A WEIGHTED UNIVERSALITY THEOREM FOR *L*-FUNCTIONS OF ELLIPTIC CURVES

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Introduction

Let *E: y² = x³ + ax + b, a,b∈* Z, be an elliptic curve. Suppose that the discriminant $\Delta = -16(4a^3 + 27b^2) \neq 0$. In this case an elliptic curve is non-singular.

Let, for a prime *p*, *v*(*p*) be the number of solutions of the congruence

$$y^2 \equiv x^3 + ax + b \pmod{p},$$

and let $\lambda(p) = p - v(p)$. Then the classical result of H. Hasse asserts that

$$|\lambda(p)| \leq 2\sqrt{p}$$

for each prime *p*.

Let $s = \sigma + it$ be a complex variable. To the curve *E* we attach the *L*-function $L_E(s)$ defined, for $\sigma > \frac{3}{2}$, by

$$L_E(s) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p\nmid\Delta}\left(1 - \frac{\lambda(p)}{p^s} + \frac{1}{p^{2s-1}}\right)^{-1}.$$

The function $L_E(s)$ is analytic continuation to an entire function and satisfies the functional equation

$$\left(\frac{\sqrt{q}}{2\pi}\right)^s \Gamma(s) L_E(s) = \eta\left(\frac{\sqrt{q}}{2\pi}\right)^{2-s} \Gamma(2-s) L_E(2-s),$$

where *q* is a positive integer composed from prime factors of the discriminant Δ, *η*=±1 is the root number, and Γ(*s*), as usual, denotes the Euler gamma-function.

Universality of the function $L_E(s)$

The papers [1], [2], [3] are devoted to the universality of the function $L_E(s)$. For example, in [3] the following statement is given. Let meas{*A*} denote the Lebesque measure of the set *A*⊂R, and let, for T > 0,

$$\nu_T(...) = \frac{1}{T}\text{meas}\{\tau\in[0,T]:...\},$$

where in place of dots a condition satisfied by *τ* is to be written.

Theorem A. *Suppose that E is a non-singular elliptic curve over the field of rational numbers. Let K be a compact subset of the strip*

$$D = \left\{s \in \mathbf{C}: 1 < \sigma < \frac{3}{2}\right\} \text{ with connected complement,}$$

and let f(s) be a continuous non-vanishing function on K, which is analytic in the interior of K. Then, for every ε > 0,

$$\liminf_{T\to\infty} \nu_T\left(\sup_{s\in K}|L_E(s+i\tau) - f(s)| < \varepsilon\right) > 0.$$

The paper [2] contains a discrete version of Theorem A.

The aim of this note is to obtain a weighted universality theorem for the function $L_E(s)$.

Let T_0 be a fixed positive number, and let $w(\tau)$ be a positive function of bounded variation on $[T_0, \infty)$. Define

$$U = U(T,w) = \int_{T_0}^{T} w(\tau)\,d\tau,$$

and suppose that $\lim_{T\to\infty} U(T,w) = +\infty$. Moreover, we need some weighted analogue of the Birkhoff-Khinchine theorem. Denote by E_ξ the mean of the random element ξ. Let $X(\tau,\omega)$, $\tau \in \mathbb{R}$, be an ergodic process defined on a certain probability space, $E|X(\tau,\omega)| < \infty$, with sample paths almost surely integrable in the Riemann sense over every finite interval. Suppose that the function $w(\tau)$ satisfies

$$\frac{1}{U}\int_{T_0}^{T} w(\tau)X(t+\tau,\omega)\,\mathrm{d}\tau = EX(0,\omega) + \mathrm{o}(1+|t|)^\delta \qquad (1)$$

almost surely for all $t \in \mathbb{R}$ with some $\delta > 0$ as $T\to\infty$.

Denote by I_A the indicator function of the set A.

Theorem 1. *Suppose that condition (1) is satisfied. Let K and f(s) be the same as in Theorem A. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\left(\tau:\ \sup_{s\in K}|L_E(s+i\tau)-f(s)|<\varepsilon\right)}\,\mathrm{d}\tau > 0.$$

Clearly, Theorem A is a partial case of Theorem 1.

The proof of Theorem 1 is based on a limit theorem in the sense of the weak convergence of probability measures in the space of analytic functions for the function $L_E(s)$. Let G be a region in the complex plane. Denote by $H(G)$ the space of functions analytic on G, equipped with the topology of uniform convergence on compacta. Let, for $V > 0$,

$$D_V = \left\{ s \in \mathbb{C}: 1 < \sigma < \frac{1}{2}, |t| < V \right\}.$$ Denote by $\mathsf{B}(S)$ the class of Borel sets of the space S, and define the probability measure

$$P_T(A) = \frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau: L_E(s+i\tau)\in A\}}\,\mathrm{d}\tau,\ A\in\mathsf{B}(H(D_V)).$$

Denote by γ the unit circle $\gamma = \{s\in\mathbb{C}:|s|=1\}$ on the complex plane, and let

$$\Omega = \prod_{p} \gamma_p,$$

where $\gamma_p = \gamma$ for each prime p. With product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological Abelian group. Therefore, there exists the probability Haar measure

m_H on $(\Omega, \mathsf{B}(\Omega))$. Thus, we obtain a probability space $(\Omega, \mathsf{B}(\Omega), m_H)$. Let $\omega(p)$ stand the projection of $\omega \in \Omega$ into the coordinate space γ_p, and the probability space $(\Omega, \mathsf{B}(\Omega), m_H)$ define the $H(D_V)$-valued random element $L_E(s,\omega)$ by the formula

$$L_E(s,\omega) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s} + \frac{\omega^2(p)}{p^{2s-1}}\right)^{-1} \times$$

$$\times \prod_{p|\Delta}\left(1 - \frac{\lambda(p)\omega(p)}{p^s}\right)^{-1},\quad \omega\in\Omega. \qquad (2)$$

Lemma 2. *Under condition (1) the probability measure P_T converges weakly to the distribution of the random element $L_E(s,\omega)$ as $T\to\infty$*

Proof. It is not difficult to see that, for $\sigma > \dfrac{3}{2}$,

$$L_E(s) = \prod_{p|\Delta}\left(1 - \frac{\lambda(p)}{p^s}\right)^{-1} \prod_{p|\Delta}\left(1 - \frac{\alpha(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta(p)}{p^s}\right)^{-1},$$

where $\lambda(p)=\alpha(p)+\beta(p)$, and

$$|\alpha(p)| \le \sqrt{p},\ \ |\beta(p)| \le \sqrt{p}.$$

Therefore, $L_E(s)$ is the Matsumoto zeta-function [4], [5] with parameters $\alpha=0$ and $\beta = \dfrac{1}{2}$. Moreover, since by [6] every *L*-function attached to a non-singular elliptic curve over the field of rational numbers is the *L*-function attached to a certain newform of weight 2 of some congruence subgroup, we have that $L_E(s)$ is an entire function, and, for $\sigma > 1$,

$$L(\sigma + it) = \mathrm{O}(|t|^{c_1}),\ \ \ |t|\ge t_0,\ c_1 > 0,$$

and [7]

$$\int_0^T |L(\sigma+it)|^2\,\mathrm{d}t = \mathrm{O}(T),\ \ T\to\infty.$$

Therefore, by Theorem 8 of [5] the probability measure

$$\frac{1}{U}\int_{T_0}^{T} w(\tau) I_{\{\tau: L_E(s+i\tau)\in A\}}\,\mathrm{d}\tau,\ A\in\mathsf{B}(H(D)),$$

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where $D_1 = \{s \in \mathbf{C}:\ \sigma > 1\}$ converges weakly to the distribution of the $H(D)$-valued random element defined by (2) as $T \to \infty$. Since the function $u : H(D_1) \to H(D)$ defined by the coordinate-wise restriction is continuous, hence the Lemma follows.

For the proof of Theorem 1 the support of the limit measure in Lemma 2 is needed. We recall that the support of a probability measure P defined on $(S, \mathsf{B}(S))$ is a minimal closed set $S_p \subset S$ such that $P(S_p) = 1$. The support S_p consists of all $x \in S$ such that for every neighbourhood G of x the inequality $P(G) > 0$ is satisfied.

Denote the limit measure in Lemma 2 by P_{L_E}, i. e. P_{L_E} is the distribution of the random element $L_E(s, \omega)$.

Lemma 3. *The support of the measure P_{L_E} is the set*

$$S_V = \{g \in H(D_V) : g(s) \neq 0 \ \text{ or } \ g(s) \equiv 0\}.$$

Proof. The proof of the lemma coincides with that of Lemma 8 in [3]. A sketch of the proof is also given in [3], Lemma 5. A more general statement for the Matsumoto zeta-function under some additional condition is contained in [5], Lemma 6.

Proof of Theorem 1

Let K be a compact subset of the strip D with connected complement. Then there exists $V > 0$ such that $K \subset D_V$.

First we suppose that the function $f(s)$ has a non-vanishing continuation to the rectangle D_V. Let G be the set of functions $g \in H(D_V)$ such that

$$\sup_{s \in K} | g(s) - f(s) | < \varepsilon.$$

Clearly, the set G is open. Moreover, by Lemma 3 we have that $G \subset S_V$. It is well known that the probability measure P_n converges weakly to P (P_n and P are given on $(S, \mathsf{B}(S))$ if and only if

$$\liminf_{n \to \infty} P_n(G) \geq P(G) \qquad (3)$$

for all open sets G of S. Therefore, in view of Lemmas 2 and 3

$$\liminf_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\left(\tau:\ \sup_{s \in K} |L_E(s+i\tau)-f(s)| < \varepsilon \right)} \, d\tau \geq P(G) > 0,$$

and in this case the theorem is proved.

The general case is reduced to the above-proved case. Let $f(s)$ be the same as in the statement of Theorem 1. Then by the Mergelyan theorem, see, for example, [8], there exists a sequence of polynomials $\{p_n(s)\}$ such that $p_n(s) \to f(s)$ as $n \to \infty$ uniformly on K. Since $f(s) \neq 0$ on K, there exists a sufficiently large n_0 such that $p_{n_0}(s) \neq 0$ on K and

$$\sup_{s \in K} | f(s) - p_{n_0}(s) | < \frac{\varepsilon}{4}.$$

However, the polynomial $p_{n_0}(s)$ has only finitely many zeros, and therefore we can find a region G with connected complement such that $K \subset G$ and $p_{n_0}(s) \neq 0$ on G. Hence, we can choose a continuous version of $\log p_{n_0}(s)$ on G, which is analytic in the interior of G. By the Mergelyan theorem again we can find a polynomial $q_n(s)$ such that

$$\sup_{s \in K} | p_{n_0}(s) - e^{q_n(s)} | < \frac{\varepsilon}{4}.$$

The latter two inequalities show that

$$\sup_{s \in K} | f(s) - e^{q_n(s)} | < \frac{\varepsilon}{2}.$$

Since, clearly, $e^{q(s)} \neq 0$, the first part of the proof implies

$$\liminf_{T \to \infty} \frac{1}{U} \int_{T_0}^{T} w(\tau) I_{\left(\tau:\ \sup_{s \in K} |L_E(s+i\tau)-e^{q(s)}| < \frac{\varepsilon}{2} \right)} \, d\tau > 0.$$

This and inequality (3) yield the assertion of the theorem.

For example, we can take $w(\tau) = \tau^{-1}$.

References

1. Garbaliauskienė V., Laurinčikas A., 2005, Some analytic properties for *L*-functions of elliptic curves. *Proc. Inst. Math. NAN Belarus*, Vol. 13, No. 1, p. 75-82.
2. Garbaliauskienė V., Laurinčikas A., 2004, Discrete value-distribution of *L*-functions of elliptic curves. *Publications de I'Institut Mathématique,* Vol. 79, No. 90, p. 65-71.
3. Laurinčikas A., Matsumoto K., Steuding J., 2003, The universality of *L*-functions associated to newforms. *Izv. Math.*, Vol. 67, p. 77-90.

4. Matsumoto K., 1989, A probabilistic study on the value-distribution of Dirichlet series attached to certain cusp forms. *Nagoya Math. J.*, Vol. 116, p. 123-138.
5. Laurinčikas A., 1998, On the Matsumoto zeta-function. *Acta Arith.*, Vol. 84, No. 1, p. 1-16.
6. Breuil C., Conrad B., Diamond F., Taylor R., 2001, On the modularity of elliptic curves over **Q**: wild 3-adic exercises. *J. Amer. Math. Soc.*, Vol. 14, p. 843-939.
7. Matsumoto K., 1990, Value distribution of zeta functions. *Lecture Notes in Math.*, Vol. 1434, p. 178-187.
8. Walsh J. L., 1960, Interpolation and Approximation by Rational Functions in the Complex Domain. *Amer. Math. Soc. Collog. Publ.,* Vol. 20.

Summary

A WEIGHTED UNIVERSALITY THEOREM FOR *L*-FUNCTIONS OF ELLIPTIC CURVES

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In the paper, a short survey on universality results for *L*-functions of elliptic curves over the field of rational numbers is given and weighted universality theorem is proven. All stated universality theorems are of continuous type. The proof of the universality for *L*-functions of elliptic curves is based on a limit theorem in the sense of weak convergence of probability measures in the space of analytic functions.

Keywords: *elliptic curve, L-function, universality.*

Santrauka

UNIVERSALUMO SU SVORIU TEOREMA ELIPSINIŲ KREIVIŲ *L*-FUNKCIJOMS

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Straipsnyje pateikiama trumpa universalumo rezultatų elipsinių kreivių *L*-funkcijoms apžvalga bei įrodoma universalumo su svoriu teorema. Visos pateiktos teoremos yra tolydaus tipo. Universalumo įrodymas remiasi ribine teorema silpno tikimybinių matų konvergavimo prasme analizinių funkcijų erdvėje.

Prasminiai žodžiai: *elipsinė kreivė, L-funkcija, universalumas.*