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Existence, uniqueness and numerical solution of a fractional PDE with integral conditions

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Abstract. This paper is devoted to the solution of one-dimensional Fractional Partial Differential Equation (FPDE) with nonlocal integral conditions. These FPDEs have been of considerable interest in the recent literature because fractional-order derivatives and integrals enable the description of the memory and hereditary properties of different substances. Existence and uniqueness of the solution of this FPDE are demonstrated. As for the numerical approach, a Galerkin method based on least squares is considered. The numerical examples illustrate the fast convergence of this technique and show the efficiency of the proposed method.

Keywords: existence and uniqueness, Galerkin scheme, fractional partial differential equation, purely integral conditions.

1 Introduction

Fractional differential equations (FDEs) can be considered as generalizations of differential equations of integer order to an arbitrary order. These types of problems appear in a large variety of areas, such as engineering, physics, and applied mathematics. Therefore, they have generated a lot of interest among engineers and scientist in recent years. Some recent developments in FDEs and in partial differential equations (PDEs) were proposed in [3, 4, 10, 13, 14, 16, 21–23, 29, 34].

Also, existence and uniqueness of solutions to initial and boundary-value problems for FDEs were studied in many articles; see [1, 2, 5, 19, 24, 35], for example. Some results of the existence and uniqueness in FDEs were obtained by using the well-known Lax–Milgram theorem, and/or by fixed point theorems [11, 24, 36].

Also, many problems in physics and technology were proposed using nonlocal conditions for PDEs, which are normally described using purely integral conditions. Nonlocal boundary conditions receive a lot of attention (see [7, 8, 25, 27, 28, 31, 32] and references therein) because of their numerous applications in blood flow models, cellular systems, chemical engineering, population dynamics, or viscoelasticity. We can also find some applications in biological fluid dynamics, control theory, diffusive transport, electrochemistry, electrical networks, electromagnetic theory, fluid flow, geology, porous media, rheology, signal processing, and many other physical processes; see [9, 15, 17, 18, 33].

In this paper, a fractional differential equation together with purely integral conditions is analyzed concerning two aspects: (i) existence and uniqueness of solutions will be studied, and (ii) we will also study the problem from a numerical point of view.

A suitable variational formulation is also the starting point of many numerical methods, such as finite element methods and spectral numerical schemes (for example, Galerkin methods). Thus, the construction of the variational formulation is essential and relies strongly on the choice of spaces and their norms. Motivated by this, we extend and generalize the study for PDEs with nonlocal conditions to the study of fractional PDEs with nonlocal conditions in Section 2. At the same time, we expand the works in classical problems of fractional PDEs to nonstandard problems. We will extend the application of the energy inequality method for obtaining uniqueness and existence of solutions in functional weighted Sobolev spaces in Section 3. Finally, we will develop some numerical schemes based on Galerkin methods in Section 4 and show the efficiency of them in the numerical Section 5.

2 Formulation of the fractional partial differential equation

2.1 Caputo and Riemann–Liouville derivatives

First, we need some definitions to explain the problem that we shall study in this work: let $\Gamma(\cdot)$ denote the gamma function. For any positive noninteger value $0 < \alpha < 1$, the Caputo derivative and the Riemann–Liouville derivative are, respectively, defined as follows:

Definition 1. The left Caputo derivatives can be expressed as

$$ {}_0^C\partial_t^\alpha u(x,t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,\tau)}{\partial \tau} \frac{1}{(t-\tau)^\alpha} \,\mathrm{d}\tau. $$

Definition 2. The left Riemann–Liouville derivatives are

$$ {}_0^R\partial_t^\alpha u(x,t) := \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial t} \int_0^t \frac{u(x,\tau)}{(t-\tau)^\alpha} \,\mathrm{d}\tau. $$

Corollary 1. *Riemann–Liouville and Caputo derivatives are linked by the following relationship*:

$$ {}_0^R\partial_t^\alpha u(x,t) = {}_0^C\partial_t^\alpha u(x,t) + \frac{u(x,0)}{\Gamma(1-\alpha)t^\alpha}. \tag{1} $$

2.2 The fractional PDE with integral conditions

In the rectangular domain $\Omega = (0,1) \times (0,T)$, with $T < \infty$, for any positive non integer $0 < \alpha < 1$, we consider the fractional partial differential equation

$$\mathcal{L}v = {}_0^C\partial_t^\alpha v(x,t) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial v}{\partial x}\right) + b(x,t)v(x,t) = F(x,t), \tag{2}$$

subject to the initial condition

$$\ell u = v(x,0) = \phi(x), \quad x \in (0,1), \tag{3}$$

and the purely integral boundary conditions

$$\int_0^1 v(x,t)\,\mathrm{d}x = \mu(t), \quad \int_0^1 xv(x,t)\,\mathrm{d}x = m(t), \quad t \in [0,T], \tag{4}$$

where F, ϕ, μ, and m are known functions.

Assumption. For each $(x,t) \in \overline{\Omega}$, we assume:

$$\sup_{\overline{\Omega}} \frac{\partial^2 b(x,t)}{\partial x^2} \geqslant 0, \qquad \inf_{\overline{\Omega}} a(x,t) \geqslant 0, \tag{5}$$

and

$$b(x,t) + 2\inf_{\overline{\Omega}} a(x,t) - \frac{1}{4}\sup_{\overline{\Omega}} \frac{\partial^2 b(x,t)}{\partial x^2} - \frac{1}{2}\frac{\partial^2 a(x,t)}{\partial x^2} - \frac{\varepsilon}{2} \geqslant M > 0. \tag{6}$$

First, we transform problem (2)–(4), with inhomogeneous conditions, to the equivalent problem with homogeneous (and purely integral, nonlocal) boundary conditions. In this way, it will be easier to demonstrate existence and uniqueness. For this purpose, we introduce a new unknown function defined by

$$w(x,t) = v(x,t) - U(x,t),$$

where

$$U(x,t) = \mu(t) + 6\bigl(2m(t) - \mu(t)\bigr)\bigl(-2x + 3x^2\bigr).$$

Problem (2)–(4) is therefore equivalent to

$$\mathcal{L}\widetilde{v} = {}_0^C\partial_t^\alpha w(x,t) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial w}{\partial x}\right) + b(x,t)w = h(x,t), \tag{7}$$

$$\ell w = w(x,0) = \varphi(x), \quad x \in (0,1), \tag{8}$$

$$\int_0^1 w(x,t)\,\mathrm{d}x = 0, \quad \int_0^1 xw(x,t)\,\mathrm{d}x = 0, \quad t \in [0,T], \tag{9}$$

where $h(x,t) = F(x,t) - \mathcal{L}U$, and $\varphi(x) = \phi(x) - \ell U$ satisfying the compatibility conditions.

Once again, we introduce a new unknown function $u(x,t) = w(x,t) - \varphi(x)$. Using relation (1), problem (7)–(9) is reformulated as follows:

$$\begin{aligned}\mathcal{L}u &= {}_0^R\partial_t^\alpha u(x,t) - \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right) + b(x,t)uh(x,t) + \frac{\partial}{\partial x}\left(a(x,t)\frac{\mathrm{d}\varphi(x)}{\mathrm{d}x}\right)\\ &= f(x,t),\end{aligned} \tag{10}$$

$$\ell u = u(x,0) = 0, \quad x \in (0,1), \tag{11}$$

$$\int_0^1 u(x,t)\,\mathrm{d}x = 0, \quad t \in (0,T), \qquad \int_0^1 xu(x,t)\,\mathrm{d}x = 0, \quad t \in [0,T]. \tag{12}$$

Thus, we transformed problem (7)–(9) with inhomogeneous initial condition to the equivalent problem with homogenous condition.

2.3 Notations and preliminary results

For the study of problem (10)–(12), we need some definitions and results on functional spaces. They will be very useful in the next section.

Definition 3. Let us denote by $C_0(0,1)$ the space of continuous functions with compact support in $(0,1)$, and its bilinear form is given by

$$(\!(u,w)\!) = \int_0^1 \Im_x^m u \cdot \Im_x^m w\,\mathrm{d}x \quad (m \in \mathbb{N}^*), \tag{13}$$

where

$$\Im_x^m u = \int_0^x \frac{(x-\xi)^{m-1}}{(m-1)!}u(\xi,t)\,\mathrm{d}\xi \quad (m \in \mathbb{N}^*).$$

For $m = 1$, we have $\Im_x u = \int_0^x u(\xi,t)\,\mathrm{d}\xi$ and $\Im_t u = \int_0^t u(x,\tau)\mathrm{d}\tau$. The bilinear form (13) is considered as a scalar product on $C_0(0,1)$ when it is not complete.

Definition 4. We denote by $B_2^m(0,1)$ the space defined by

$$B_2^m(0,1) = \begin{cases} L^2(0,1) & \text{for } m = 0,\\ u/\Im_x^m u \in L^2(0,1) & \text{for } m \in \mathbb{N}^*,\end{cases}$$

the completion of $C_0(0,1)$ for the scalar product defined by (13). The associated norm to the scalar product is

$$\|u\|_{B_2^m(0,1)} = \|\Im_x^m u\|_{L^2(0,1)} = \left(\int_0^T \left(\Im_x^m u\right)^2\mathrm{d}x\right)^{1/2}.$$

Lemma 1. (*See* [6].) *For all $m \in \mathbb{N}^*$, we obtain*

$$\|u\|_{B_2^m(0,1)} \leqslant \left(\frac{1}{2}\right)^m \|u\|^2_{L^2(0,1)}. \tag{14}$$

Definition 5. Let X be a Banach space with the norm $\|u\|_X$, and let $u : (0,T) \to X$ be an abstract functions, by $\|u(\cdot,t)\|_X$ we denote the norm of the element $u(\cdot,t) \in X$ at a fixed t. We denote by $L^2(0,T;X)$ the set of all measurable abstract functions $u(\cdot,t)$ from $(0,T)$ into X such that

$$\|u\|_{L^2(0,T;X)} = \left(\int_0^T \big\|u(\cdot,t)\big\|_X \,\mathrm{d}t\right)^{1/2} < \infty.$$

Lemma 2 [Cauchy inequality with ε]. *For all ε and arbitrary variables $a, b \in \mathbb{R}$, we have the following inequality*:

$$|ab| \leqslant \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2. \tag{15}$$

3 Existence and uniqueness of the solution

3.1 Uniqueness of solution (a priori estimates)

A priori estimate method is one efficient functional analysis technique (it is also called the energy-integral method by many researchers). This is an important technique for studying PDEs in general and with some purely integral conditions in particular. It has been successfully used when proving the existence, uniqueness, and continuous dependence of the solutions of PDEs; see [30] and references therein. It is essentially based on the construction of multipliers for each specific problem. In this way a priori estimate is provided, from which it is possible to establish the solvability of the problem.

Our proof is based on an energy inequality and the density of the range of the operator generated by the abstract formulation of the problem. First, we introduce the needed function spaces. Later, we will prove the uniqueness of solution (if it exists), and finally, the existence of the solution for equations (10)–(12).

Problem (10)–(12) is equivalent to the operational equation

$$Lu = F,$$

where $F = (f)$, and $L = (\mathcal{L}, \ell)$ is considered as an operator from X to H, X is a Banach space consisting of all functions $u \in L^2(0,T;B_2^1(0,1))$ with the finite norm

$$\|u\|_X = \int_0^\tau \int_0^1 \big({}_0^R\partial_t^\alpha \Im_x u(x,t)\big) \cdot \Im_x u(x,t) \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_0^1 \big(\Im_x u(x,t)\big)^2 \,\mathrm{d}x\,\mathrm{d}t,$$

and H is the Hilbert space consisting of all elements $F \in L^2(0,T;B_2^1(0,1))$ with the finite norm

$$\|F\|_H = \int_0^\tau \int_0^1 \big(\Im_x f(x,t)\big)^2 \, \mathrm{d}x \, \mathrm{d}t.$$

The domain of definition $D(L)$ is the set of all functions $u \in L^2(0,T;B_2^1(0,1))$ for which ${}_0^R\partial_t^\sigma u$, $\partial u/\partial x$, $\partial^2 u/\partial x^2 \in L^2(0,T;B_2^1(0,1))$ and satisfying integral conditions (12).

Definition 6. The operator L from X into H has a closure $\overline{L}$. The solution of the operational equation

$$\overline{L}u = F$$

is called strong solution of problem (10)–(12).

Theorem 1. *Under assumptions* (5)–(6)*, the solution of problem* (10)–(12) *satisfies a priori estimate*

$$\|u\|_X \leqslant k\|Lu\|_H \quad \forall u \in D(L),$$

where k is a positive constant independent of u.

Proof. We take the scalar product in $B_2^1(0,1)$ of equation (10) and $u(x,t)$. Integrating over $(0,\tau)$, where $0 < \tau \leqslant T$, we obtain

$$\begin{aligned}
&\int_0^\tau \big\langle \mathcal{L}u, u(x,t) \big\rangle_{B_2^1(0,1)} \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = \int_0^\tau \int_0^1 \Im_x\big({}_0^R\partial_t^\alpha u(x,t)\big) \cdot \Im_x u(x,t) \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad - \int_0^\tau \int_0^1 \Im_x\left(\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\right) \cdot \Im_x u(x,t) \, \mathrm{d}x \, \mathrm{d}t \\
&\qquad + \int_0^\tau \int_0^1 \Im_x\big(b(x,t)u\big) \cdot \Im_x u(x,t) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad = \int_0^\tau \int_0^1 \big(\Im_x f(x,t)\big) \cdot \Im_x u(x,t) \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{16}$$

Integration by parts of equation (16) and using conditions (12) give

$$\int_0^\tau \int_0^1 \Im_x\big({}_0^R\partial_t^\alpha u(x,t)\big) \cdot \Im_x u(x,t) \, \mathrm{d}x \, \mathrm{d}t = \int_0^\tau \int_0^1 {}_0^R\partial_t^\alpha \Im_x \, u(x,t) \cdot \Im_x u(x,t) \, \mathrm{d}x \, \mathrm{d}t, \tag{17}$$

$$\begin{aligned}
&-\int_0^\tau\int_0^1 \Im_x\left(\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\right)\cdot \Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= -\int_0^\tau\int_0^1 \Im_x\left(\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\right)\cdot \frac{\partial}{\partial x}\Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= -\int_0^\tau\left[\Im_x\left(\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\right)\cdot \Im_x^2 u(x,t)\bigg|_0^1 - \int_0^1 \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\cdot \Im_x^2 u(x,t)\,\mathrm{d}x\right]\mathrm{d}t\\
&\quad= \int_0^\tau\int_0^1 \frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\cdot \Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_0^\tau\left[a(x,t)\frac{\partial u}{\partial x}\cdot \Im_x^2 u(x,t)\bigg|_0^1 - \int_0^1 a(x,t)\frac{\partial u}{\partial x}\cdot \Im_x u(x,t)\,\mathrm{d}x\right]\mathrm{d}t\\
&\quad= -\int_0^\tau\int_0^1 a(x,t)\frac{\partial u}{\partial x}\cdot \Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= -\int_0^\tau\left[a(x,t)u(x,t)\cdot \Im_x u(x,t)|_0^1 - \int_0^1 u(x,t)\frac{\partial}{\partial x}\big(a(x,t)\cdot \Im_x u(x,t)\big)\,\mathrm{d}x\right]\mathrm{d}t\\
&\quad= \int_0^\tau\int_0^1 u(x,t)\frac{\partial}{\partial x}\big(a(x,t)\cdot \Im_x u(x,t)\big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_0^\tau\int_0^1 u(x,t)a(x,t)\cdot u(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_0^1 u(x,t)\frac{\partial a}{\partial x}\cdot \Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t. \qquad (18)
\end{aligned}$$

We calculate the integral

$$\begin{aligned}
&\int_0^\tau\int_0^1 u(x,t)\frac{\partial a}{\partial x}\cdot \Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_0^\tau\int_0^1 \frac{\partial}{\partial x}\big(\Im_x u(x,t)\big)\frac{\partial a}{\partial x}\cdot \Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad= \int_0^\tau\left[\Im_x u(x,t)\frac{\partial a}{\partial x}\cdot \Im_x u(x,t)\bigg|_0^1 - \int_0^1 \Im_x u(x,t)\frac{\partial}{\partial x}\left(\frac{\partial a}{\partial x}\cdot \Im_x u(x,t)\right)\mathrm{d}x\right]\mathrm{d}t
\end{aligned}$$

$$= -\int_0^\tau \int_0^1 \Im_x u(x,t) \frac{\partial}{\partial x}\left(\frac{\partial a}{\partial x} \cdot \Im_x u(x,t)\right) \mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^\tau \int_0^1 \Im_x u(x,t) \frac{\partial a}{\partial x} \cdot \Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_0^1 \Im_x u(x,t) \frac{\partial a}{\partial x} \cdot u(x,t)\,\mathrm{d}x\,\mathrm{d}t,$$

then

$$2\int_0^\tau \int_0^1 u(x,t) \frac{\partial a}{\partial x} \cdot \Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^\tau \int_0^1 \frac{\partial^2 a(x,t)}{\partial x^2} \big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t.$$

We conclude

$$\int_0^\tau \int_0^1 u(x,t) \frac{\partial a}{\partial x} \cdot \Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\frac{1}{2}\int_0^\tau \int_0^1 \frac{\partial^2 a(x,t)}{\partial x^2} \big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t.$$

Finally, we obtain

$$-\int_0^\tau \int_0^1 \Im_x\left(\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\right) \cdot \Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^\tau \int_0^1 \frac{\partial a}{\partial x} \cdot \big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_0^\tau \int_0^1 \frac{\partial^2 a(x,t)}{\partial x^2} \big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t$$

and

$$\int_0^\tau \int_0^1 \Im_x\big(b(x,t)u\big) \cdot \Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^\tau \int_0^1 \Im_x\big(b(x,t)u\big) \cdot \frac{\partial}{\partial x}\Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^\tau \left[\Im_x\big(b(x,t)u\big) \cdot \Im_x^2 u(x,t)\big|_0^1 - \int_0^1 b(x,t)u \cdot \Im_x^2 u(x,t)\,\mathrm{d}x\right] \mathrm{d}t$$
$$= -\int_0^\tau \int_0^1 b(x,t)u \cdot \Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^\tau \int_0^1 b(x,t)\frac{\partial}{\partial x}\Im_x u \cdot \Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^\tau \left[b(x,t)\Im_x u \cdot \Im_x^2 u(x,t)\big|_0^1 - \int_0^1 \Im_x u \cdot \frac{\partial}{\partial x}\big(b(x,t)\Im_x^2 u(x,t)\big)\,\mathrm{d}x\right] \mathrm{d}t$$

$$
\begin{aligned}
&= \int_0^\tau \int_0^1 \Im_x u \cdot \frac{\partial}{\partial x}\bigl(b(x,t)\Im_x^2 u(x,t)\bigr)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^\tau \int_0^1 \Im_x u \cdot \frac{\partial b}{\partial x}\Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_0^1 \Im_x u(x,t) \cdot b(x,t)\Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t. \qquad (19)
\end{aligned}
$$

We calculate the integral

$$
\begin{aligned}
&\int_0^\tau \int_0^1 \Im_x u \cdot \frac{\partial b}{\partial x}\Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^\tau \int_0^1 \frac{\partial}{\partial x}\Im_x^2 u \cdot \frac{\partial b}{\partial x}\Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^\tau \left[\Im_x^2 u \cdot \frac{\partial b}{\partial x}\Im_x^2 u(x,t)\bigg|_0^1 - \int_0^1 \Im_x^2 u \cdot \frac{\partial}{\partial x}\left(\frac{\partial b}{\partial x}\Im_x^2 u(x,t)\right)\mathrm{d}x\right]\mathrm{d}t \\
&\quad = -\int_0^\tau \int_0^1 \Im_x^2 u \cdot \frac{\partial}{\partial x}\left(\frac{\partial b}{\partial x}\Im_x^2 u(x,t)\right)\mathrm{d}x\,\mathrm{d}t \\
&\quad = -\int_0^\tau \int_0^1 \Im_x^2 u \cdot \frac{\partial^2 b}{\partial x^2}\Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_0^1 \Im_x^2 u \cdot \frac{\partial b}{\partial x}\Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Then

$$
2\int_0^\tau \int_0^1 \Im_x u \cdot \frac{\partial b}{\partial x}\Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\int_0^\tau \int_0^1 \Im_x^2 u \cdot \frac{\partial^2 b}{\partial x^2}\Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t,
$$

and we can conclude that

$$
\int_0^\tau \int_0^1 \Im_x u \cdot \frac{\partial b}{\partial x}\Im_x^2 u(x,t)\,\mathrm{d}x\,\mathrm{d}t = -\frac{1}{2}\int_0^\tau \int_0^1 \frac{\partial^2 b}{\partial x^2}\bigl(\Im_x^2 u(x,t)\bigr)^2\,\mathrm{d}x\,\mathrm{d}t.
$$

Finally, we obtain

$$
\begin{aligned}
&\int_0^\tau \int_0^1 \Im_x\bigl(b(x,t)u\bigr) \cdot \Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^\tau \int_0^1 \Im_x u(x,t) \cdot b(x,t)\Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_0^\tau \int_0^1 \frac{\partial^2 b}{\partial x^2}\bigl(\Im_x^2 u(x,t)\bigr)^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Now, we can apply the Cauchy inequality with ε, (15), in the right-hand side of equation (16) as follows:

$$\begin{aligned}\int_0^\tau\int_0^1 \big(\Im_x f(x,t)\big)\cdot\big(\Im_x u(x,t)\big)\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant \frac{1}{2\varepsilon}\int_0^\tau\int_0^1 \big(\Im_x f(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{2}\int_0^\tau\int_0^1 \big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned} \tag{20}$$

and we replace equations (17)–(20) in equation (16), which yields

$$\begin{aligned}&\int_0^\tau\int_0^1 \big({}_0^R\partial_t^\alpha \Im_x u(x,t)\big)\cdot\Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_0^1 a(x,t)\big(u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^\tau\int_0^1 b(x,t)\big(\Im_x u(x,t)\big)^2 \\ &\leqslant \frac{1}{2\varepsilon}\int_0^\tau\int_0^1 \big(\Im_x f(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{2}\int_0^\tau\int_0^1 \big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{1}{2}\int_0^\tau\int_0^1 \frac{\partial^2 a(x,t)}{\partial x^2}\big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \frac{1}{2}\int_0^\tau\int_0^1 \frac{\partial^2 b(x,t)}{\partial x^2}\big(\Im_x^2 u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{21}$$

According to inequality (14) and assumptions (5)–(6), we can conclude that

$$\inf_{\overline{\Omega}} a(x,t)\int_0^\tau\int_0^1 \big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2}\int_0^\tau\int_0^1 a(x,t)\big(u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t$$

and

$$\begin{aligned}&\int_0^\tau\int_0^1 \frac{\partial^2 b(x,t)}{\partial x^2}\big(\Im_x^2 u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad \leqslant \frac{1}{2}\sup_{\overline{\Omega}}\frac{\partial^2 b(x,t)}{\partial x^2}\int_0^\tau\int_0^1 \big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{22}$$

If we combine equations (21)–(22), and taking into account assumptions (5)–(6), then we get

$$\begin{aligned}\int_0^\tau\int_0^1 \partial_t^\alpha\big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_0^1 \big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t \\ \leqslant \frac{1}{2\varepsilon\min(M,\frac{1}{2})}\int_0^\tau\int_0^1 \big(\Im_x f(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Finally, we obtain a priori estimate (1), where

$$\|u\|_X = \int_0^\tau\int_0^1 \big({}_0^R\partial_t^\alpha \Im_x u(x,t)\big)\cdot\Im_x u(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau\int_0^1 \big(\Im_x u(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t,$$

$$\|Lu\|_H = \int_0^\tau\int_0^1 \big(\Im_x f(x,t)\big)^2\,\mathrm{d}x\,\mathrm{d}t$$

with $k = 1/(2\varepsilon\min(M,1/2))$. □

Since $\overline{L}$ is the closure of L, we can extend inequality (1) as follows:

$$\|u\|_X \leqslant k\|\overline{L}u\|_H \quad \forall u \in D(\overline{L}) \tag{23}$$

Hence, inequality (23) leads to the following corollaries:

Corollary 2. *A strong solution of* (10)–(12) *is unique if it exists, and depends continuously on $F = (f)$.*

Corollary 3. *The range of $\overline{L}$ is closed in H and $R(\overline{L}) = \overline{R(L)}$.*

3.2 Existence of solutions

We proved the uniqueness of solution, if there is a solution. However, we have not demonstrated the existence yet. To do it, we will just prove that $R(L)$ is dense in H.

Theorem 2. *Let us suppose that the conditions of Theorem* 1 *and assumptions* (5)–(6) *are filled, and for $\omega \in L^2(0,\tau;B_2^1(0,1))$, where $0 < \tau \leqslant T$, we have*

$$(\mathcal{L}u,\omega)_{L^2(0,\tau;B_2^1(0,1))} = 0 \quad \forall u \in D(L). \tag{24}$$

Then ω vanishes almost everywhere in Ω.

Proof. Let us consider that there exists ω that satisfies (24). We express ω in terms of a function z as

$$\omega = \Im_t z(x,\tau).$$

We can rewrite equation (24) as follows:

$$\int_0^\tau \int_0^1 \Im_x\bigl({}_0^R\partial_t^\alpha u(x,t)\bigr) \cdot \Im_x\omega \,\mathrm{d}x\,\mathrm{d}t - \int_0^\tau \int_0^1 \Im_x\biggl(\frac{\partial}{\partial x}\biggl(a(x,t)\frac{\partial u}{\partial x}\biggr)\biggr) \cdot \Im_x\omega \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_0^1 \Im_x\bigl(b(x,t)u\bigr) \cdot \Im_x\omega \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{25}$$

Because (24) holds for any function u, it can expressed in a particular form. Assume that there is the function

$$u(x,t) = \omega(x,t) = \Im_t\bigl(z(x,\tau)\bigr) = \int_0^t z(x,\tau)\mathrm{d}\tau,$$

where $z(x,t)$ satisfies conditions(8), (9) such that ${}_0^R\partial_t^\sigma z$, $\partial z/\partial x$, $\partial^2 z/\partial x^2 \in L^2(0,T;$ $B_2^1(0,1))$. Replacing u in equation (25), we obtain

$$\int_0^\tau \int_0^1 \Im_x\bigl({}_0^R\partial_t^\alpha \Im_t z(x,\tau)\bigr) \cdot \Im_x\,\omega \,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_0^\tau \int_0^1 \Im_x\biggl(\frac{\partial}{\partial x}\biggl(a(x,t)\frac{\partial \Im_t z(x,\tau)}{\partial x}\biggr)\biggr) \cdot \Im_x\omega \,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau \int_0^1 \Im_x\bigl(b(x,t)\Im_t z(x,\tau)\bigr) \cdot \Im_x\omega \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{26}$$

Let us calculate all integrals in equation (26), it yields

$$\int_0^\tau \int_0^1 \Im_x\bigl({}_0^R\partial_t^\alpha \Im_t z(x,\tau)\bigr) \cdot \Im_x\,\Im_t z(x,\tau) \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^\tau \int_0^1 t_0^R\partial_t^\alpha\bigl(\Im_x\Im_t z(x,\tau)\bigr) \cdot \Im_x\Im_t z(x,\tau) \,\mathrm{d}x\,\mathrm{d}t,$$

$$- \int_0^\tau \int_0^1 \Im_x\biggl(\frac{\partial}{\partial x}\biggl(a(x,t)\frac{\partial \Im_t z(x,\tau)}{\partial x}\biggr)\biggr) \cdot \Im_x\,\Im_t z(x,\tau) \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^\tau \int_0^1 a(x,t)\bigl(\Im_t z(x,\tau)\bigr)^2 \,\mathrm{d}x\,\mathrm{d}t - \frac{1}{2}\int_0^\tau \int_0^1 \frac{\partial^2 a(x,t)}{\partial x^2}\bigl(\Im_x\Im_t z(x,\tau)\bigr)^2 \,\mathrm{d}x\,\mathrm{d}t,$$

and

$$\int_0^\tau \int_0^1 \Im_x \big(b(x,t)\Im_t z(x,\tau)\big) \cdot \Im_x \Im_t z(x,\tau)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^\tau \int_0^1 b(x,t)\big(\Im_x \Im_t z(x,\tau)\big)^2 - \frac{1}{2}\int_0^\tau \int_0^1 \frac{\partial^2 b(x,t)}{\partial x^2}\big(\Im_x^2 \Im_t z(x,\tau)\big)^2\,\mathrm{d}x\,\mathrm{d}t.$$

According to the previous identities (17), (18), and (19), we conclude that

$$\int_0^\tau \int_0^1 {}_0^R\partial_t^\alpha \big(\Im_x \Im_t z(x,\tau)\big) \cdot \Im_x \Im_t z(x,\tau) \leqslant -M \int_0^\tau \int_0^1 \big(\Im_x \Im_t z(x,\tau)\big)^2\,\mathrm{d}x\,\mathrm{d}t,$$

hence

$$\int_0^\tau \int_0^1 {}_0^R\partial_t^\alpha \big(\Im_x \Im_t z(x,\tau)\big) \cdot \Im_x \Im_t z(x,\tau)\,\mathrm{d}x\,\mathrm{d}t = 0,$$

therefore $z = 0$ in Ω, then $\omega = 0$ in Ω. □

4 Galerkin method

Many traditional results can be obtained for Galerkin or generalized Galerkin methods and similar problems to the one above (see [37]). In this paper, our goal is the numerical study of the spectral collocation method (also called the pseudo-spectral method). This collocation scheme can be considered as a generalized Galerkin scheme, and therefore the stability and convergence is similar to any of these methods.

As in [26, 30], we will work in the Hilbertian basis of the Cartesian product of polynomials on t and x (and y for the two-dimensional case, z in the three-dimensional case if it is necessary), i.e., we seek a solution in the form

$$u(x,t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} a'_{k,l} t^k x^l,$$

where the coefficients $a'_{kl}s$ are to be determined, or for two-dimensional problems,

$$u(x,y,t) = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} a'_{k,l,m} t^k x^l y^m,$$

and similarly, it can be done with three-dimensional problems. In this paper, we will study only the one-dimensional equation (2) subject to conditions (3) and (4). Obviously, the infinite series can be approximated at the Nth order by the dot product

$$u(x,t) = \sum_{k=0}^{N-1}\sum_{l=0}^{N-1} a_{k,l} t^k x^l,$$

(there might be two scalars N_1, N_2, but in this work, we will consider that $N = N_1 = N_2$).

These coefficients $a_{k,l}$ can be determined by solving a linear system or a least-squares linear problem $A \cdot x \approx b$ using equations (2)–(4) in a mesh. This is how we will proceed in the numerical section:

- Making a choice of the nodes $(x,t)_i$. When Ω is any general domain, equispaced points inside the domain are usually employed:

$$(x,t)_i = (x_{i_1}, t_{i_2}) = \left(\frac{i_1 - 1}{M-1}, \frac{i_2 - 1}{M-1}\right), \quad i_1, i_2 = 1, \dots, M. \tag{27}$$

  It is well known that interpolating polynomials based on equispaced grid may suffer the Runge phenomenon, or have some problems to approximate with a high accuracy oscillatory functions. However when Ω is an interval or a square, as it is the case, other sets of points can be employed, such as the Chebyshev nodes

$$x_{i_1}, t_{i_2} = \frac{1 + \cos\frac{(2i_j - 1)\pi}{2M-2}}{2}, \quad i_1, i_2 = 2, \dots, M-1,$$

  and $x_1 = t_1 =, x_M = t_M = 1$.

  In any case, it is important that $M \geqslant N$ to obtain accurate solutions. In this manuscript, equispaced and Chebyshev nodes were chosen (we will compare the results) with $M = N + 1$, and as we will show below, superalgebraical convergence is usually obtained, demonstrating the efficiency of the proposed schemes.

- Finding the values $a_{k,l}$ such that minimize the sum of the residuals

$$\begin{aligned}
\min\Biggl(&\sum_{(x,t)_i \in (0,1)\times(0,1]} \biggl\| {}_0^C\partial_t^\alpha w_N\bigl((x,t)_i\bigr) - \frac{\partial}{\partial x}\biggl(a\bigl((x,t)_i\bigr)\frac{\partial w_N}{\partial x}\bigg|_{(x,t)_i}\biggr) \\
&\qquad + b\bigl((x,t)_i\bigr)u - F\bigl((x,t)_i\bigr)\biggr\|_2 \\
&+ \sum_{(x,t)_i \in (0,1)\times(0,1]} \biggl\| \biggl(\int_0^1 w_M(x,t)\biggr)\bigg|_{(x,t)_i} - \mu\bigl(t_{i_2}\bigr)\biggr\|_2 \\
&+ \sum_{(x,t)_i \in (0,1)\times(0,1]} \biggl\| \biggl(\int_0^1 x w_M(x,t)\biggr)\bigg|_{(x,t)_i} - m\bigl(t_{i_2}\bigr)\biggr\|_2 \\
&+ \sum_{(x,t)_i \in (0,1)\times\{t=0\}} \bigl\| w_M\bigl((x,t)_i\bigr) - \phi(x_{i_1})\bigr\|_2 \Biggr)
\end{aligned}$$

  for $w_M(x,t) = t^k x^l$, $k, l = 0, \dots, M-1$.

The sums

$$\sum_{(x,t)_i\in(0,1)\times(0,1]}\left\|\left(\int\limits_0^1 xw_M(x,t)\right)\bigg|_{(x,t)_i}-\mu(t_{i_2})\right\|_2$$

and

$$\sum_{(x,t)_i\in(0,1)\times(0,1]}\left\|\left(\int\limits_0^1 xw_M(x,t)\right)\bigg|_{(x,t)_i}-m(t_{i_2})\right\|_2$$

have M terms each one (one term for each value t_{i_2}).

Matrices associated to collocation methods are usually ill-conditioned (see [38]), and therefore, in general, it is better to solve the least squares problem. To calculate the coefficients in matrix A, it is very important to understand the properties of the Caputo operator, and also, we need to calculate the Caputo fractional derivative of polynomials [20]

$${}_0^C\partial_t^\alpha t^p=\begin{cases}\frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}t^{p-\alpha} & \text{if } p>n-1,\\ 0 & \text{if } p\leqslant n-1\end{cases}$$

for $n-1<\alpha<n$.

5 Numerical examples

Example 1. Let us first consider problem

$$\begin{aligned}&{}_0^C\partial_t^{1/2}v(x,t)-\frac{\partial}{\partial x}\left(4xt\frac{\partial v}{\partial x}\right)-4tv(x,t)=F(x,t),\\ &\ell u=v(x,0)=0,\quad x\in(0,1),\end{aligned}$$

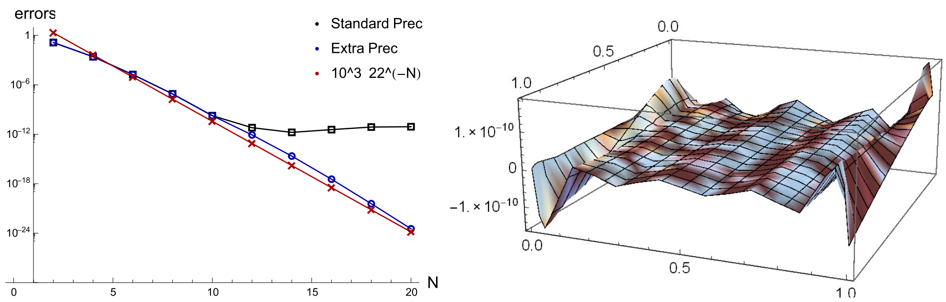
with the integral conditions

$$\int\limits_0^1 v(x,t)\,\mathrm{d}x=-\frac{1}{2}\sin\frac{\pi t}{2},\qquad \int\limits_0^1 xv(x,t)\,\mathrm{d}x=-\frac{1}{6}\sin\frac{\pi t}{2},\quad t\in(0,1),$$

and with $F(x,t)$ generated by the solution $v(x,t)=(x-1)\sin(\pi t/2)$. Without loss of generality, we consider that $T=1$ in our numerical experiments.

As reader can check, the functions in Examples 1 and 2 do not satisfy assumptions given by equations (5) and (6). However, solution exists, and it is unique in both numerical tests. This shows that these equations are sufficient to guarantee existence and uniqueness of the solution, but they are not necessary to guarantee solution.

We used equispaced nodes (Eq. (27)) for different N values and $M=N+1$ (there is an oversampling since the number of variables is N^3 and the number of equations is $(N+1)^3$).



(a) (b)

Figure 1. (a) Error in infinity norm for different values of N for equispaced nodes with standard and extra precision. (b) Three-dimensional plot of the errors is shown with $N = 10$, equispaced nodes and standard precision are used $(u(x_i, t_j) - u_N(x_i, t_j))$.

In Fig. 1, the errors are calculated with the infinity norm (maximum norm) at a uniform and dense grid. We used the `LeastSquares` solver (a direct method) of *Mathematica* to solve the least squares problem.

Standard and extra precision were considered, the figure shows how the method is very efficient for both, even for small values of $N \sim 10$. For larger values, the rounding-off errors appear with standard precision, and it is difficult to obtain errors under $O(10^{-10})$. However, when extra precision is considered, we check the expected super-algebraical convergence, errors decrease in a similar way as $c\rho^{-N}$ with $\rho \sim 22$.

Example 2. For the second example, we will consider mildly oscillatory functions to test the different choices of nodes and explain why extra precision might be required in some cases.

Now, let us consider

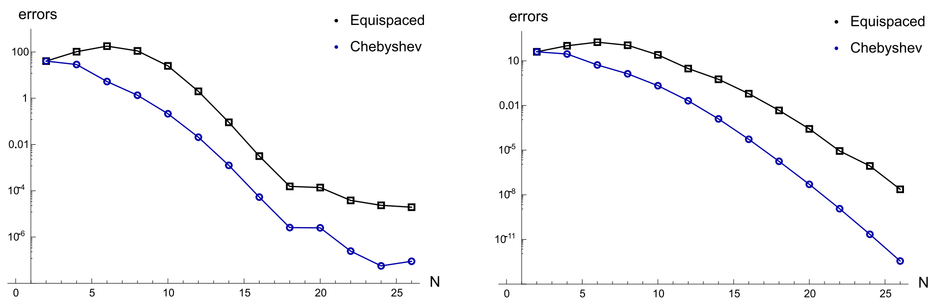
$$
\begin{aligned}
&{}_0^C\partial_t^{1/2} v(x,t) - \frac{\partial}{\partial x}\left(x\frac{\partial v}{\partial x}\right) - \cos\frac{\pi t}{2} v(x,t) = F(x,t),\\
&\ell u = v(x,0) = \sin\frac{24\pi x + \pi}{6}, \quad x \in (0,1),
\end{aligned}
$$

with the integral conditions

$$
\int\limits_0^1 v(x,t)\,\mathrm{d}x = 0, \qquad \int\limits_0^1 xv(x,t)\,\mathrm{d}x = \frac{-\sqrt{3}\cos\frac{\pi t}{2}\sin\frac{\pi t}{2}}{8\pi}, \quad t \in (0,1),
$$

and with $F(x,t)$ such that the solution $v(x,t) = \sin((24\pi x + 3\pi t + \pi)/6)$.

In Fig. 2, we show the errors obtained with equispaced and also Chebyshev nodes: first, with finite precision, and on the right-hand side of the figure, with extra precision. As explained in other articles (see [26], for example), it is essentially necessary that the nodes cluster quadratically near both endpoints (as is the case with Chebyshev nodes) [12] to avoid some difficulties with similar oscillatory functions or others that suffer the Runge phenomenon.



(a) (b)

Figure 2. The errors for different values of N for equispaced (squares) and Chebyshev nodes (circles) with normal precision (a) and extra precision (b).

In this numerical test, the Galerkin method obtained clearly more accurate results with Chebyshev nodes. With equispaced points and standard precision, the schemes have some problems and are not able to obtain errors under $O(10^{-5})$. If it is necessary to obtain smaller errors, then we can either use Chebyshev nodes or extra precision.

6 Conclusions

In the present paper, the existence and uniqueness of the solution of a fractional partial differential equation subject to purely integral conditions were demonstrated. Additionally, a Galerkin method based on least squares was proposed. The numerical examples illustrated the technique and showed the efficiency of the proposed method.

This manuscript also opens a procedure to demonstrate existence and uniqueness of similar FPDEs subject to nonlocal conditions. Thus, in the future, we want to obtain similar results for other fractional partial differential equations with analogous initial and boundary conditions.

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