Stability of reaction–diffusion systems with stochastic switching*

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Abstract. In this paper, we investigate the stability for reaction systems with stochastic switching. Two types of switched models are considered: (i) Markov switching and (ii) independent and identically distributed switching. By means of the ergodic property of Markov chain, Dynkin formula and Fubini theorem, together with the Lyapunov direct method, some sufficient conditions are obtained to ensure that the zero solution of reaction–diffusion systems with Markov switching is almost surely exponential stable or exponentially stable in the mean square. By using Theorem 7.3 in [R. Durrett, *Probability: Theory and Examples*, Duxbury Press, Belmont, CA, 2005], we also investigate the stability of reaction–diffusion systems with independent and identically distributed switching. Meanwhile, an example with simulations is provided to certify that the stochastic switching plays an essential role in the stability of systems.

Keywords: reaction–diffusion system, Markov switching, ergodic theory, stability.

1 Introduction

Random disturbance exists in the natural world owing to various environmental noise. Such phenomena can be described by stochastic differential systems, which have been successfully applied to problems in mechanics, engineering, electronics, automation, economics, etc. Therefore, the dynamics of stochastic systems have become a hot topic in recent years [9, 11, 20–23, 25, 29]. However, a simple color noise is said to be telegraph

noise, which is demonstrated as a switching between two or more environmental regimes. If the switching is no memory and the switched times follow an exponential distribution, the switched regime can be modelled by a finite state Markov chain. If the switched times are independent random variables and the time intervals have the same expectations, the switched regime can be modelled by independent and identically distributed switching. Hajnal [8] investigated the behavior of finite nonhomogenous Markov chain having regular transition matrices in 1956. Salehi and Jadbabaie [27] provided consensus algorithms under ergodic stationary graph process. In [12], Kim extended Markov switching model to a more general state space model and proposed a new algorithm. In [3–5,10,24], the authors studied the stability of a linear systems with Markov switching or independent and identically distributed process. Gray et al. [7] examined the effects of telegraph noise on the classic SIS model and proved the extinction and persistence for a finite state Markov chain. In [17], the authors investigated the synchronization of complex networks with stochastic switching. In practical application, influence of diffusion is inevitable. Therefore, we must consider the state variables varying with time and space variables. With respect to reaction–diffusion systems, there are many reports of the stability in the literature [6, 14–16, 18, 19, 26, 28, 30]. For instance, Luo and Zhang [26] investigated the asymptotical stability in probability and almost sure exponential stability of stochastic reaction–diffusion systems by using the Lyapunov method. In [30], Zhu et al. proved the stability for stochastic bidirectional associative memory neural networks with reaction–diffusion term. In [15], Li et al. investigate the synchronization problem for delayed reaction–diffusion neural networks (RDNNs) with unknown time-varying coupling strengths by an adaptive learning control strategy. However, to the best of our knowledge, none of the authors have considered the stability of reaction–diffusion systems with stochastic switching, which motivates our current research.

The aim of this paper is to study stability for reaction–diffusion systems with stochastic switching of finite state space. Two types of switched models are considered:

(i) Markov switching and
(ii) independent and identically distributed switching.

If the switched sequence is Markov chain, by means of ergodic property for Markov chain [1] and the Lyapunov method, sufficient condition is obtained to confirm that the zero solution of switched system can achieve almost sure stability. If the switched sequence is an independent and identically distributed process, we also derive the stability by the theorem in Durrent [2]. It is interesting that if some subsystems are not stable, but the other subsystems are stable, eventually the overall system will reach stability, which means that Markov switching, as well as independent and identically distributed switching, play an essential role in the stable behavior of reaction–diffusion systems. In addition, an example with simulations is provided to demonstrate the applicability of our results. The rest of this paper is organized as follows. Reaction–diffusion system model with stochastic switching is presented in Section 2 together with some definitions of stability for the zero solution. In Section 3, almost surely exponential stability and exponential stability in the mean square of switching systems are derived. A numerical example is given to demonstrate our results in Section 4.

Notations. Let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. The symbol $^{\mathrm{T}}$ represents the transpose. I_n stands for the $n \times n$ identity matrix. For $v = (v_1, v_2, \ldots, v_n)^{\mathrm{T}} \in \mathbb{R}^n$, we define the norm as $\|v\| = (\sum_{i=1}^{n} v_i^2)^{1/2}$. Let $y = (y_1, y_2, \ldots, y_m)^{\mathrm{T}}$, $U_i = (u_{i1}, u_{i2}, \ldots, u_{im})^{\mathrm{T}}$, $i = 1, 2, \ldots, n$, and matrix $U = (U_1, U_2, \ldots, U_n)^{\mathrm{T}}$, we denote $\nabla \cdot U_i = \partial u_{i1}/\partial y_1 + \partial u_{i2}/\partial y_2 + \cdots + \partial u_{im}/\partial y_m$, $\nabla \cdot U = (\nabla \cdot U_1, \nabla \cdot U_2, \ldots, \nabla \cdot U_n)^{\mathrm{T}}$.

2 System description and definitions

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ be a probability space related to an increasing right-continuous filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$. $\mathbf{E}[\cdot]$ denotes mathematical expectation. We present two types of stochastic switchings in probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$.

2.1 Markov switching process

Let $r(t)$ be a right-continuous Markov chain on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ taking values in the state space $\mathbb{S} = \{1, 2, \ldots, M\}$ with generator $\Gamma = (\delta_{ij})_{M \times M}$ generated by $\mathbf{P}\{r(t + \epsilon) = j \mid r(t) = i\} = \delta_{ij}\epsilon + o(\epsilon)$, where $\epsilon > 0$, δ_{ij} is the transition rate from i to j satisfying $\delta_{ij} > 0$, $i \neq j$, and $\delta_{ii} = -\sum_{1 \leqslant j \leqslant M, j \neq i} \delta_{ij}$. Let $\{\tau_k\}_{k \geqslant 0}$ be a sequence of finite-valued $\mathcal{F}_t$ stopping times satisfying $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$, and let $\lim_{k \to +\infty} \tau_k = +\infty$. $r(t) = \sum_{k=0}^{+\infty} r(\tau_k)\mathcal{I}_{[\tau_k, \tau_{k+1})}(t)$, where $\mathcal{I}_{[\tau_k, \tau_{k+1})}$ represents the indicator function of set $[\tau_k, \tau_{k+1})$. Given that $r(\tau_k) = i$, the exponential distribution of the random variable $\tau_{k+1} - \tau_k$ is defined as $\mathbf{P}(\tau_{k+1} = j \mid \tau_k = i) = -\delta_{ij}/\delta_{ii}$, $j \neq i$, $\mathbf{P}(\tau_{k+1} - \tau_k \geqslant t \mid r(\tau_k) = i) = \mathrm{e}^{\delta_{ii}t}$ for all $t \geqslant 0$. Moreover, the Markov chain has a unique stationary distribution $\Pi = (\pi_1, \pi_2, \ldots, \pi_M)^{\mathrm{T}}$ satisfying $\Pi\Gamma = 0$ and $\sum_{i=1}^{M} \pi_i = 1$.

Let X be a compact set with smooth boundary ∂X and measure $\mu(X) > 0$ in $\mathbb{R}^m$; $L^2(\mathbb{R} \times X)$ denotes the space of real Lebesgue measurable functions of $\mathbb{R} \times X$. It is a Banach space for the 2-norm $\|u(t)\|_2 = (\int_X \|u(t, y)\|^2 \, \mathrm{d}y)^{1/2}$, where $\|\cdot\|$ is Euclid norm.

Consider the following reaction–diffusion with Markovian switching:

$$
\begin{aligned}
\frac{\partial u(t, y)}{\partial t} &= \nabla \cdot \big(D(t, y, u(t, y)) \circ \nabla u(t, y)\big) + f\big(t, y, u(t, y), r(t)\big), \\
&\quad (t, y) \in [0, +\infty) \times X, \\
u(0, y) &= \phi(y), \quad y \in X, \qquad \frac{\partial u(t, y)}{\partial N} = 0, \quad (t, y) \in [0, +\infty) \times \partial X,
\end{aligned}
\tag{1}
$$

where $u(t, y) = (u_1(t, y), u_2(t, y), \ldots, u_n(t, y))^{\mathrm{T}}$, $y = (y_1, y_2, \ldots, y_m)^{\mathrm{T}}$, $D(t, y, u)$ is a smooth diffusion operator satisfying that $D(t, y, u) = (D_{ik}(t, y, u))_{n \times m} \geqslant 0$, $D_{ik}(t, y, u) \geqslant 0$, $\nabla u = (\nabla u_1, \nabla u_2, \ldots, \nabla u_n)^{\mathrm{T}}$, $\nabla u_i = (\partial u_i/\partial y_1, \partial u_i/\partial y_2, \ldots, \partial u_i/\partial y_m)^{\mathrm{T}}$, $i = 1, 2, \ldots, n$, $D \circ \nabla u = (D_{ik}\partial u_i/\partial y_k)_{n \times m}$ is a Hadamard product of matrix D and ∇u, $f : [0, +\infty) \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{S} \to \mathbb{R}^n$ is Borel measurable function.

2.2 Independent and identically distributed switching process

Let $r(t)$ be independent and identically distributed sequence on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$ taking value in $\mathbb{S} = \{1, 2, \ldots, M\}$. Let $\{\tau_k\}_{k \geqslant 0}$ be a sequence of finite-valued $\mathcal{F}_t$ stopping time $0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{k \to +\infty} \tau_k = +\infty$. Denote $\Delta\tau_k = \tau_k - \tau_{k-1}$. It is assumed that $\{\Delta\tau_k\}$ is the sequence of independent and identically distributed random variables satisfying $\mathbf{E}[\Delta\tau_k] = \mu > 0$ and $\mathbf{E}[\Delta\tau_k^2] < \infty$, which implies that $\{\tau_k\}$ forms a renewal process. For $t \in [\tau_{k-1}, \tau_k)$, $r(t) = r_k$ is a constance with probability distribution

$$\mathbf{P}(r_k = l) = \rho_l, \quad l \in \mathbb{S},$$

where $\rho_l \geqslant 0$ and $\sum_{l=1}^{M} \rho_l = 1$. Then the reaction diffusion system with independent and identically distributed switching can be written as

$$
\begin{aligned}
\frac{\partial u(t, y)}{\partial t} &= \nabla \cdot \big(D\big(t, y, u(t, y)\big) \circ \nabla u(t, y)\big) + f\big(t, y, u(t, y), r_k\big), \\
&(t, y) \in [\tau_{k-1}, \tau_k) \times X, \\
u(0, y) &= \phi(y), \quad y \in X, \qquad \frac{\partial u(t, y)}{\partial N} = 0, \quad (t, y) \in [\tau_{k-1}, \tau_k) \times \partial X,
\end{aligned}
\tag{2}
$$

Throughout this paper, we suppose that there exists a constant $L > 0$ such that

$$\big\| f\big(t, y, u_1, r(t)\big) - f\big(t, y, u_2, r(t)\big) \big\| \leqslant L \|u_1 - u_2\|.$$

Only a limited number of switching occurs for each finite time interval, which precludes the possibility of infinitely fast switching. In view of the Lipschitz condition of f, we see that there exists a unique solution for (1) or (2). Also we suppose that $f(t, y, 0, r(t)) = 0$ for any $t \geqslant 0$, $y \in X$, which implies that $u(t, y) \equiv 0$ is a trivial solution.

Definition 1. The zero solution of system (1) or (2) is said to be almost surely exponentially stable if there exists $\lambda > 0$ such that for any initial value u_0 and $t \geqslant 0$,

$$\limsup_{t \to +\infty} \frac{1}{t} \log \big\| u(t, y, u_0) \big\|_2 \leqslant -\lambda \quad \text{a.s.}$$

Definition 2. The zero solution of system (1) or (2) is said to be exponentially stable in the mean square if there exist $\lambda > 0$, $d > 0$ such that for any initial value u_0 and $t \geqslant 0$,

$$\mathbf{E}\big[\big\| u(t, y, u_0) \big\|_2^2\big] \leqslant d \|u_0\|_2^2 \mathrm{e}^{-\lambda t}.$$

3 Almost surely exponential stability and exponential stability in the mean square

In this section, some criteria on almost surely exponential stability are established. The following lemma is important for almost surely exponential stability and exponential stability in the mean square of switched systems (1) or (2).

Lemma 1. *For any given initial value $u(0, y) = u_0$, $u_0 \neq 0$, $u_0 \in \mathbb{R}^n$, there exists a unique solution $u(t, x)$ on $[0, +\infty)$ for (1) or (2) such that*

$$\mathbf{P}\left(\int_X u(t, y) \, \mathrm{d}y \neq 0, \ t \geqslant 0 \right) = 1. \tag{3}$$

Proof. Without loss of generality, for the switching of the Markov chain or independent and identically distributed process, we may suppose that initial switching is $r(0) = 1$. If $r(0) = 2, \ldots, M$, we can prove it in the same way. Then we deduce that $r(t) = 1$ for $t \in [0, \tau_1)$. Hence, (1) or (2) can be written as

$$\frac{\partial u(t, y)}{\partial t} = \nabla \cdot \big(D(t, y, u(t, y)) \circ \nabla u(t, y) \big) + f\big(t, y, u(t, y), 1\big),$$
$$(t, y) \in [0, \tau_1) \times X,$$
$$u(0, y) = \phi(y), \quad y \in X, \qquad \frac{\partial u(t, y)}{\partial N} = 0, \quad (t, y) \in [0, \tau_1) \times \partial X.$$

Obviously, (3) has a unique solution on $[0, +\infty)$, which implies that the solution $u(t, x)$ of (1) or (2) is uniquely determined on $[0, \tau_1)$. In the following, we shall prove that $\mathbf{P}(\int_X u(t, y) \, \mathrm{d}y \neq 0, \ t \in [0, \tau_1)) = 1$. If this is not true, for given $u_0 \neq 0$, we have $\mathbf{P}(\vartheta < \tau_1) > 0$, where

$$\vartheta = \inf\left\{ 0 \leqslant t < \tau_1, \ \int_X u(t, y) \, \mathrm{d}y = 0 \right\}.$$

Therefore, there exist a pair of constants $0 \leqslant t_1 < \tau_1$ and $\alpha > \max\{1, \|u_0\|\}$ such that $\mathbf{P}(\Lambda) > 0$, where $\Lambda = \{\vartheta \leqslant t_1, \|u(t, y)\| \leqslant \alpha \ \forall t \in [0, \vartheta)\}$. It follows from the Lipschitz condition of f that there exists $l_\alpha > 0$ such that for any $t \in [0, t_1]$, $\|u(t, y)\| \leqslant \alpha$,

$$\big\| f\big(t, y, u(t, y), 1\big) \big\| \leqslant l_\alpha \big\| u(t, y) \big\|.$$

Let $V(t, u) = \|u(t, y)\|^{-1}$, we derive that for $0 \leqslant t \leqslant t_1$,

$$\frac{\mathrm{d} \int_X V(t, u) \, \mathrm{d}y}{\mathrm{d}t} = \int_X \Big[-\|u(t, y)\|^{-3} u^{\mathrm{T}}(t, y) \nabla \cdot \big(D\big(t, y, u(t, y)\big) \circ \nabla u(t, y) \big)$$
$$- \|u(t, y)\|^{-3} u^{\mathrm{T}}(t, y) f\big(t, y, u(t, y), 1\big) \Big] \, \mathrm{d}y.$$

By Neumann value condition and integration by parts, we have

$$-\int_X \|u(t, y)\|^{-3} u^{\mathrm{T}}(t, y) \nabla \cdot \big(D\big(t, y, u(t, y)\big) \circ \nabla u(t, y) \big) \, \mathrm{d}y$$
$$= -\sum_{i=1}^n \int_X \|u\|^{-3} u_i \nabla \cdot \left(\sum_{k=1}^m D_{ik}(t, y, u) \frac{\partial u_i}{\partial y_k} \right) \mathrm{d}y$$

$$
\begin{aligned}
= & -\sum_{i=1}^{n}\sum_{k=1}^{m}\|u\|^{-3}u_i D_{ik}(t,y,u)\frac{\partial u_i}{\partial y_k}\bigg|_{\partial X} \\
& + \sum_{i=1}^{n}\sum_{k=1}^{m}\int_X D_{ik}(t,y,u)\frac{\partial(\|u\|^{-3}u_i)}{\partial u_i}\left(\frac{\partial u_i}{\partial y_k}\right)^2 \mathrm{d}y \\
= & -2\sum_{k=1}^{m}\int_X \|u\|^{-3}D_{ik}(t,y,u)\left(\frac{\partial u_i}{\partial y_k}\right)^2 \mathrm{d}y \leqslant 0.
\end{aligned}
$$

Then

$$
\frac{\mathrm{d}\int_X V(t,u)\,\mathrm{d}y}{\mathrm{d}t} \leqslant \int_X \big\|u(t,y)\big\|^{-2}\big\|f\big(t,y,u(t,y),1\big)\big\|\,\mathrm{d}y \leqslant \mathbf{1}_{\iota_\alpha}\int_X V(t,u)\,\mathrm{d}y.
$$

Define stop time $\iota_\varepsilon \triangleq \inf\{t \in [0,\tau_1),\ \|u(t,y)\| \in (\varepsilon,\alpha)\}$ for rall $\varepsilon \in (0,\|u_0\|)$, then

$$
\mathrm{e}^{-l_\alpha(\iota_\varepsilon \wedge t_1)}\int_X V\big(\iota_\varepsilon \wedge t_1,\ u(\iota_\varepsilon \wedge t_1,\ y)\big)\,\mathrm{d}y = \int_X V(0,u_0)\,\mathrm{d}y.
$$

Noting $\|u(\iota_\varepsilon,y)\| = \varepsilon$, $\iota_\varepsilon \leqslant t_1$, we get $\|u_0\| \leqslant \varepsilon\mathrm{e}^{\iota_\alpha t_1}$. Letting $\varepsilon \to 0$ causes a contradiction. Therefore, we can conclude that $\mathbf{P}\big(\int_X u(t,y)\,\mathrm{d}y \neq 0,\ t \in [0,\tau_1)\big) = 1$. Together with continuity, for $t = \tau_1$, we get $u(\tau_1,y) \neq 0$. Repeating this procedure means that there exists a unique solution $u(t,y)$ on $[0,+\infty)$ for (1) or (2) such that (3) holds. $\qquad\square$

Remark 1. The existence and uniqueness of the solution can guarantee that the zero solution of system (1) or (2) is almost surely unique equilibrium point, which are the precondition for studying the behavior of solution for the system with stochastic switching.

Let $\mathcal{C}^1([0,+\infty) \times \mathbb{R}^n, [0,+\infty))$ be the family of all nonnegative continuous functions $V(t,\xi)$ on $[0,+\infty) \times \mathbb{R}^n$ and $V_t(t,\xi)$, $V_\xi(t,\xi)$ are continuous on $[0,+\infty) \times \mathbb{R}^n$. For each $V \in \mathcal{C}^1([0,+\infty) \times \mathbb{R}^n, [0,+\infty))$, we define an operator $\mathcal{L}V : [0,+\infty) \times \mathbb{R}^n \to [0,+\infty)$ associated with system (1) or system (2) as follows:

$$
\mathcal{L}V(t,u) = \frac{\partial V(t,u)}{\partial t} + \left(\frac{\partial V(t,u)}{\partial u}\right)^{\mathrm{T}}f\big(t,y,u,r(t)\big).
$$

Theorem 1. *Let $r(t)$ be a right-continuous Markov chain taking values in the state space $\mathbb{S} = \{1,2,\ldots,M\}$ and $V \in \mathcal{C}^1([0,+\infty) \times \mathbb{R}^n, [0,+\infty))$. Assume that there exist constants $d_1, d_2 > 0$, η_i, $i = 1,2,\ldots,M$, such that:*

(i) $d_1\|u(t,y)\|_2 \leqslant \int_X V(t,u(t,y))\,\mathrm{d}y \leqslant d_2\|u(t,y)\|_2$;
(ii) *$V(t,u)$ is separated as to variables u_i, $i = 1,2,\ldots,n$;*
(iii) $\partial^2 V(t,u)/\partial u_i^2 \geqslant 0$, $i = 1,2,\ldots,n$, $(t,u) \in [0,+\infty) \times \mathbb{R}^n$;
(iv) $\int_X \mathcal{L}V(t,u(t,y))\,\mathrm{d}y \leqslant \eta_{r(t)}\int_X V(t,u(t,y))\,\mathrm{d}y$;
(v) $\sum_i^M \pi_i\eta_i < 0$.

Then the zero solution of system (1) *is almost surely exponentially stable.*

Proof. For any $u_0 \neq 0$, it follows from Lemma 1 that $\int_X u(t, y) \, \mathrm{d}y \neq 0$ a.s. for $t \geqslant 0$. Thus,

$$
\frac{\mathrm{d}[\log \int_X V(t, u(t, y)) \, \mathrm{d}y]}{\mathrm{d}t}
$$

$$
= \frac{1}{\int_X V(t, u(t, y)) \, \mathrm{d}y} \left[\int_X \frac{\partial V(t, u)}{\partial t} + \left(\frac{\partial V(t, u)}{\partial u} \right)^{\mathrm{T}} f\big(t, y, u, r(t)\big) \, \mathrm{d}y \right.
$$

$$
\left. + \int_X \left(\frac{\partial V(t, u)}{\partial u} \right)^{\mathrm{T}} \nabla \cdot D\big(t, y, u(t, y)\big) \circ \nabla u(t, y) \, \mathrm{d}y \right]
$$

$$
= \frac{1}{\int_X V(t, u(t, y)) \, \mathrm{d}y} \left[\int_X \mathcal{L}V\big(t, u(t, y)\big) \, \mathrm{d}y \right.
$$

$$
\left. + \int_X \left(\frac{\partial V(t, u)}{\partial u} \right)^{\mathrm{T}} \nabla \cdot D\big(t, y, u(t, y)\big) \circ \nabla u(t, y) \, \mathrm{d}y \right].
$$

By (ii), we see that $\partial^2 V(t, u)/\partial u_i \partial u_j = 0$, $i \neq j$, $i, j \in \{1, 2, \ldots, n\}$. From Neumann value condition and integration by parts, together with (iii), we get

$$
\int_X \left(\frac{\partial V(t, u)}{\partial u} \right)^{\mathrm{T}} \nabla \cdot \big(D\big(t, y, u(t, y)\big) \circ \nabla u(t, y)\big) \, \mathrm{d}y
$$

$$
= \sum_{i=1}^n \int_X \left(\frac{\partial V}{\partial u_i} \nabla \cdot \left(\sum_{k=1}^m D_{ik}(t, y, u) \frac{\partial u_i}{\partial y_k} \right) \right) \mathrm{d}y
$$

$$
= \sum_{i=1}^n \sum_{k=1}^m \frac{\partial V}{\partial u_i} D_{ik}(t, y, u) \frac{\partial u_i}{\partial y_k} \bigg|_{\partial X} - \sum_{i=1}^n \sum_{k=1}^m \int_X D_{ik}(t, y, u) \frac{\partial^2 V}{\partial u_i^2} \left(\frac{\partial u_i}{\partial y_k} \right)^2 \mathrm{d}y
$$

$$
= -\sum_{i=1}^n \sum_{k=1}^m \int_X D_{ik}(t, y, u) \frac{\partial^2 V}{\partial u_i^2} \left(\frac{\partial u_i}{\partial y_k} \right)^2 \mathrm{d}y \leqslant 0. \tag{4}
$$

By (iv), we have

$$
\frac{\mathrm{d}[\log \int_X V(t, u(t, y)) \, \mathrm{d}y]}{\mathrm{d}t} \leqslant \frac{1}{\int_X V(t, u(t, y)) \, \mathrm{d}y} \int_X \mathcal{L}V\big(t, u(t, y)\big) \, \mathrm{d}y \leqslant \eta_{r(t)},
$$

which means that for any $t > 0$,

$$
\frac{\log \int_X V(t, u(t, y)) \, \mathrm{d}y}{t} = \frac{\log \int_X V(t, u(t, y)) \, \mathrm{d}y}{t} + \frac{1}{t} \int_0^t \eta_{r(s)} \, \mathrm{d}s.
$$

Letting $t \to \infty$, we get

$$\limsup_{t\to\infty} \frac{1}{t} \log \int_X V\big(t, u(t, y)\big)\, \mathrm{d}y \leqslant \limsup_{t\to\infty} \frac{1}{t} \int_0^t \eta_{r(s)}\, \mathrm{d}s. \tag{5}$$

It follows from the ergodic property of Markov chain [6] that

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t \eta_{r(s)}\, \mathrm{d}s = \sum_{i=1}^M \eta_i \pi_i \quad \text{a.s.}$$

Thus, by (5), (i) and (v), we have

$$\limsup_{t\to\infty} \frac{1}{t} \log \big\| u(t, y) \big\|_2 \leqslant \sum_{i=1}^M \eta_i \pi_i < 0 \quad \text{a.s.} \qquad \square$$

Remark 2. Condition (i) is a general condition for the stability of a reaction–diffusion system. Conditions (ii)–(iv) are required due to stochastic switching of the system. Further, in view of the ergodicity of the Markov process, we can choose condition (v).

Theorem 2. *Let $r(t)$ be a right-continuous Markov chain taking values in the state space $\mathbb{S} = \{1, 2, \ldots, M\}$ and $V \in \mathcal{C}^1([0, +\infty) \times \mathbb{R}^n, [0, +\infty))$. Assume that there exist constants $d_1, d_2, \theta_i > 0$, γ_i, $i = 1, 2, \ldots, M$, such that:*

(i) *$d_1 \| u(t, y) \|_2 \leqslant \int_X V(t, u(t, y))\, \mathrm{d}y \leqslant d_2 \| u(t, y) \|_2$;*
(ii) *$V(t, u)$ is separated as to variables u_i, $i = 1, 2, \ldots, n$;*
(iii) *$\partial^2 V(t, u)/\partial u_i^2 \geqslant 0$, $i = 1, 2, \ldots, n$, $(t, u) \in [0, +\infty) \times \mathbb{R}^n$;*
(iv) *$\int_X \mathcal{L}V(t, u(t, y))\, \mathrm{d}y \leqslant \gamma_{r(t)} \int_X V(t, u(t, y))\, \mathrm{d}y$;*
(v) *$\max_{1 \leqslant i \leqslant M} (\gamma_i + \sum_{j=1}^M \delta_{ij}\theta_j/\theta_i) < 0$.*

Then the zero solution of system (1) is exponentially stable in the mean square.

Proof. For $t \geqslant 0$, we get

$$\frac{\mathrm{d}[\int_X V(t, u(t, y))\, \mathrm{d}y]}{\mathrm{d}t}$$
$$= \int_X \left[\frac{\partial V(t, u)}{\partial t} + \left(\frac{\partial V(t, u)}{\partial u} \right)^{\mathrm{T}} f\big(t, y, u, r(t)\big) \right.$$
$$\left. + \left(\frac{\partial V(t, u)}{\partial u} \right)^{\mathrm{T}} \nabla \cdot D\big(t, y, u(t, y)\big) \circ \nabla u(t, y) \right] \mathrm{d}y$$
$$= \int_X \left[\mathcal{L}V\big(t, u(t, y)\big) + \left(\frac{\partial V(t, u)}{\partial u} \right)^{\mathrm{T}} \nabla \cdot D\big(t, y, u(t, y)\big) \circ \nabla u(t, y) \right] \mathrm{d}y.$$

By (4), (ii) and (iv), we deduce

$$\frac{\mathrm{d}[\int_X V(t,u(t,y))\,\mathrm{d}y]}{\mathrm{d}t} \leqslant \int_X \mathcal{L}V\big(t,u(t,y)\big)\,\mathrm{d}y \leqslant \gamma_{r(t)}\int_X V\big(t,u(t,y)\big)\,\mathrm{d}y.$$

We consider the following switched system:

$$\frac{\mathrm{d}\Xi(t)}{\mathrm{d}t} = \gamma_{r(t)}\Xi(t), \quad t \geqslant 0, \qquad \Xi(0) = \int_X V\big(0,u(0,y)\big)\,\mathrm{d}y,$$

where $\Xi(t) = (\Xi_1(t),\Xi_2(t),\ldots,\Xi_n(t))^{\mathrm{T}}$. It is obvious that $\{(\Xi(t),r(t)) \mid t \geqslant 0\}$ is a Markov process. We define an infinitesimal operator as follows:

$$\mathrm{L} = \Gamma + \mathrm{diag}\left\{\gamma_1\Xi^{\mathrm{T}}\frac{\partial}{\partial\Xi},\ \gamma_2\Xi^{\mathrm{T}}\frac{\partial}{\partial\Xi},\ \ldots,\ \gamma_n\Xi^{\mathrm{T}}\frac{\partial}{\partial\Xi}\right\}. \tag{6}$$

Consider a Lyapunov function $\overline{V}(\Xi(t),i) = \theta_i(\Xi^{\mathrm{T}}(t)\Xi(t))^{1/2}$ for $\Xi(t) = (\Xi_1(t),\Xi_2(t),\ldots,\Xi_n(t))^{\mathrm{T}}$ and $i \in \mathbb{S}$. Calculating the differential operator along (6), we have

$$\big(\mathrm{L}\overline{V}\big)\big(\Xi(t),i\big) = \sum_{j=1}^{M}\delta_{kij}\theta_j\big(\Xi^{\mathrm{T}}(t)\Xi(t)\big)^{1/2} + \theta_i\gamma_i\Xi^{\mathrm{T}}(t)\frac{1}{2}\big(\Xi^{\mathrm{T}}(t)\Xi(t)\big)^{-1/2}\big(2\Xi(t)\big)$$

$$\leqslant \max_{1\leqslant i\leqslant M}\left(\gamma_i + \sum_{j=1}^{M}\delta_{ij}\frac{\theta_j}{\theta_i}\right)\overline{V}\big(\Xi(t),i\big).$$

Let $-\vartheta = \max_{1\leqslant i\leqslant M}(\gamma_i + \sum_{j=1}^{M}\delta_{ij}\theta_j/\theta_i)$. By (v), we see that $\vartheta > 0$, which means that

$$\big(\mathbf{L}\overline{V}\big)\big(\Xi(t),i\big) \leqslant -\vartheta\overline{V}\big(\Xi(t),i\big).$$

According to Dynkin formula and Fubini theorem [13], we have

$$\mathbf{E}\big\{\overline{V}\big(\Xi(t),r(t)\big) \mid \Xi(0),r(0) = i\big\} - \overline{V}\big(\Xi(0),i\big)$$

$$= \mathbf{E}\left\{\int_0^t (\mathbf{L}\overline{V})\big(\Xi(s),r(s)\big)\,\mathrm{d}s \ \Big| \ \Xi(0),r(0) = i\right\}$$

$$\leqslant -\vartheta\int_0^t \mathbf{E}\big\{\overline{V}\big(\Xi(s),r(s)\big) \mid \Xi(0),r(0) = i\big\}\,\mathrm{d}s.$$

It follows from Gronwall–Bellman that

$$\mathbf{E}\big\{\overline{V}\big(\Xi(t),r(t)\big) \mid \Xi(0),r(0) = i\big\} \leqslant \overline{V}\big(\Xi(0),i\big)\exp(-\vartheta t) \quad \forall t \geqslant 0.$$

Let $\theta = \min_{1 \leqslant i \leqslant M}\{\theta_i\}$. By comparison principle, we have

$$\mathbf{E}\left\{ \int\limits_X V\big(t, u(t,y)\big)\, \mathrm{d}y \;\Big|\; Y(0), r(0) = i \right\} \leqslant \frac{\overline{V}(\Xi(0), i)}{\theta} \exp(-\vartheta t) \quad \forall t \geqslant 0.$$

This implies that system (1) is exponentially stable in the mean square. $\qquad\square$

Theorem 3. *Let $r(t)$ be independent and identically distributed sequence taking value in $\mathbb{S} = \{1, 2, \ldots, M\}$ and $V \in \mathcal{C}^1([0, +\infty) \times \mathbb{R}^n, [0, +\infty))$. Assume that there exist constants $d_1, d_2 > 0$, ξ_i, $i = 1, 2, \ldots, M$, such that:*

  (i) $d_1 \|u(t,y)\|_2 \leqslant \int_X V(t, u(t,y))\, \mathrm{d}y \leqslant d_2 \|u(t,y)\|_2$;
  (ii) $V(t,u)$ *is separated as to variables* u_i, $i = 1, 2, \ldots, n$;
  (iii) $\partial^2 V(t,u)/\partial u_i^2 \geqslant 0$, $i = 1, 2, \ldots, n$, $(t, u) \in [0, +\infty) \times \mathbb{R}^n$;
  (iv) $\int_X \mathcal{L}V(t, u(t,y))\, \mathrm{d}y \leqslant \xi_{r_k} \int_X V(t, u(t,y))\, \mathrm{d}y$ *for all* $t \in [\tau_{k-1}, \tau_k)$;
  (v) $\sum_{i=1}^M \rho_i \xi_i < 0$.

Then the zero solution of system (2) is almost surely exponentially stable.

Proof. For $t \in [\tau_{k-1}, \tau_k)$, by (ii), we have

$$\frac{\mathrm{d}[\log \int_X V(t, u(t,y))\, \mathrm{d}y]}{\mathrm{d}t} \leqslant \frac{1}{\int_X V(t, u(t,y))\, \mathrm{d}y} \int\limits_X \mathcal{L}V\big(t, u(t,y)\big)\, \mathrm{d}y \leqslant \xi_{r_k}. \qquad (7)$$

Let $L_t = \sup\{n | \tau_n \leqslant t\}$. By Theorem 7.3 in Durrett [16], we can obtain

$$\lim_{t \to +\infty} \frac{L_t}{t} = \frac{1}{\mu} \quad \text{a.s.},$$

which implies that

$$\lim_{t \to +\infty} L_t = +\infty \quad \text{a.s.}$$

It follows from (7) that

$$\log \int\limits_X V\big(t, u(t,y)\big)\, \mathrm{d}y \leqslant \log \int\limits_X V\big(0, u(0,y)\big)\, \mathrm{d}y + \sum_{i=1}^{L_t} \int\limits_{\tau_{i-1}}^{\tau_i} \xi_{r_i}\, \mathrm{d}s + \int\limits_{t_{L_t}}^{t} \xi_{r_{L_t+1}}\, \mathrm{d}s.$$

Then

$$\frac{1}{t} \log \int\limits_X V\big(t, u(t,y)\big)\, \mathrm{d}y$$

$$\leqslant \frac{1}{t} \log \int\limits_X V\big(0, u(0,y)\big)\, \mathrm{d}y + \frac{1}{t} \sum_{i=1}^{L_t} \int\limits_{\tau_{i-1}}^{\tau_i} \xi_{r_i}\, \mathrm{d}s + \frac{1}{t} \int\limits_{t_{L_t}}^{t} \xi_{r_{L_t+1}}\, \mathrm{d}s.$$

Letting $t \to +\infty$, we get

$$\limsup_{t \to +\infty} \frac{1}{t} \log \int_X V\big(t, u(t,y)\big)\, dy \leqslant \limsup_{t \to +\infty} \frac{1}{t}\left[\sum_{i=1}^{L_t} \int_{\tau_{i-1}}^{\tau_i} \xi_{r_i}\, ds + \int_{t_{L_t}}^{t} \xi_{r_{L_t+1}}\, ds\right]. \quad (8)$$

By the strong law of large numbers for independent and identically distributed sequence, we have

$$\limsup_{t \to +\infty} \frac{1}{t}\left[\sum_{i=1}^{L_t} \int_{\tau_{i-1}}^{\tau_i} \xi_{r_i}\, ds + \int_{t_{L_t}}^{t} \xi_{r_{L_t+1}}\, ds\right]$$

$$= \limsup_{t \to +\infty} \frac{L_t}{t}\frac{1}{L_t}\left[\sum_{i=1}^{L_t} \int_{\tau_{i-1}}^{\tau_i} \xi_{r_i}\, ds + \int_{t_{L_t}}^{t} \xi_{r_{L_t+1}}\, ds\right]$$

$$= \limsup_{t \to +\infty} \frac{L_t}{t}\frac{1}{L_t}\sum_{i=1}^{L_t} \int_{t\tau_{i-1}}^{\tau_i} \xi_{r_i}\, ds = \frac{1}{\mu}\sum_{i=1}^{M} \rho_i \xi_i < 0 \quad \text{a.s.} \quad (9)$$

It follows from (8), (9) and (i) that

$$\limsup_{t \to +\infty} \frac{1}{t} \log\big\|u(t,y)\big\|_2 \leqslant \frac{1}{\mu}\sum_{i=1}^{M} \rho_i \xi_i < 0 \quad \text{a.s.}$$

This completes the proof. □

Remark 3. The reaction–diffusion system (1) or (2) can be considered as the following subsystems:

$$\frac{\partial u(t,y)}{\partial t} = \nabla \cdot \big(D\big(t,y,u(t,y)\big) \circ \nabla u(t,y)\big) + f\big(t,y,u(t,y),i\big),$$

$$y \in X,\ i = 1,2,\ldots,M,$$

$$u(0,y) = \phi(y), \quad y \in X, \qquad \frac{\partial u(t,y)}{\partial N} = 0, \quad y \in \partial X,$$

switching from one to the other by the law of Markov chain or independent and identically distributed process. If some of η_i or ξ_i are not negative, i.e., the zero solution for some of subsystem achieve stability, but the zero solution for the other subsystems do not achieve stability. However, if the rate of stochastic switching from unstable state to stable state is faster than that from stable state to unstable state, so that $\sum_{i=1}^{M} \pi_i \eta_i < 0$ or $\sum_{i=1}^{M} \rho_i \xi_i < 0$, then the overall system will achieve stability, which means that stochastic switching play an essential role in the stability of reaction–diffusion system.

Remark 4. Theorems 1–3 establish a general framework for analyzing the stable behaviors of reaction diffusion systems with stochastically switching. Sufficient conditions are derived under which system (1) or (2) can achieve stability in the two kinds of stochastic switchings sense.

Remark 5. In Theorem 2, the constants θ_k, $k = 1, 2, \ldots, M$, are very important. If all of $\gamma_k < 0$, all the subsystems are stable. However, if there are some subsystem are unstable, but the other subsystems are stable, the overall system can eventually achieve stability under $\gamma_k + \sum_{j=1}^{M} \delta_{kj}\theta_j/\theta_k < 0$.

Corollary 1. *Let $r(t)$ be a right-continuous Markov chain taking values in the state space* $\mathbb{S} = \{1, 2, \ldots, M\}$. *Assume that there exist constants β_i, $i = 1, 2, \ldots, M$, such that*:

   (i) $u^{\mathrm{T}} f(t, y, u, r(t)) \leqslant \beta_{r(t)} \|u\|^2$;
   (ii) $\sum_{i=1}^{M} \pi_i \beta_i < 0$.

Then the zero solution of system (1) *is almost surely exponentially stable.*

Proof. Let $V(t, u(t, y)) = \|u(t, y)\|^2$. Then

$$\int\limits_X \mathcal{L}V\big(t, u(t, y)\big) \, \mathrm{d}y = 2 \int\limits_X u^{\mathrm{T}}(t, y) f\big(t, y, u(t, y), r(t)\big) \, \mathrm{d}y$$

$$\leqslant 2\beta_{r(t)} \int\limits_X \big\|u(t, y)\big\|^2 \, \mathrm{d}y$$

$$= 2\beta_{r(t)} \int\limits_X V\big(t, u(t, y)\big) \, \mathrm{d}y.$$

Consequently, the conclusion follows from Theorem 1. $\qquad\qquad\square$

Corollary 2. *Let $r(t)$ be a right-continuous Markov chain taking values in the state space* $\mathbb{S} = \{1, 2, \ldots, M\}$. *Assume that there exist constants β_i, θ_i, $i = 1, 2, \ldots, M$, such that*:

   (i) $u^{\mathrm{T}} f(t, y, u, r(t)) \leqslant \beta_{r(t)} \|u\|^2$;
   (ii) $\max_{1 \leqslant i \leqslant M} \big(2\beta_i + \sum_{j=1}^{M} \delta_{ij}\theta_j/\theta_i\big) < 0$.

Then the zero solution of system (1) *is exponentially stable in the mean square.*

Corollary 3. *Let $r(t)$ be independent and identically distributed sequence taking value in* $\mathbb{S} = \{1, 2, \ldots, M\}$. *Assume that there exist constants $\mu_i, i = 1, 2, \ldots, M$, such that*:

   (i) $u^{\mathrm{T}} f(t, y, u, r_k) \leqslant \mu_{r_k} \|u\|^2$, $t \in [t_{k-1}, t_k)$;
   (ii) $\sum_{i=1}^{M} \rho_i \mu_i < 0$.

Then the zero solution of system (2) *is almost surely exponentially stable.*

Corollary 4. *Let $r(t)$ be a right-continuous Markov chain taking values in the state space* $\mathbb{S} = \{1, 2, \ldots, M\}$ *and $f(t, y, u(t, y), r(t)) = A_{r(t)} u(t, y)$, $A_{r(t)} \in \mathbb{R}^{n \times n}$. Assume that there exist positive definite matrix Q and constants ν_i, $i = 1, 2, \ldots, M$, such that*:

   (i) $QA_i + A_i^{\mathrm{T}}Q - \nu_i Q \leqslant 0$;
   (ii) $\sum_{i=1}^{M} \pi_i \nu_i < 0$.

Then the zero solution of system (1) *is almost surely exponentially stable.*

Proof. Let $V(t, u(t, y)) = u^{\mathrm{T}}(t, y) Q u(t, y)$. Then for $t \geqslant 0$,

$$
\begin{aligned}
\int_X \mathcal{L} V\big(t, u(t, y)\big)\,\mathrm{d}y &= \int_X \big[2u^{\mathrm{T}}(t, y) Q f\big(t, y, u(t, y), r(t)\big)\big]\,\mathrm{d}y \\
&= \int_X \big[u^{\mathrm{T}}(t, y)\big(Q A_{r(t)} + A_{r(t)}^{\mathrm{T}} Q\big) u(t, y)\big]\,\mathrm{d}y \\
&\leqslant \nu_{r(t)} \int_X u^{\mathrm{T}}(t, y) Q u(t, y)\,\mathrm{d}y.
\end{aligned}
$$

By Theorem 1, we can see that the conclusion of Corollary 3 holds. $\qquad\square$

Corollary 5. *Let $r(t)$ be a right-continuous Markov chain taking values in the state space $\mathbb{S} = \{1, 2, \ldots, M\}$ and $f(t, y, u(t, y), r(t)) = A_{r(t)} u(t, y)$, $A_{r(t)} \in \mathbb{R}^{n \times n}$. Assume that there exist positive definite matrix Q and constants ν_i, θ_i, $i = 1, 2, \ldots, M$, such that:*

(i) $Q A_i + A_i^{\mathrm{T}} Q - \nu_i Q \leqslant 0$;
(ii) $\max_{1 \leqslant i \leqslant M} (\nu_i + \sum_{j=1}^M \delta_{ij} \theta_j / \theta_i) < 0$.

Then the zero solution of system (1) *is exponentially stable in the mean square.*

Corollary 6. *Let $r(t)$ be independent and identically distributed sequence taking value in $\mathbb{S} = \{1, 2, \ldots, M\}$ and $f(t, y, u(t, y), r_k) = A_{r_k} u(t, y)$, $A_{r_k} \in \mathbb{R}^{n \times n}$. Assume that there exist positive definite matrix U and constants ω_i, $i = 1, 2, \ldots, M$, such that:*

(i) $U A_i + A_i^{\mathrm{T}} U - \omega_i U \leqslant 0$;
(ii) $\sum_{i=1}^M \rho_i \omega_i < 0$.

Then the zero solution of system (2) *is almost surely exponentially stable.*

Remark 6. Let $Q = I$. We can take value $\nu_i = \omega_i = \lambda_{\max}(A_i + A_i^{\mathrm{T}})$.

4 Numerical example

In this section, an example with numerical simulations are presented to demonstrate our results.

Example 1. Consider the following 2-dimensional reaction–diffusion neural networks with Markov switching:

$$
\begin{aligned}
\frac{\partial u_1(t, y)}{\partial t} &= \Delta u_1(t, y) + a_{r(t)} u_1(t, y) + b_{r(t)} f_1\big(u_2(t, y)\big), \\
\frac{\partial u_2(t, y)}{\partial t} &= \Delta u_2(t, y) + c_{r(t)} u_2(t, y) + d_{r(t)} f_2\big(u_1(t, y)\big), \quad t \geqslant 0, \ y \in (0, 1), \\
\frac{\partial u(t, y)}{\partial N} &= 0, \quad t \geqslant 0, \ y \in \{0, 1\},
\end{aligned}
$$

where Markov chain $r(t)$ takes values in $\mathbb{S} = \{1, 2\}$ with generator $\Gamma = \begin{bmatrix} -3 & 3 \\ 1 & -1 \end{bmatrix}$, then $\delta_{12} = 3$, $\delta_{21} = 1$, $\pi_1 = 1/4$, $\pi_2 = 3/4$, the parametric coefficients $a_1 = 1/2$, $b_1 = 1/2$,
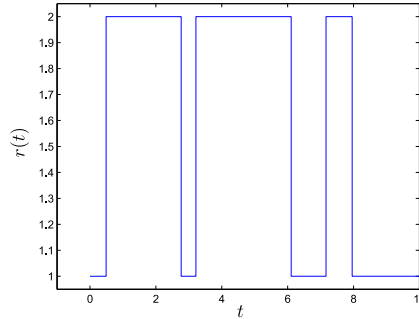
Figure 1. Markov chain $r(t)$.

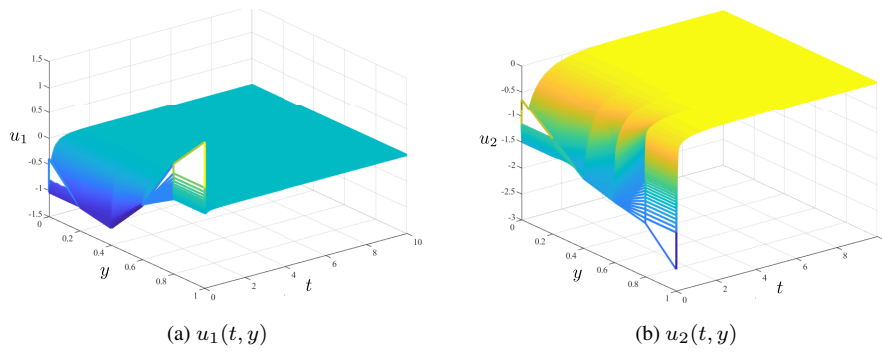(a) $u_1(t, y)$ (b) $u_2(t, y)$

Figure 2. Trajectories of the states of (10).

$c_1 = 1/2$, $d_1 = 1/4$, $a_2 = -2$, $b_2 = 1/4$, $c_2 = -2$, $d_2 = 1/4$, the activation function $f_1(u) = f_2(u) = \arctan u$. We can easily tain that $|f_1(u)| \leqslant |u|$, $|f_2(u)| \leqslant |u|$. The switching times of $r(t)$ follow the exponential distribution with $r(0) = 1$, as shown by Fig. 1.

Construct the Lyapunov function $V(t, u) = u_1^2 + u_2^2$, then

$$
\begin{aligned}
\int_X \mathcal{L}V(t, u)\, \mathrm{d}y &= \int_X \left[2a_{r(t)}u_1^2 + 2c_{r(t)}u_2^2 + 2b_{r(t)}u_1 f_1(u_2) + 2d_{r(t)}u_2 f_2(u_1) \right] \mathrm{d}x \\
&\leqslant \int_X \left[2a_{r(t)}u_1^2 + 2c_{r(t)}u_2^2 + 2|b_{r(t)}||u_1(u_2)| + 2|d_{r(t)}||u_2 u_1| \right] \mathrm{d}x \\
&\leqslant \eta_{r(t)} \int_X V(t, u)\, \mathrm{d}y,
\end{aligned}
$$

where $\eta_1 = 7/4$, $\eta_2 = -7/2$, which implies that $\sum_1^2 \pi_i \eta_i = 1/4 \cdot 7/4 + 3/4 \cdot (-7/2) = -35/16 < 0$. By Theorem 1, the zero solution of the switched system (10) is almost surely exponentially stable. Taking the initial conditions $u_1(0, y) = -\cos y/2$, $u_2(0, y) = -3\cos y$. Figure 2(a) shows the trajectory of the state $u_1(t, y)$ of system (10). Figure 2(b) depicts the trajectory of the state $u_2(t, y)$ of system (10).
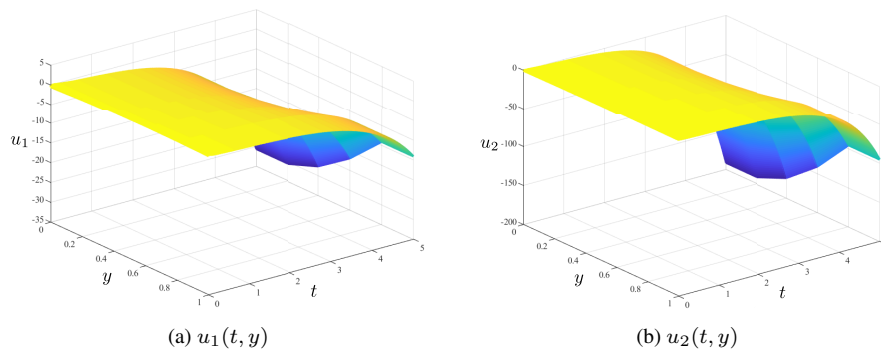
(a) $u_1(t,y)$ (b) $u_2(t,y)$

Figure 3. Trajectories of the states of (10) for $r(t) = 1$.

Remark 7. Figures 3(a) and 3(b) show the trajectory of state $u_1(t,y)$ and $u_2(t,y)$ for $r(t) = 1$. We can see that the zero solution of the first subsystem is not stable, but the overall system (10) can achieve stability almost surely. It means that condition (i)–(v) play an important role in the stability of switched systems.

5 Conclusions

In this paper, stability analysis of reaction diffusion systems with stochastically switched parameter has been studied. The switched model includes two kinds of stochastic switchings. Switched process takes values in finite state space. By method of stochastic analysis and Lyapunov function, some new stability criteria have been derived. Finally, a standard numerical package illustrate that the new results are practical. Our future work will focus on the stability of delayed reaction diffusion systems with stochastic switching.

References

1. W. Anderson, *Continuous-Time Markov Chains*, Springer-Verlag, Berlin, Heidelberg, 1991.

2. W.R. Durrett, *Probability: Theory and Examples*, Duxbury Press, Belmont, CA, 2005.

3. Y. Fang, *Stability Analysis of Linear Control Systems with Uncertain Parameters*, PhD thesis, Case Western Reserve University, Cleveland, OH, 1994.

4. Y. Fang, A new general sufficient condition for almost sure stability of jump linear systems, *IEEE Trans. Autom. Control*, **42**:378–382, 1997.

5. Y. Fang, K.A. Loparo, Stabilization of continuous-time jump linear systems, *IEEE Trans. Autom. Control*, **47**:1590–1603, 2002.

6. C. Fu, A. Wu, Global exponential stability of reaction–diffusion delayed BAM neural networks with Dirichlet boundary conditions, in W. Yu, H. He, N. Zhang (Eds.), *Advances in Neural*

Networks – ISNN 2009, Lect. Notes Comput. Sci., Vol. 5551, Springer, Berlin, Heidelberg, 2009, pp. 303–312.

7. A. Gray, D. Greenhalgh, X. Mao, J. Pan, The SIS epidemic model with Markovian switching, *J. Math. Anal. Appl.*, **394**:496–516, 2012.

8. J. Hajnal, M. Bartlett, The ergodic properties of non-homogenous finite Markov chains, *Math. Proc. Camb. Philos. Soc.*, **52**:67–77, 1956.

9. K. Itǒ, *On Stochastic Differential Equations*, Mem. Am. Math. Soc., Vol. 4, AMS, Providence, RI, 1951.

10. Y. Ji, H.J. Chizeck, Controllability, stabilizability and continuous-time Markovian jump linear quadratic control, *IEEE Trans. Autom. Control*, **35**:777–788, 1990.

11. I. Karatzas, S.E. Shreve, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, Berlin, 1991.

12. C. Kim, Dynamic linear models with markov switching, *J. Econometrics*, **60**:1–22, 1994.

13. H. Kushner, *Stochastic Stability and Control*, Academic Press, New York, 1967.

14. J. Li, C. He, W. Zhang, M. Chen, Adaptive synchronization of delayed reaction–diffusion neural networks with unknown nonidentical time-varying coupling strengths, *Neurocomputing*, **219**:144–153, 2017.

15. J. Li, W. Zhang, M. Chen, Synchronization of delayed reaction–diffusion neural networks via adaptive learning control approach, *Comput. Math. Appl.*, **65**:1775–1785, 2013.

16. J. Li, W. Zhang, M. Chen, pth moment exponential stability of impulsive stochastic reaction–diffusion Cohen–Grossberg neural networks with mixed time delays, *Neural Process. Lett.*, **46**:83–111, 2017.

17. B. Liu, W. Lu, T. Chen, Synchronization in complex networks with stochastically switching coupling structures, *IEEE Trans. Autom. Control*, **57**:754–760, 2012.

18. X. Lou, B.Cui, Boundedness and exponential stability for nonautonomous cellar neural networks with reaction–diffusion terms, *Chaos Solitons Fractals*, **33**:653–662, 2007.

19. Q. Luo, Y. Zhang, Exponential stability of stochastic reaction diffusion systems, *Nonlinear Anal., Theory Methods Appl.*, **71**:487–493, 2009.

20. X. Mao, *Exponential Stability of Stochastic Differential Equations*, Marcel Dekker, New York, 1994.

21. X. Mao, Razumikhin-type theorems on exponential stability of stochastic functional differential equations, *Stochastic Processes Appl.*, **65**:233–250, 1996.

22. X. Mao, *Stochastic Differential Equations and Applications*, Ellis Horwood, Chichester, 2007.

23. X. Mao1, *Stability of Stochastic Differential Equations with Respect to Semimartingales*, Longman Scientific and Technical, London, 1991.

24. M. Mariton, *Jump Linear Systems in Automatic Control*, Marcel Dekker, New York, 1990.

25. L. Pan, J. Cao, Exponential stability of impulsive stochastic functional differential equations, *J. Math. Anal. Appl.*, **382**:672–685, 2011.

26. Q. Song, Z. Zhao, Y. Li, Global exponential stability of BAM neural networks with distributed delays and reaction–diffusion terms, *Phys. Let. A*, **335**:213–225, 2005.

27. A. Tahbaz-Salehi, A. Jadbabaie, Consensus over ergodic stationary graph processes, *IEEE Trans. Autom. Control*, **55**:225–230, 2010.

28. L. Wan, Q. Zhou, Exponential stability of stochastic reaction–diffusion Cohen–Grossberg neural networks with delays, *Appl. Math. Comput.*, **206**:818–824, 2008.

29. E. Xu, X. Yang, Adaptive synchronization of coupled nonidentical chaotic systems with complex variables and stochastic perturbations, *Nonlinear Dyn.*, **84**:261–269, 2016.

30. Q. Zhu, X. Li, X. Yang, Exponential stability of stochastic reaction–diffusion BAM neural networks with time-varying and distributed delays, *Appl. Math. Comput.*, **217**:6078–6091, 2011.