ON THE GENERALIZATIONS OF UNIVERSALITY THEOREM FOR *L*-FUNCTIONS OF ELLIPTIC CURVES

Virginija Garbaliauskienė
Šiauliai University, Šiauliai State College

Introduction

Elliptic curves are one of the most important objects in algebraic geometry and, in general, in mathematics. They have many practical applications, for example, in cryptography, or for factoring positive integers and primary testing. To study the properties of elliptic curves H. Hasse introduced *L*-functions attached to these curves.

Let E be an elliptic curve over the field of rational numbers Q defined by the Weierstrass equation

$$y^2 = x^3 + ax + b, \quad a,b \in \mathbf{Z}.$$

Denote by $\Delta = -16(4a^3 + 27b^2)$ the discriminant of the curve E, and suppose that $\Delta \neq 0$. Then the roots of the cubic $x^3 + ax + b$ are distinct, and the curve E is non-singular.

For each prime p, denote by $\nu(p)$ the number of solutions of the congruence

$$y^2 \equiv x^3 + ax + b (\text{mod}\, p),$$

and let $\lambda(p) = p - \nu(p)$. Let $s = \sigma + it$ be a complex variable. Then the *L*-function of the elliptic curve E is defined by the Euler product

$$L_E(s) = \prod_{p|\Delta}\left(1-\frac{\lambda(p)}{p^s}\right)^{-1}\prod_{p\nmid\Delta}\left(1-\frac{\lambda(p)}{p^s}+\frac{1}{p^{2s-1}}\right)^{-1}.$$

In view of the Hasse estimate

$$|\lambda(p)| < 2\sqrt{p},$$

the infinite product for $L_E(s)$ converges absolutely and uniformly on compact subsets of the half-plane $D_a = \left\{s \in C : \sigma > \frac{3}{2}\right\}$, and defines there an analytic function with no zeros. Moreover, since the Shimura-Taniyama conjecture was proven in [1], the function $L_E(s)$ is analytically continuable to an entire function, and it satisfies the functional equation

$$\left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s) L_E(s) = \eta \left(\frac{\sqrt{q}}{2\pi}\right)^{2-s} \Gamma(2-s) L_E(2-s),$$

where q is a positive integer composed from prime factors of the discriminant Δ, $\eta = \pm 1$ is the root number, and $\Gamma(s)$ denotes the Euler gamma-function.

The function $L_E(s)$also can be written in the form of Dirichlet series

$$L_E(s) = \sum_{m=1}^{\infty} \frac{\lambda(m)}{m^s},$$

where

$$\lambda(m) = \prod_{p^\alpha \| m} \lambda(p^\alpha),$$

and $p^\alpha \| m$ means that $p^\alpha \mid m$ but $p^{\alpha+1} \nmid m$, and the series also converges absolutely in D_a.

Universality theorem of *L*-functions of elliptic curves

The universality is a very interesting property of zeta and *L*-functions. J. Marcinkiewicz was the first who used the name of the universality in 1935.

The first universality theorem for the Riemann zeta-function $\varsigma(s)$ defined, for $\sigma > 1$, by

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s},$$

And by analytic continuation elsewhere, was discovered by S. M. Voronin in 1975 [6]. Let $0 < r < \frac{1}{4}$, and let $f(s)$ be a continuous non-vanishing function on the disc $|s| \leq r$ which is analytic in the interior of this disc. Then S. M. Voronin proved that for every $\varepsilon > 0$ there exists a real number $\tau = \tau(\varepsilon)$ such that

$$\max_{|s|\leq r}\left|\zeta\left(s+\frac{3}{4}+i\tau\right)-f(s)\right| < \varepsilon.$$

Later, S. M. Gonek, A. Reich, B. Bagchi, A. Laurinčikas, K. Matsumoto, R. Garunkštis, J. Steuding, W. Schwarz, H. Mishou, R. Kačinskaitė, R. Šleževičienė, R. Macaitienė, J. Ignatavičiūtė, J. Genys and others generalized and improved the Voronin theorem. It turns out that a given analytic function $f(s)$ can be approximated by translations of $\zeta(s)$ uniformly on more general sets than a disc. Denote by meas$\{A\}$ the Lebesgue measure of a measurable set $A \subset \mathbf{R}$, and let for $T > 0$,

$$\nu_T(...) = \frac{1}{T}\text{meas}\{\tau \in [0,T]:...\},$$

where in the place of dots a condition satisfied by τ is to be written. Then the last version of the Voronin theorem is contained in the following statement, see, for example, [5]. Let K be a compact subset of the strip $D = \left\{s \in \mathbf{C}: \frac{1}{2} < \sigma < 1\right\}$ with connected complement. Let $f(s)$ be a continuous and non-vanishing on K function which is analytic in the interior of K. Then, for every $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}|\zeta(s+i\tau) - f(s)| < \varepsilon\right) > 0. \quad (1)$$

The later theorem shows that there exist many translations $\zeta(s + i\tau)$ which approximate a given analytic function $f(s)$: the set of τ in (1) has a positive lower density.

The majority of classical zeta and L-functions are universal in the Voronin sense. The Linnik-Ibragimov conjecture says that all functions in some half-plane, given by Dirichlet series, analytically continuable to the left of the absolute convergence half-plane and satisfying some natural growth conditions are universal in the Voronin sense. All recent results on the universality of the Dirichlet series support that conjecture.

The aim of this paper is to give a survey on the generalizations of the continuous type's universality theorem of L-functions of elliptic curves.

The universality of L-functions of new forms has been proved in [5]. From this the universality of $L_E(s)$ follows. Let $D = \left\{s \in \mathbf{C}: 1 < \sigma < \frac{3}{2}\right\}$.

Theorem 1 [3]. *Suppose that E is a non-singular elliptic curve over the field of rational numbers. Let K be a compact subset of the strip D with connected complement, and let be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}|L_E(s+i\tau) - f(s)| < \varepsilon\right) > 0.$$

Some generalizations of Theorem 1

Theorem 1 of L-functions of elliptic curves can be generalized for powers of $L_E(s)$ as well as the universality theorem of the derivative $L'_E(s)$, and a weighted universality theorem can be obtained.

The first mentioned result is given in [3].

Theorem 2. *Suppose that E is a non-singular elliptic curve over the field of rational numbers. Let K be a compact subset of the strip D with connected complement, and let f(s) be a continuous non-vanishing function on K which is analytic in the interior of K. Then, for every* $\varepsilon > 0$ *and* $k \in \mathbf{N}$,

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}|L_E^k(s+i\tau) - f(s)| < \varepsilon\right) > 0.$$

The universality for the derivative $L'_E(s)$ is contained in the following statement.

Theorem 3 [4]. *Let K be a compact subset of the strip D with connected complement, and let f(s) be a continuous function on K which is analytic in the interior of K. Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}|L'_E(s+i\tau) - f(s)| < \varepsilon\right) > 0.$$

Note that, differently from Theorem 3, in this case the function $f(s)$ can be vanishing on K.

Now let T_0 be a fixed positive number, and let $w(t)$ be a positive function of bounded variation on $[T_0, \infty)$. Define

$$U = U(T,w) = \int_{T_0}^{T} w(t)\,\mathrm{d}t$$

and suppose that

$$\lim_{T\to\infty} U(T,w) = +\infty. \quad (2)$$

For the proof of Theorems 1 and 3, limit theorems in the sense of weak convergence of probability measures in the space of analytic functions for the considered functions are applied. On the other hand, the identification of the limit measures in these theorems is based on the ergodic theory, more precisely, on the Birkhoff-Khinchine theorem.

As we have already mentioned, a weighted analogue of Theorem 1 is known. Therefore, to obtain a weighed universality theorem for the function $L_E(s)$ we need a certain additional condition.

Denote by E_η the expectation of the random variable η. Let $X(\tau,\omega)$ be an ergodic process, $E|X(\tau,\omega)|<\infty$, with sample paths almost surely integrable in the Riemann sense over every finite interval. Suppose that

$$\frac{1}{U}\int_{T_0}^{T} w(\tau)\,X(t+\tau,\omega)\,\mathrm{d}\tau = EX(0,\omega) + o\left(1+|t|\right)^{\delta} \quad (3)$$

almost surely for all $t \in \mathbf{R}$ with some $\delta > 0$ as $T \to \infty$. Let I_A denote the indicator function of the set A.

Then, in [2] the following result was given.

Theorem 4. *Let the function* $w(t)$ satisfy conditions (2) and (3), and let K and $f(s)$ be the same as in Theorem 1. *Then, for every* $\varepsilon > 0$,

$$\liminf_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\left(\sup\limits_{s\in K}\left|L_E(s+i\tau)-f(s)\right|<\varepsilon\right)} \, d\tau > 0.$$

Proof of Theorem 4 is given in [2].

References

1. Breuil C., Conrad B., Diamond F., Taylor R., 2001, On the modularity of elliptic curves over **Q**: wild 3-adic exercises. *J. Amer. Math. Soc.* Vol. 14, pp. 843-939.
2. Garbaliauskienė V., 2004, A weighted universality theorem for zeta-functions of elliptic curves. *Liet. Matem. Rink*. Vol. 44, Spec. issue, pp. 43-47.
3. Garbaliauskienė V., Laurinčikas A., 2005, Some analytic properties for *L*-functions of elliptic curves. *Proc. Inst. Math. NAN Belarus*. Vol. 13, No. 1, pp. 75-82.
4. Garbaliauskienė V., Laurinčikas A., 2007, The universality of the derivatives of *L*-functions of elliptic curves. *Analytic and Probab. Methods in Number Theory, Proceed. the 4th Palanga Confer.* pp. 24-29. TEV.
5. Laurinčikas A., 1996, *Limit Theorems for the Riemann Zeta-Function*. Kluwer, Dordrecht.
6. Voronin S. M., 1975, Theorem on the "universality" of the Riemann zeta-function. *Math. USSR Izv.* Vol. 9. pp. 443-453.

Summary

ON THE GENERALIZATIONS OF UNIVERSALITY THEOREMS FOR *L*-FUNCTIONS OF ELLIPTIC CURVES

V. Garbaliauskienė

In the paper, the continuous type's universality theorem for *L*-functions of elliptic curves is discussed and its generalizations in three directions – for positive integer powers and derivatives of *L*-functions of elliptic curves as well as the weighted universality theorem of *L*-functions of elliptic curves – are given. The proofs of the universality are based on limit theorems in the sense of weak convergence of probability measures in functional spaces.

Keywords: elliptic curve, *L*-function, universality.

Santrauka

ELIPSINIŲ KREIVIŲ *L* FUNKCIJŲ UNIVERSALUMO TEOREMOS APIBENDRINIMAI

V. Garbaliauskienė

Straipsnyje pateikiama tolydaus tipo elipsinių kreivių *L* funkcijų universalumo teorema, kuri apibendrinama trimis aspektais: nagrinėjamas elipsinių kreivių *L* funkcijos laipsnių bei jos išvestinės universalumas ir pateikiama universalumo teorema su svoriu elipsinių kreivių *L* funkcijai. Universalumo įrodymai remiasi ribinėmis teoremomis silpno tikimybinių matų konvergavimo prasme funkcinėse erdvėse.

Prasminiai žodžiai: elipsinė kreivė, *L* funkcija, universalumas.

Įteikta 2015-09-30
Priimta 2015-10-12