Nonlinear Analysis: Modelling and Control, Vol. 24, No. 3, 353–367 ISSN: 1392-5113
https://doi.org/10.15388/NA.2019.3.3 eISSN: 2335-8963

On leaderless consensus of fractional-order nonlinear multi-agent systems via event-triggered control*

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Received: March 7, 2018 / **Revised:** September 3, 2018 / **Published online:** April 19, 2019

Abstract. The consensus problem of fractional-order multi-agent systems is investigated by event-triggered control in this paper. Based on the graph theory and the Lyapunov functional approach, the conditions for guaranteeing the consensus are derived. Then, according to some basic theories of fractional-order differential equation and some properties of Mittag–Leffler function, the Zeno behavior could be excluded. Finally, a simulation example is given to check the effectiveness of the theoretical result.

Keywords: fractional oder, multi-agent systems, consensus, event-triggered protocol.

1 Introduction

Fractional-order calculus could trace back to three hundred years ago [18], but the applications of fractional-order calculus in engineering field just in the last decades. The concept of a fractional-order $PI^{\lambda}D^{\mu}$-controller, involving fractional-order integrator and fractional-order differentiator, was proposed in [21]. After that, more and more results about fractional-order control systems have been published; it could be seen in [16] and references therein. As the research further develops, people have found that the fractional-order systems show great memory and hereditary properties, and many results have proved that some phenomena can be explained better by the fractional-order systems, especially, the viscoelastic systems [1]. Thus, fractional-order systems has gradually been in a hot topic [8, 15, 28, 36, 39, 40].

Cooperative control of multi-agent systems has been widely investigated for its easy implementation, strong robustness. In the past decades, multi-agent systems have been applied to many fields, such as unmanned aerial vehicle [17], formation control [13],

*This work was jointly supported by the Natural Science Foundation of Jiangsu Province of China under grant No. BK20161126.

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target tracking [11] and so on. As a significant problem for the coordination control, consensus has received many attentions, which means that all the agents would converge to a desired target via communicating with their local neighbors. Consensus of multi-agent system has gotten many results in recent years, which could be seen in [7, 13, 19, 22] and references therein.

However, most of results about multi-agent systems were concerned with integer-order dynamics for agents. Indeed, lots of agents often work in complex environments, for example, vehicles moving on sand, muddy road, or grass, high-speed aircraft traveling in rain, dust storm, or snow environment, then, integer-order dynamics cannot well describe behaviors of agents. Under these cases, the dynamics for agents can be described as the fractional-order systems. Recently, fractional-order multi-agent systems have been gotten many results. Distributed coordination algorithms were studied for fractional-order systems firstly in [5]. Some further studies about consensus for the fractional-order multi-agent systems by the authors of [5] were published [6]. Then, lots of control laws for fractional-order multi-agent systems have been designed based on different approaches. For example, output feedback control for uncertain fractional-order multi-agent systems has been investigated in [33]. Adaptive pinning control for the leader-following consensus of fractional-order multi-agent systems has been studied in [37]. Distributed consensus tracking for the fractional-order multi-agent systems based on the sliding mode control method has been studied in [3].

All these aforementioned works were considered based on continuous control. It is obviously that unnecessary communication will lead to a waste of energy. Continuous communication would also cause the communication resource competition among agents. More recently, results based on impulsive control and sampled-data control methods have also been proposed in [26] and [38], respectively. However, data of them are updated period, which may also contain unnecessary communication. Event-triggered control strategy under which the information of agents were updated when a pre-set thing was triggered. There lots of results based on event-triggered control law were investigated for integer-order multi-agent systems [12, 14, 24, 27, 29, 31, 32, 41]. To the best of our knowledge, there were just few results about event-triggered control for fractional-order multi-agent systems [25, 30]. On the other hand, the leader-following consensus problem can be conveniently converted into a stabilization problem. However, as for the leaderless consensus problem, there is no specified leader. Consequently, the leaderless consensus problem is more challenging. Thus, more and more results about leaderless consensus have been published [2, 10, 34].

Based on the above analysis, this paper discusses consensus of fractional-order nonlinear multi-agent via event-triggered control. Firstly, a event-triggered control law will be introduced, the consensus can be achieved under some simple conditions formed as matrix inequalities. Then, for the proposed event-triggered scheme, we have proved that the inter-event-time intervals are strictly positive, i.e., there is no Zeno behavior. The rest of the manuscript is organized as follows: In Section 2, the graph theory and Caputo fractional operator have been introduced, and some basic lemmas have been given, which would be used in the later sections. In Section 3, we have given a statement for the problem. The main results are shown in Section 4. A simulation is presented in Section 5 to illustrate the effectiveness of the proposed method. Finally, conclusions and the further research directions are made in Section 6.

Notations. Let $\mathbb{R}$ be the set of real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n_1\times n_2}$ refer to the n-dimensional real vector and $n_1 \times n_2$ real matrices. The superscript "T" denotes matrix transposition. I_n denotes the n-dimensional identify matrix. For a vector $x \in \mathbb{R}^n$, $\|x\|$ is defined as $\|x\| = \sqrt{x^{\mathrm{T}}x}$. For $P \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(P)$ represents the maximum eigenvalue of P. $\otimes$ denotes Kronecker product.

2 Preliminaries

In this section, some basic theories of graph theory and Caputo fractional-order operator would be introduced.

2.1 Graph theory

Let $\mathcal{G} = \{\Delta, E, W\}$ be a undirected graph of order N, where $\Delta = \{v_1, v_2, \dots, v_N\}$, $E \subseteq \Delta \times \Delta$ denote the set of nodes and edges respectively. For any i, $w_{ii} = 0$. $e_{ij} = (v_i, v_j) \in E$ is a edge from i to j, where $i \neq j$, which means that v_j can receive message from v_i. v_i is a neighbor of v_j if $e_{ij} \in E$. The set of all neighbors of v_i can be denoted as $\mathcal{N}_i = \{v_j\colon e_{ij} \in E,\ j = 1, 2, \dots, N\}$. $W = \{w_{ij}\} \in \mathbb{R}^{N\times N}$ denotes weighted adjacency matrix, where w_{ij} is weight, which satisfies $w_{ij} = w_{ji} \neq 0$ if $e_{ij} \in E$ and $w_{ij} = 0$ otherwise. We assume $w_{ii} = 0$ for all $i = 1, 2, \dots, N$. The Laplacian matrix $L = (L_{ij}) \in \mathbb{R}^{N\times N}$ is defined by $L_{ij} = -w_{ij}$ for $i \neq j$ and $L_{ii} = -\sum_{j\neq i} L_{ij}$.

2.2 Caputo fractional derivative

Caputo fractional operator plays an important role in the fractional systems, since the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer-order differential, which have well-understood physical meanings [20]. Thus, we use Caputo derivatives as main tool in this paper. The formula of the Caputo fractional derivative is defined as follows.

Definition 1. (See [20].) The Caputo fractional derivative of function $x(t)$ is defined as

$$
{}_C D^{\alpha}_{0,t} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} x^{(m)}(\tau)\,\mathrm{d}\tau,
$$

where $m - 1 < \alpha < m$, $m \in Z^+$.

Let $m = 1$, $0 < \alpha < 1$, then

$$
{}_C D^{\alpha}_{0,t} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} x'(\tau)\,\mathrm{d}\tau.
$$

For simply, denote $D^\alpha x(t)$ as the ${}_C D^\alpha_{0,t} x(t)$. The following properties of Caputo operators are specially provided.

Lemma 1. (*See* [20].) *If* $w(t), u(t) \in C^1[t_0, b]$ *and* $\alpha, \beta > 0$, *then*

(i) $D^\alpha D^{-\beta} w(t) = D^{\alpha-\beta} w(t)$;
(ii) $D^\alpha(w(t) \pm u(t)) = D^\alpha w(t) \pm D^\alpha u(t)$.

The Mittag–Leffler function is defined by

$$E_{\alpha,\beta}(z) := \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\alpha i + \beta)},$$

where $\alpha > 0$, $\beta > 0$, $\Gamma(\cdot)$ is the gamma function. For short, $E_\alpha(z) := E_{\alpha,1}(z)$. The following lemma of Mittag–Leffler function will be used in the later.

Lemma 2. (*See* [20].) *Let* $0 < \alpha < 1$ *and* $p \in \mathbb{R}$. *The solution of the initial value problem*

$$D^\alpha x(t) = px(t) + q(t),$$

where $q(t)$ *is a given continuous function, can be expressed in the form*

$$x(t) = x(t_0) E_\alpha\bigl(p(t-t_0)^\alpha\bigr) + \alpha \int_{t_0}^{t} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\bigl(p(t-\tau)^\alpha\bigr) q(\tau)\,\mathrm{d}\tau.$$

3 Problem formulation

Each considered agent is modeled by a generic nonlinear system dynamics taking up the following form:

$$D^\alpha x_i(t) = Ax_i(t) + Df\bigl(x_i(t)\bigr) + Bu_i(t), \quad i = 1, 2, \dots, N, \tag{1}$$

where $x_i(t) = [x_{i1}(t), x_{i2}(t), \dots, x_{in}(t)]^{\mathrm{T}} \in \mathbb{R}^n$ is the state of agent i, $f : \mathbb{R}^n \to \mathbb{R}^r$, $u_i(t) \in \mathbb{R}^m$ is the distributed control law for agent i, which uses only the local information from its neighboring agents. A, B and D are constant matrices with appropriate dimensions. In this paper, the following control input for ith agent would be considered:

$$u_i(t) = \gamma K \sum_{j=1}^{N} w_{ij}\bigl(x_j(t_k) - x_i(t_k)\bigr), \quad t \in [t_k, t_{k+1}), \tag{2}$$

where t_k is kth event-triggered moment. Let

$$y_i(t) = \sum_{j=1}^{N} w_{ij}\bigl(x_j(t) - x_i(t)\bigr), \quad i = 1, 2, \dots, N,$$

and $e_i(t) = y_i(t_k) - y_i(t)$, then, one has

$$D^\alpha x_i(t) = Ax_i(t) + Df\bigl(x_i(t)\bigr) + \gamma BKy_i(t) + \gamma BKe_i(t), \quad i = 1, 2, \dots, N.$$

This formula can be written in the following compact matrix form:

$$\begin{aligned}D^\alpha x(t) &= (I_N \otimes A)x(t) + (I_N \otimes D)F\bigl(x(t)\bigr) - \gamma(I_N \otimes BK)y(t)\\ &\quad + \gamma(I_N \otimes BK)e(t),\end{aligned}$$

where $x(t) = [x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \dots, x_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $F(x(t)) = [f(x_1(t))^{\mathrm{T}}, f(x_2(t))^{\mathrm{T}}, \dots, f(x_N(t))^{\mathrm{T}}]^{\mathrm{T}}$, $y(t) = [y_1^{\mathrm{T}}(t), y_2^{\mathrm{T}}(t), \dots, y_N^{\mathrm{T}}(t)]^{\mathrm{T}}$, $e(t) = [e_1^{\mathrm{T}}(t), e_2^{\mathrm{T}}(t), \dots, e_N^{\mathrm{T}}(t)]^{\mathrm{T}}$. Note that $y(t) = -(L \otimes I_n)x(t)$, then

$$\begin{aligned}D^\alpha x(t) &= \bigl(I_N \otimes A - \gamma(L \otimes BK)\bigr)x(t) + (I_N \otimes D)F\bigl(x(t)\bigr)\\ &\quad + \gamma(I_N \otimes BK)e(t).\end{aligned} \tag{3}$$

In this paper, the next trigger moment is determined as

$$t_{k+1} = \inf\bigl\{t > t_k\colon \bigl\|e(t)\bigr\| = \eta\bigl\|y(t)\|\bigr)\bigr\} \quad \forall k \in \mathbb{N}. \tag{4}$$

Assumption 1. For the nonlinear function $f(\cdot)$, we assume that $f(0_{n\times 1}) = 0_{r\times 1}$, and there exists a constant l such that

$$\bigl\|f(x) - f(y)\bigr\| \leqslant l\|x - y\| \quad \forall x, y \in \mathbb{R}^n.$$

Assumption 2. The undirected communication graph is connected.

Under Assumption 2, the corresponding Laplacian matrix L is symmetric and positive semi-definite; hence, the eigenvalues of the matrix L are real and can be labelled as $0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \dots \leqslant \lambda_N$.

Lemma 3. *If $x, y \in \mathbb{R}^n$ are real vectors, then, for any positive constant ρ and positive matrix Θ,*

$$2x^{\mathrm{T}}y \leqslant \rho x^{\mathrm{T}}\Theta x + \frac{1}{\rho}y^{\mathrm{T}}\Theta^{-1}y.$$

4 Main results

Now the main results for system (1) can be given as follows.

Theorem 1. *Under Assumptions* 1 *and* 2, *the consensus of multi-agent system* (1) *under controller* (2) *with $K = B^{\mathrm{T}}P$ can be reached if there exists real constants ε_1, ε_2, β and a positive definite matrix P such that*

$$\begin{aligned}&PA + A^{\mathrm{T}}P + \varepsilon_2\widehat{D} - 2\gamma\lambda_2\widehat{B} + \varepsilon_1\gamma\mu_{\widehat{B}}\lambda_N I_n + \frac{l^2\lambda_N}{\varepsilon_2}I_n\\ &\quad < -\lambda_N\beta I_n < -\frac{\eta^2\gamma\mu_{\widehat{B}}}{\varepsilon_1}I_n,\end{aligned}$$

where $\widehat{B} = PBB^{\mathrm{T}}P$, $\widehat{D} = PDD^{\mathrm{T}}P$, $\mu_{\widehat{B}} = \|\widehat{B}\|$.

Proof. Consider the following Lyapunov function: $V(t) = x^{\mathrm{T}}(t)(L \otimes P)x(t)$. Calculating the derivatives of $V(t)$ along the solutions of system (3), one has

$$\begin{aligned} D^{\alpha}V(t) &\leqslant 2x^{\mathrm{T}}(t)(L \otimes P)\big(I_N \otimes A - \gamma(L \otimes BK)\big)x(t) \\ &\quad + 2\gamma x^{\mathrm{T}}(t)(L \otimes P)(I_N \otimes BK)e(t) + 2x^{\mathrm{T}}(t)(L \otimes P)(I_N \otimes D)F\big(x(t)\big) \\ &\leqslant x^{\mathrm{T}}(t)\big(L \otimes \big(PA + A^{\mathrm{T}}P\big) - 2\gamma\big(L^2 \otimes (PBK)\big)\big)x(t) \\ &\quad + 2\gamma x^{\mathrm{T}}(t)(L \otimes P)(I_N \otimes BK)e(t) + 2x^{\mathrm{T}}(t)(L \otimes P)(I_N \otimes D)F\big(x(t)\big). \end{aligned}$$

Note that L is a symmetric matrix. Then one can find a orthogonal matrix $Q \in \mathbb{R}^{N\times N}$ such that $QLQ^{-1} = QLQ^{\mathrm{T}} = \Lambda = \operatorname{diag}\{\lambda_1, \lambda_2, \dots, \lambda_N\}$. By letting $z(t) = (Q \otimes I_n) \times x(t)$, $\hat{e}(t) = (Q \otimes I_n)e(t)$, for any positive constant ε_1, ε_2 and positive matrix Θ, we can get

$$\begin{aligned} &x^{\mathrm{T}}(t)\big(L \otimes \big(PA + A^{\mathrm{T}}P\big) - 2\gamma\big(L^2 \otimes (PBK)\big)\big)x(t) \\ &\quad = z^{\mathrm{T}}(t)\big(\Lambda \otimes \big(PA + A^{\mathrm{T}}P\big) - 2\gamma\big(\Lambda^2 \otimes \widehat{B}\big)\big)z(t) \\ &\quad = \sum_{i=2}^{N} z_i^{\mathrm{T}}(t)\big(\lambda_i\big(PA + A^{\mathrm{T}}P\big) - 2\gamma\lambda_i^2\widehat{B}\big)z_i(t), \end{aligned}$$

$$\begin{aligned} &2\gamma x^{\mathrm{T}}(t)(L \otimes P)(I_N \otimes BK)e(t) \\ &\quad = 2\gamma z^{\mathrm{T}}(t)(\Lambda \otimes \widehat{B})\hat{e}(t) = 2\gamma \sum_{i=2}^{N} \lambda_i z_i^{\mathrm{T}}(t)\widehat{B}\hat{e}_i(t) \\ &\quad \leqslant \gamma \sum_{i=2}^{N} \Big(\varepsilon_1 \mu_{\widehat{B}} \lambda_i^2 \big\|z_i(t)\big\|^2 + \frac{\mu_{\widehat{B}}}{\varepsilon_1}\big\|\hat{e}_i(t)\big\|^2\Big), \end{aligned}$$

$$\begin{aligned} &2x^{\mathrm{T}}(t)(L \otimes P)(I_N \otimes D)F\big(x(t)\big) \\ &\quad = 2z^{\mathrm{T}}(t)(\Lambda \otimes PD)\big(Q^{\mathrm{T}} \otimes I_n\big)F\big(x(t)\big) \\ &\quad \leqslant \varepsilon_2 z^{\mathrm{T}}(t)(\Lambda\Theta\Lambda \otimes \widehat{D})z(t) \\ &\qquad + \frac{1}{\varepsilon_2}F^{\mathrm{T}}\big(x(t)\big)\big(Q^{\mathrm{T}} \otimes I_n\big)\big(\Theta^{-1} \otimes I_n\big)(Q \otimes I_n)F\big(x(t)\big). \end{aligned}$$

Now, choosing $\Theta = \Lambda^{-1}$ and combining with the Assumption 1, we have

$$\begin{aligned} &2x^{\mathrm{T}}(t)(L \otimes P)(I_N \otimes D)F\big(x(t)\big) \\ &\quad \leqslant \varepsilon_2 z^{\mathrm{T}}(t)(\Lambda \otimes \widehat{D})z(t) + \frac{l^2}{\varepsilon_2}z^{\mathrm{T}}(t)(\Lambda \otimes I_n)z(t) \\ &\quad \leqslant \sum_{i=2}^{N} z_i^{\mathrm{T}}(t)\left(\varepsilon_2\widehat{D}\lambda_i + \frac{l^2\lambda_i^2}{\varepsilon_2}I_n\right)z_i(t). \end{aligned}$$

Due to $Q^{\mathrm{T}}Q = I_N$, from the condition in the Theorem 1 we have

$$\begin{aligned}
&\frac{PA + A^{\mathrm{T}}P + \varepsilon_2 \widehat{D}}{\lambda_i} - 2\gamma \widehat{B} + \varepsilon_1 \gamma \mu_{\widehat{B}} I_n + \frac{l^2}{\varepsilon_2} I_n \\
&\quad < \frac{1}{\lambda_i}\left(PA + A^{\mathrm{T}}P + \varepsilon_2 \widehat{D} - 2\gamma\lambda_2 \widehat{B} + \varepsilon_1 \gamma \mu_{\widehat{B}} \lambda_N I_n + \frac{l^2 \lambda_N}{\varepsilon_2} I_n\right) \\
&\quad < -\beta I_n.
\end{aligned}$$

Then, based on the event-triggered conditions, it is easy to get

$$\begin{aligned}
D^\alpha V(t) &\leqslant \sum_{i=2}^{N} \left(\frac{PA + A^{\mathrm{T}}P + \varepsilon_2 \widehat{D}}{\lambda_i} - 2\gamma \widehat{B} + \varepsilon_1 \mu_{\widehat{B}} I_n + \frac{l^2}{\varepsilon_2} I_n\right) \lambda_i^2 \left\|z_i(t)\right\|^2 \\
&\quad + \frac{\gamma \mu_{\widehat{B}}}{\varepsilon_1} \sum_{i=2}^{N} \left\|\hat{e}_i(t)\right\|^2 \\
&= \sum_{i=2}^{N} \left(\frac{PA + A^{\mathrm{T}}P + \varepsilon_2 \widehat{D}}{\lambda_i} - 2\gamma \widehat{B} + \varepsilon_1 \mu_{\widehat{B}} I_n + \frac{l^2}{\varepsilon_2} I_n\right) \lambda_i^2 \left\|z_i(t)\right\|^2 \\
&\quad + \frac{\gamma \mu_{\widehat{B}}}{\varepsilon_1} \sum_{i=1}^{N} \left\|e_i(t)\right\|^2 \\
&\leqslant -\beta \sum_{i=1}^{N} \lambda_i^2 \left\|z_i(t)\right\|^2 + \frac{\gamma \eta^2 \mu_{\widehat{B}}}{\varepsilon_1} \sum_{i=1}^{N} \left\|y_i(t)\right\|^2 \\
&= -\beta z^{\mathrm{T}}(t) \left(\Lambda^2 \otimes I_n\right) z(t) + \frac{\eta^2 \gamma \mu_{\widehat{B}}}{\varepsilon_1} \sum_{i=1}^{N} \left\|y_i(t)\right\|^2 \\
&= -\left(\beta - \frac{\eta^2 \gamma \mu_{\widehat{B}}}{\varepsilon_1}\right) \sum_{i=1}^{N} \left\|y_i(t)\right\|^2
\end{aligned}$$

Let $Y(t) = \sum_{i=1}^{N} \|y_i(t)\|^2$, then, similar to the analysis in [4, 35], one can conclude that $Y(t) \to 0$ as $t \to +\infty$, which completes our proof. □

Remark 1. Under the event-triggered control law introduced in this paper, the information of agents would be updated just when states of them have changed much. Thus, compared with traditional continuous control strategies and periodic sampling control methods, this lowers the cost of information transmission. In the proof of Theorem 1, the symmetry of L has been used to diagonalization, which required topological structure of the multi-agent network is undirected. The case of directed topological structure would be our future work.

Noting that the matrix inequalities in the conditions of Theorem 1 are not linear, one cannot solve it by using the LMI toolbox of the MATLAB. The following corollary would overcome this difficulty.

Corollary 1. *Under Assumptions* 1 *and* 2*, the consensus of multi-agent system* (1) *under controller* (2) *with* $K = B^{\mathrm{T}}P$ *can be reached if there exists real constants* ε_1*,* ε_2*,* ε_3*,* β *and a positive definite matrix* P *such that*

$$\begin{pmatrix} PA + A^{\mathrm{T}}P + \gamma\lambda_N I_n + l^2\lambda_N\varepsilon_3 I_n + \lambda_N\beta I_n & \varepsilon_2 PD & \sqrt{2\gamma\lambda_2}PB \\ * & -\varepsilon_2 I_n & 0 \\ * & * & -I_n \end{pmatrix} < 0, \tag{5}$$

$$\eta^2 < \frac{\varepsilon_1^2\lambda_N\beta}{\gamma}, \tag{6}$$

$$\varepsilon_2\varepsilon_3 > 1, \tag{7}$$

$$\varepsilon_1\mu_{\widehat{B}} < 1, \tag{8}$$

where $\widehat{B}$ *and* $\mu_{\widehat{B}}$ *are the same as in Theorem 1.*

This corollary can be proved directly from that of Theorem 1 above by using Schur complement lemma, so the proof is omitted for brevity.

Remark 2. In fact, the condition of unknown parameter ε_1 in the Corollary 1 are not linear, and the parameter $\mu_{\widehat{B}}$ depends on the unknown matrix P, which is also linear in the $\widehat{B}$. However, to solve this problem, one can give a small ε_1 at first. Then (5) and (6) could be solved by using LMI toolbox of the MATLAB. Afterthat, we can check whether (7) and (8) are true or not. Consequently, the matrix inequalities of Corollary 1 could be solved.

Theorem 2. *Consider the leaderless multi-agent system with fractional-order nonlinear dynamics* (1)*, the control protocol* (2)*, the triggering condition* (4)*. The Zeno behavior can also be excluded, which means that the minimum inter-event interval is lower bounded by a positive scalar.*

Proof. Based on the definition of $e_i(t)$ and $y_i(t)$, one has

$$\begin{aligned} &D^{\alpha}\|e_i(t)\| \\ &\quad\leqslant \|D^{\alpha}e_i(t)\| = \|-D^{\alpha}y_i(t)\| = \Bigg\|-\sum_{j=1}^{N} w_{ij}\big(D^{\alpha}x_j(t) - D^{\alpha}x_i(t)\big)\Bigg\| \\ &\quad= \Bigg\|-Ay_i(t) - D\sum_{j=1}^{N} w_{ij}\big(f\big(x_j(t)\big) - f\big(x_i(t)\big)\big) + \gamma BK\sum_{j=1}^{N} w_{ij}\big(y_i(t_k) - y_j(t_k)\big)\Bigg\| \\ &\quad\leqslant \epsilon\|e_i(t)\| + l\|D\|\|y_i(t_k)\| + \Bigg\|Ay_i(t_k) - \gamma BK\sum_{j=1}^{N} w_{ij}\big(y_i(t_k) - y_j(t_k)\big)\Bigg\| \\ &\quad\leqslant \epsilon\|e_i(t)\| + \psi_k^i, \end{aligned}$$

where $\epsilon = \|A\| + l\|D\|$, $\psi_k^i = l\|D\|\|y_i(t_k)\| + \|Ay_i(t_k) - \gamma BK\sum_{j=1}^{N} w_{ij}(y_i(t_k) - y_j(t_k))\|$. When $t \in [t_k, t_{k+1})$, it is easy to get $\|e_i(t_k)\| = 0$, then based on Lemma 2 and comparison principle of fractional-order differential equations, one can get

$$\|e_i(t)\| \leqslant \alpha\psi_k^i \int_{t_k}^{t} (t-t_k)^{\alpha-1} E_{\alpha,\alpha}\big(\epsilon(t-t_k)^\alpha\big)\,\mathrm{d}s.$$

Recall the property of Mittag–Leffler function $E_{\alpha,\beta}(\cdot)$ that [9, 23]

$$t^{\beta-2}E_{\alpha,\beta-1}\big(\lambda t^\alpha\big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(t^{\beta-1}E_{\alpha,\beta}\big(\lambda t^\alpha\big)\big).$$

Then, letting $\beta = \alpha + 1$, we have

$$\begin{aligned}
\|e_i(t)\| &\leqslant \alpha\psi_k^i \int_{t_k}^{t} (t-t_k)^{\alpha-1} E_{\alpha,\alpha}\big(\epsilon(t-t_k)^\alpha\big)\,\mathrm{d}s \\
&= \alpha\psi_k^i \int_{0}^{t-t_k} s^{\alpha-1} E_{\alpha,\alpha}\big(\epsilon s^\alpha\big)\,\mathrm{d}s \\
&= \alpha\psi_k^i \int_{0}^{t-t_k} \frac{\mathrm{d}}{\mathrm{d}s}\big(t^\alpha E_{\alpha,\alpha+1}\big(\epsilon s^\alpha\big)\big)\,\mathrm{d}s \\
&= \alpha\psi_k^i (t-t_k)^\alpha E_{\alpha,\alpha+1}\big(\epsilon(t-t_k)^\alpha\big).
\end{aligned}$$

According to the other property of Mittag–Leffler function $E_{\alpha,\beta}(\cdot)$ [9, 23]

$$E_{\alpha,\beta}(t) = tE_{\alpha,\alpha+\beta}(t) + \frac{1}{\Gamma(\beta)},$$

we have

$$\|e_i(t)\| \leqslant \frac{\alpha\psi_k^i}{\epsilon}\big(E_\alpha\big(\epsilon(t-t_k)^\alpha\big) - 1\big).$$

According to the definition of event-triggered instants (4), the next event will not be generated before $\|e_i(t_{k+1})\| = \eta\|y(t_{k+1})\|$, which implies that $\|e_i(t)\| \leqslant \eta\|y_i(t)\|$ when $t \in [t_k, t_{k+1})$. By using $\|e_i(t)\| \geqslant |\|y_i(t_k)\| - \|y_i(t)\||$, one has

$$\frac{\|y_i(t_k)\|}{1+\eta} \leqslant \|y_i(t)\| \leqslant \frac{\|y_i(t_k)\|}{1-\eta}. \tag{9}$$

Then, we have $\|e_i(t_{k+1})\| \geqslant (\eta/(1+\eta))\|y_i(t_k)\|$, one can conclude that

$$\frac{\eta}{1+\eta}\|y_i(t_k)\| \leqslant \frac{\alpha\psi_k^i}{\epsilon}\big(E_\alpha(\epsilon(t_{k+1}-t_k)^\alpha) - 1\big).$$

To prove that the inter-event interval $t_{k+1} - t_k$ is strictly positive, we first consider the case when $y_i(t_k) \neq 0$. It is easy to get

$$\frac{\alpha\psi_k^i}{\epsilon}\big(E_\alpha\big(\epsilon(t_{k+1}-t_k)^\alpha\big) - 1\big) \geqslant \frac{\eta}{1+\eta}\|y_i(t_k)\| > 0,$$

which implies that $t_{k+1} - t_k > 0$. Next, based on Theorem 1, we have $y_i(t_k) = 0$ as $k \to \infty$. Then, according to (9), we have $y_i(t) = 0$ when $t \in [t_k, t_{k+1})$ as $k \to \infty$. One has

$$\begin{aligned} D^\alpha y_i(t) = Ay_i(t) + D\sum_{j=1}^N w_{ij}\big(f\big(x_j(t)\big) - f\big(x_i(t)\big)\big) \\ - \gamma BK \sum_{j=1}^N w_{ij}\big(y_i\big(t_k\big) - y_j(t_k)\big) = 0. \end{aligned}$$

Then, based on the definition of ψ_k^i, we have

$$\psi_k^i \leqslant \epsilon\big\|y_i(t_k)\big\| + \epsilon\big\|y_i(t*)\big\|, \quad t* \in [t_k, t_{k+1}).$$

At the same time, one can get from (9) that

$$1 - \eta \leqslant \lim_{k\to\infty} \frac{\|y_i(t_k)\|}{\|y_i(t)\|}, \quad t \in [t_k, t_{k+1}].$$

Consequently, we have

$$\lim_{k\to\infty} \frac{\|y_i(t_k)\|}{\psi_k^i} \geqslant \frac{1}{\epsilon + \epsilon(1-\eta)} > 0.$$

As a result,

$$\lim_{k\to\infty} \big(E_\alpha\big(\epsilon(t_{k+1} - t_k)^\alpha\big) - 1\big) \geqslant \frac{\eta}{\alpha(1+\eta)(2-\eta)} > 0,$$

which implies that $t_{k+1} - t_k$ is strictly positive. The proof is thus completed with the case $y_i(t_k) \neq 0$. □

Remark 3. The existence of lower bound of the minimum inter-event interval has been proved above. In fact, the lower bound can be further determined. Let τ be the lower bound of the minimum inter-event interval. It is easy to show that

$$E_\alpha\big(\epsilon\tau^\alpha\big) \geqslant 1 + \frac{\eta}{\alpha(1+\eta)(2-\eta)}.$$

Thanks to the monotone increasing characteristics for the function $E_\alpha(t)$, one can conclude that the inverse of the $E_\alpha(t)$ must be existed. Then, one has

$$\tau \geqslant \left(\frac{E_\alpha^{-1}(1 + \frac{\eta}{\alpha(1+\eta)(2-\eta)})}{\epsilon}\right)^\alpha.$$

5 Numerical simulations

In this section, a simulation example is provided to check the effectiveness of the above theoretical results.

Consider a 3-dimension multi-agent system with the following matrices

$$A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & -1.2 & 0 \\ 1.8 & 1.71 & 1.15 \\ -4.75 & 0 & 1.1 \end{pmatrix},$$

$$B = \begin{pmatrix} -0.1 & 0 & 0 \\ 0 & -0.1 & 0 \\ 0 & 0 & -0.1 \end{pmatrix}$$

and the nonlinear function $f(x_i(t)) = (2\tanh(x_{i1}(t)), 2\tanh(x_{i2}(t)), 2\tanh(x_{i3}(t)))^{\mathrm{T}}$. When the order $\alpha = 0.98$, under the initial value $x_i(0) = [1, 1, -1]^{\mathrm{T}}$, every agent has a chaotic behavior, which could be seen in Fig. 1.

Now, four agents considered in the multi-agent system has a communication topology with the Laplacian

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}.$$

The initial values of the four agents are given as $x_1(0) = [-1; 2; 3]$, $x_2(0) = [3; -1.5; 0.5]$, $x_3(0) = [-2; -3; 1]$ and $x_4(0) = [1.5; -2.5; 0.2]$. Without control, the state trajectories of them have been shown in Fig. 2.

By resorting to some standard software in MATLAB, according to Remark 2, let $\varepsilon_1 = 0.1$ and $\eta = 0.1$, $r = 0.5$ at first. Then, the matrix inequalities in (5) and (6) are solvable

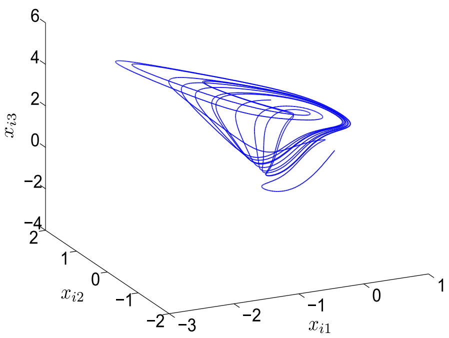

Figure 1. Chaotic behavior of isolated agent with initial value $x_i(0) = [1, 1, -1]^{\mathrm{T}}$.

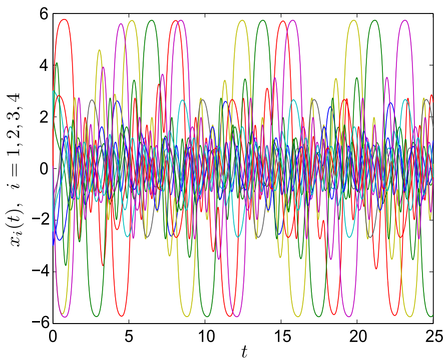

Figure 2. The state trajectories of four agents without control.

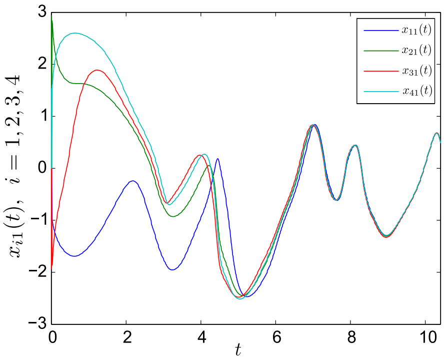

Figure 3. The state trajectories $x_{i1}(t)$ of four agents under the control protocol (2).

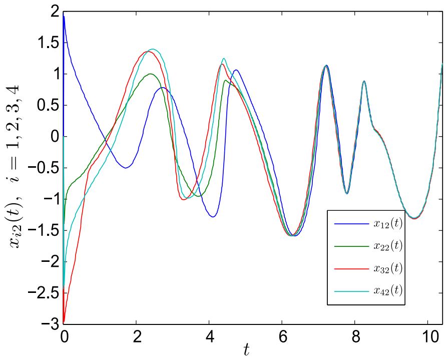

Figure 4. The state trajectories $x_{i2}(t)$ of four agents under the control protocol (2).

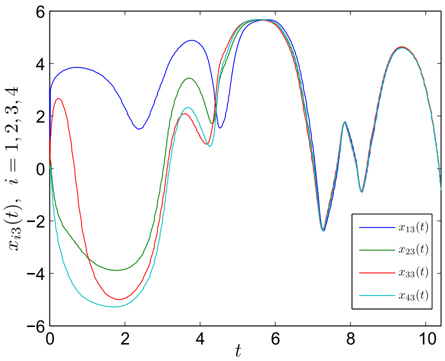

Figure 5. The state trajectories $x_{i3}(t)$ of four agents under the control protocol (2).

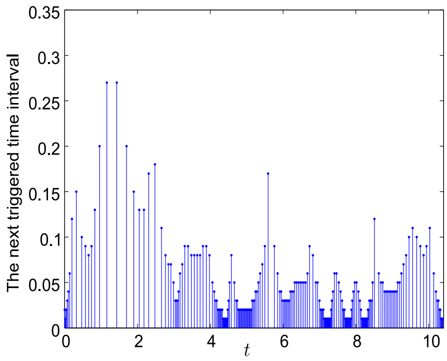

Figure 6. The release instants and release interval.

with a feasible solution as follows: $\varepsilon_2 = 614.9215$, $\varepsilon_3 = 0.1305$, $\beta = 4.0092$, and

$$P = \begin{pmatrix} 21.9410 & -0.3789 & 0.9716 \\ -0.3789 & 22.1133 & 0.8548 \\ 0.9716 & 0.8548 & 19.9148 \end{pmatrix}$$

Then, one can get $\varepsilon_1 \mu_{\widehat{B}} = 0.5029$, $\varepsilon_1 \varepsilon_2 = 80.2412 > 1$. Thus, all conditions of Corollary 1 could be satisfied. Consequently, one has

$$K = \begin{pmatrix} -2.1941 & 0.0379 & -0.0972 \\ 0.0379 & -2.2113 & -0.0855 \\ -0.0972 & -0.0855 & -1.9915 \end{pmatrix}.$$

Under the control protocol (2), the state trajectories can be seen in Figs. 3–5.

Remark 4. For the above example, we also can select

$$B = \begin{pmatrix} -0.1 & 0 \\ 0 & -0.1 \\ 0 & 0 \end{pmatrix}.$$

By resorting to some standard software in MATLAB, according to Remark 2, let $\varepsilon_1 = 0.1$ and $\eta = 0.1$, $r = 0.5$ at first. Then, the matrix inequalities in (5) and (6) are solvable with a feasible solution with

$$P = \begin{pmatrix} 22.2395 & -0.3934 & 1.0262 \\ -0.3934 & 22.4176 & 0.8993 \\ 1.0262 & 0.8993 & 20.1405 \end{pmatrix}, \qquad K = \begin{pmatrix} -2.2239 & 0.0393 & -0.1026 \\ 0.0393 & -2.2418 & -0.0899 \end{pmatrix}.$$

Indeed, the theoretical results of this paper could deal with the case that control input with different dimension.

6 Conclusion

In this paper, we considered the consensus of fractional-order multi-agent systems via the centralized event-triggered protocols. The main theoretical results were derived based on the fractional-order Lyapunov stability theory and properties of graph's Laplacian matrix. Moreover, the conditions can be changed to some simple linear matrix inequalities. In addition, the Zeno behavior can effectively avoid. The given numerical examples illustrate the corresponding theoretical analysis. Note that this paper has considered the centralized mechanism, however, the distributed mechanism has a better robustness. Thus, it is an interesting topic of our future work. On the other hand, the consensus of fractional-order multi-agent systems with switched topologies by using the proposed event-triggered control will be considered in the future work.

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