this to zero distribution of these functions.

The universality of general Dirichlet series

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s},$$

where $a_m \in \mathbf{C}$ and $\{\lambda_m\}$ is an increasing sequence of real numbers, $\lim_{m \to \infty} \lambda_m = +\infty$, has been considered in [17]. We note that in the case of general Dirichlet series many additional conditions in hypotheses of universality theorems are involved. The most important of them is the linear independence over the field of rational numbers of the system of exponents $\{\lambda_m\}$.

The problem of effectivization of universality theorems for Dirichlet series has been investigated in [18] and [19]. The universality theorems are non-effective, since it is impossible to indicate at least one value of τ.

The universality of zeta-functions implies their functional independence. Note that the problem of independence of functions comes back to D. Hilbert. During the International Congress of Mathematicians in 1900 he raised a problem of algebraic-differential independence for functions given by Dirichlet series. D. Hilbert noted that an algebraic-differential independence of the Riemann zeta-function $\varsigma(s)$ can be proven using the algebraic-differential independence of the function $\Gamma(s)$ and the functional equation for $\varsigma(s)$, He also conjectured that there is no algebraic-differential equation with partial derivatives which can be satisfied by the function

$$\varsigma(s, x) = \sum_{m=1}^{\infty} \frac{x^m}{m^s}.$$

This conjecture was proven independently by D. D. Mordukhai-Boltovskoi and by A. Ostrowski [20]. A. G. Postnikov generalized the Hilbert problem for the function

$$L(x, s, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^s} x^m,$$

where $\chi(m)$ is a Dirichlet character modulo q.

All universality theorems stated above are of continuous type: in them translations of the imaginary part of the complex variable vary continuously in the interval $[0, T]$. Also, a discrete version of universality theorems exists. In this case, the translations of the imaginary part of the complex variable take values from some discrete sets, for example arithmetical progression.

Universality in Physics

Riemann zeta function has many interesting properties and has proven to be useful for many applications in physics. In [21] K.M. Bitar attempted to use one such property, described by Voronin's theorems, where a new characteristic distribution was discovered numerically and then calculated analytically. This distribution allows the use of the Riemann zeta function as a generator of pseudo random numbers.

K. Bitar [20] claims, that the Riemann zeta function is complex and according to Voronin's theorems can approximate any complex number when evaluated for an argument whose real part is in the region $\frac{1}{2}$ to 1 and for a very large imaginary part. As one increases the imaginary part, one will encounter other values where this property will repeat itself. In this critical strip if one then chooses a value for the real part of the argument, say $-\frac{3}{4}$, and then varies the imaginary part from say 10^6 upwards to larger values, these theorems assure us that one will generate for some such value any pre-chosen number with arbitrary precision and that this will happen more than once. K. Bitar has implemented this process with the aim of studying this characteristic property.

K. Bitar, N.N. Khuri and H.C. Ren [22] present a new formulation of Feynman's path integral, based on Voronin's theorems on the universality of the Riemann zeta function. The result is a discrete sum over "paths", each given by a zeta function.

In quantum mechanics or quantum field theory one has to evaluate path integrals of the following from:

$$(P(\phi)) = \frac{1}{\Omega} \int \prod_{j=1}^{v} d\phi(j) e^{-S(\phi(j))} \cdot P(\phi(l_1)...\phi(l_m)),$$

with

$$\Omega = \int \prod_{j=1}^{v} d\phi(j) e^{-S(\phi(j))}.$$

Here v is the number of lattice points, $\varphi(j)$ is the field at the j'th lattice point, $S(\varphi)$ is the Euclidean action, and $P(\phi(l_1)... (\phi(l_m))$ is a polynomial in the fields.

In paper [23] K. M. Bitar states that the path integral can write by series

$$\Omega(v) = \sum_{n=N_0}^{N} \frac{e^{-S(n;v)}}{p_v(n)} + O\left(N^{-\frac{1}{v}}\right)$$

where N is large, $N >> N_0$ and $p_v(n)$ is a measure, or density function. The probability density is the product of the independent probabilities for each component

$$p_v(n) = \prod_{j=1}^{v} W_\sigma(\gamma_\sigma(j;n)).$$

For $v=1$, the density $p_1(n)$ is simply given by $p_1(n) = W_\sigma(\gamma(1; n))$, with $\gamma(1; n)$ defined in equality

$$\gamma_\sigma(j;n) = \log|\varsigma(\sigma + ijh + in\Delta)| \equiv \phi^{(n)}(x_j),$$

here $j=1,..., v$, $n=1,..., N$, $\frac{1}{2} < \sigma < 1$, any real $\Delta > 0$, $\Delta >> h$, and $\Delta > hv$. For $v>1$, and $h>1$, the values of $\gamma_\sigma(j; n)$ and $\gamma_\sigma(j + l; n)$ are uncorrelated.

By summing over n we sum over all "paths", but the density $p_v(n)$ insures that we have the correct Jacobian for quantum mechanics.

A new measure that leads to the correct quantum mechanics is explicitly given.

Thus, a path integral can be easily approximated by summation. Khalil M. Bitar [23] notes, that the continuous form of Voronin's theorem, leads us to contemplate a far-reaching conjecture. This concerns taking the limit a → 0, where a is the lattice spacing. If in this limit a measure, $p_\infty(n)$, exists, then essentially any quantum mechanical problem can be reduced to quadratures. The important thing to remember is that $p_\infty(n)$ depends only on the properties of the Riemann zeta function. All the physics enters through S(n). Clearly, the existence of measure $p_\infty(n)$ and an explicit formula for it would be a remarkable achievement.

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Summary

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The paper presents universality of some analytic functions, which is a very exceptional and useful property of zeta and *L*-functions. The property for the Riemann zeta-function was discovered by S. M. Voronin. Later, many mathematicians, such as S. M. Gonek, A. Reich, B. Bagchi, A. Laurinčikas, K. Matsumoto, R. Garunkštis, J. Steuding and others improved and generalized Voronin's theorem. Physicists examine numerous models from different branches of physics (from classical mechanics to statistical physics) where this function plays an integral role.

Keywords: *Rymann zeta-function, L-function, universality, applications in physics.*

Santrauka

UNIVERSALUMO TEOREMOS FIZIKOJE

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Darbe pateikiamos kai kurių analizinių funkcijų tolydaus tipo universalumo teoremos. Universalumas – įdomi ir reikšminga dzeta ir *L*-funkcijų savybė. Šią savybę Rymano dzeta funkcijai pirmasis įrodė S. Voroninas. Po to daug matematikų (S. M. Gonekas, A. Reichas, B. Bagchi, A. Laurinčikas, K. Matsumoto, R. Garunkštis, J. Steudingas ir kt.) apibendrino ir pagerino universalumo rezultatus. Rymano dzeta funkcija yra svarbus instrumentas atliekant klasikinės mechanikos, kvantinės mechanikos, statistinės fizikos tyrimus.

Prasminiai žodžiai: *Rymano dzeta funkcija, L-funkcija, universalumas, taikymai fizikoje.*