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Distributions on the circle group

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Abstract. In this paper, we extend the definition of a random angle and the definition of a probability distribution of a random angle. We expand P. Lévy's researches related to wrapping the probability distributions defined on $\mathbb{R}$. We determine a relation between quasi-lattice probability distributions on $\mathbb{R}$ and lattice probability distributions on the unit circle $\mathbb{S}$. We use the Bergström identity for comparison of a convolution of probability distributions of random angles. We also prove an inverse formula for lattice probability distributions on $\mathbb{S}$.

Keywords: probability distributions on the unit circle, wrapped lattice and quasi-lattice distributions, Bergström identity on the unit circle.

1 Preface

Von Mises [17], Perrin [15] and Fisher [6] have made the start of the new trend in statistics. In present, it is called a statistical analysis of directional observations. In 1939, Lévy [11] published methods useful analyzing distributions on torus $\mathbb{T}$ using the results of probability theory distributions on the real line $\mathbb{R}$.

In the encyclopedic publication "Probability measures on locally compact groups", in 1977, Heyer has noticed that the theory of Gaussian distribution on the unit circle was developed by von Mises [17], Lévy [11] and it was widely discussed in Mardia's monograph [13], which was translated into Russian in 1978 [12]. Mardia's book is useful to solve statistical problems when probability distributions are on the unit circle. The developed theory is of great significance in various applications, for example, in spectroscopy, geodesy, navigation etc. Mardia book is a comprehensive monograph useful to practitioners and for all whose researches are related to probability distributions on the unit circle. Later, in 1999, Mardia and Jupp published a monograph [14] in which one can find more well-founded statements about directional statistics. In 2013, Pewsey, Neuhauser, Ruxton [16] published a book useful in working with software environment R.

In the translation of Mardia book [12], editor L.N. Bolshev submitted notes about definitions of random variables and their distributions. In this paper, definitions that did not cause discussion questions are used. We also consider the subjects related to probability distributions on locally compact Abelian (LCA) groups [4, 7, 18].

In the articles of statisticians and probability theory specialists, distributions on the unit circle

$$\mathbb{S} = \{(\cos\theta, \sin\theta)\colon\ 0 \leqslant \theta < 2\pi\},$$

the centre of which is at the origin, are constructed not only directly on the unit circle, but also using the following methods: wrapping, offsetting, characterizing and stereographic projection [1, 6, 9]. The methods are based on probability distributions on the real line $\mathbb{R}$ or on the space $\mathbb{R}^2$. In this paper, we do not discuss the offsetting method. The main results are related to the wrapped probability distributions and probability distributions constructed directly on $\mathbb{S}$.

2 Preliminaries

Suppose, $\{\Omega, \mathcal{F}\}$ is a measurable space.

Definition 1. We say $\Theta = \Theta(\omega)$ is a random angle given on measurable space $\{\Omega, \mathcal{F}\}$ if for every Borel set $B \in \mathcal{B}(\mathbb{S})$,

$$\{\omega\colon\ \Theta(\omega) \in B\} \in \mathcal{F}.$$

Thus, the random angle Θ generates measurable space $\{\mathbb{S}, \mathcal{B}(\mathbb{S})\}$, where $\mathbb{S}$ is the unit circle, and $\mathcal{B}(\mathbb{S})$ is σ-algebra of Borel sets generated by $\mathbb{S}$. Suppose, $\mathbf{P}$ denotes a probability on $\{\Omega, \mathcal{F}\}$.

Definition 2. We call the function

$$P_\Theta(B) = \mathbf{P}\{\omega\colon\ \Theta(\omega) \in B\}$$

of all Borel sets $B \in \mathcal{B}(\mathbb{S})$ a probability distribution of random angle Θ given on the space $\{\mathbb{S}, \mathcal{B}(\mathbb{S})\}$.

Definition 3. The function

$$F_\Theta(\theta) = \mathbf{P}\{\omega\colon\ 0 < \Theta(\omega) \leqslant \theta\}, \quad \theta \in [0, 2\pi),$$

which satisfies the equality

$$F_\Theta(\theta + 2\pi) - F_\Theta(\theta) = 1, \quad -\infty < \theta < \infty,$$

is a distribution function of a random angle Θ.

One can find the definition of a probability function of a random angle and its properties in [13, 14]

For $-\infty < \alpha < \beta < \infty$ and $\beta - \alpha < 2\pi$,

$$P_\Theta([\alpha; \beta)) = F_\Theta(\beta) - F_\Theta(\alpha) = \int_\alpha^\beta \mathrm{d}F_\Theta(x),$$

where the integral is a Lebesgue–Stieltjes integral.

The main results of the paper are related to distributions of a lattice random angle. Before introducing its definition, we remind the definition of lattice and quasi-lattice random variables.

Definition 4. A random variable ξ is lattice if it takes values in

$$L_{a,h} = \{a + h\nu\colon\ \nu = 0, \pm1, \pm2, \dots\},$$

i.e.

$$\sum_{\nu=-\infty}^{\infty} \mathbf{P}\{\xi = a + h\nu\} = 1,$$

where $a \in \mathbb{R}$ and $h > 0$.

Esseen [5] was the first who widely used lattice random variables in the theory of sums of independent random variables. He proves an inverse formula

$$\mathbf{P}\{\xi = a + h\nu\} = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} \mathrm{e}^{-\mathrm{i}t(a+h\nu)} \mathbf{E}\mathrm{e}^{\mathrm{i}t\xi}\, \mathrm{d}t, \tag{1}$$

where

$$\mathbf{E}\mathrm{e}^{\mathrm{i}t\xi} = \sum_{\nu=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t(a+h\nu)} \mathbf{P}\{\xi = a + h\nu\}.$$

In [2], a quasi-lattice random variable is defined. One can also find an inverse formula of the same type as (1) for the quasi-lattice random variable in [2].

Definition 5. A random variable η is quasi-lattice if it takes its values in

$$L_{\beta_1,\beta_2} = \{\beta_1\nu_1 + \beta_2\nu_2\colon\ \nu_1, \nu_2 = 0, \pm1, \pm2, \dots\}$$

with probabilities

$$\sum_{\nu_1=-\infty}^{\infty} \sum_{\nu_2=-\infty}^{\infty} \mathbf{P}\{\eta = \beta_1\nu_1 + \beta_2\nu_2\} = 1,$$

where $\beta_1, \beta_2 > 0$ are rationally independent, i.e.

$$\beta_1\nu_1 + \beta_2\nu_2 = 0$$

if and only if $\nu_1 = \nu_2 = 0$.

If η is quasi-lattice, then its characteristic function is

$$\mathbf{E}\mathrm{e}^{\mathrm{i}t\eta} = \sum_{\nu_1=-\infty}^{\infty} \sum_{\nu_2=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t(\beta_1\nu_1+\beta_2\nu_2)} \mathbf{P}\{\eta = \beta_1\nu_1 + \beta_2\nu_2\}.$$

In [2], the inverse formula

$$\mathbf{P}\{\eta = \beta_1\nu_1 + \beta_2\nu_2\} = \frac{\beta_1\beta_2}{(2\pi)^2} \int_{-\pi/\beta_1}^{\pi/\beta_1} \int_{-\pi/\beta_2}^{\pi/\beta_2} \mathrm{e}^{-\mathrm{i}(t_1\beta_1\nu_1 + t_2\beta_2\nu_2)} \mathbf{E}\mathrm{e}^{\mathrm{i}(\gamma_1 t_1 + \gamma_2 t_2)}\, \mathrm{d}t_1\, \mathrm{d}t_2 \tag{2}$$

obtained, where (γ_1, γ_2) is a two-dimensional lattice random vector, and its characteristic function is

$$\mathbf{E}\mathrm{e}^{\mathrm{i}t_1\gamma_1 + \mathrm{i}t_2\gamma_2} = \sum_{\nu_1=-\infty}^{\infty} \sum_{\nu_2=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t_1\beta_1\nu_1 + \mathrm{i}t_2\beta_2\nu_2} \mathbf{P}\{\gamma_1 = \beta_1\nu_1,\, \gamma_2 = \beta_2\nu_2\}.$$

The random vector (γ_1, γ_2) is related to the random variable η:

$$\mathbf{P}\{\gamma_1 = \beta_1\nu_1,\, \gamma_2 = \beta_2\nu_2\} = \mathbf{P}\{\eta = \beta_1\nu_1 + \beta_2\nu_2\}.$$

One can find the definition of distributions of lattice random angles in Mardia book [13] on page 54.

Definition 6. The distribution of a random angle Θ is called lattice if for some $l \geqslant 1$,

$$\sum_{r=0}^{l-1} \mathbf{P}\left\{\Theta = \frac{2\pi r}{l} \pmod{2\pi}\right\} = 1.$$

We write $\mathbf{P}\{\Theta = 2\pi r/l\}$ instead of $\mathbf{P}\{\Theta = 2\pi r/l \pmod{2\pi}\}$.

It follows from the definition above that a characteristic function of a lattice random angle Θ is

$$\mathbf{E}\mathrm{e}^{ip\Theta} = \sum_{r=0}^{l-1} \mathrm{e}^{i\frac{2\pi}{l}rp} \mathbf{P}\left\{\Theta = \frac{2\pi}{l}r\right\}, \quad p = 0, \pm 1, \pm 2, \dots . \tag{3}$$

For instance, Mardia [13, p. 54] calls the random angle Θ as Poisson if it takes the values

$$\Theta := \frac{2\pi}{l}r, \quad r = 0, 1, \dots, l-1,$$

where l is integer and $l \geqslant 1$, with probabilities

$$\mathbf{P}\left\{\Theta = \frac{2\pi}{l}r\right\} = \sum_{\rho=0}^{\infty} \frac{\lambda^{(r+l\rho)}}{(r+l\rho)!}\mathrm{e}^{-\lambda}, \quad \lambda > 0.$$

Let ξ be a random variable defined on the real line $\mathbb{R}$, and let $F_\xi(x)$, $x \in \mathbb{R}$, be its probability function.

Definition 7. We say that the distribution function $F_\Theta(\theta)$ of a random angle

$$\Theta_w = \xi \pmod{2\pi}$$

is wrapped if its distribution function is

$$F_w(\theta) = \sum_{k=-\infty}^{\infty} \big[F_\xi(\theta + 2k\pi) - F_\xi(2k\pi)\big], \tag{4}$$

where $0 \leqslant \theta < 2\pi$ (see [14, pp. 47–48].

Now we present some properties of wrapping that can be found in [13, 14].

(i) Wrapping is a homomorphism from $\mathbb{R}$ to $\mathbb{S}$:

$$(x + y)_w = x_w + y_w,$$

operation $+$ in the right-hand side of the equality denotes addition modulo 2π.

(ii) If ξ is a random variable and $\Theta_w = \xi \pmod{2\pi}$ is a random angle, then their characteristic functions are related by

$$\mathbf{E}\mathrm{e}^{\mathrm{i}p\Theta_w} = \mathbf{E}\mathrm{e}^{\mathrm{i}t\xi}\big|_{t=p}, \quad p = 0, \pm1, \pm2, \dots .$$

(iii) If a random variable ξ is infinitely divisible, then a random angle $\Theta_w = \xi \pmod{2\pi}$ is also infinitely divisible.

One can also wrap the probabilities of a random variable, which takes integer values. If ξ is a random variable, which takes values $m = 0, \pm1, \pm2, \dots$ with probabilities $\mathbf{P}\{\xi = m\}$, then the random angle $\Theta_w = 2\pi\xi/l \pmod{2\pi}$ takes its values in the lattice

$$\left\{\frac{2\pi}{l}r\colon r = 0, 1, \dots, l-1\right\}, \quad l \geqslant 1,$$

with probabilities

$$\mathbf{P}\left\{\Theta_w = \frac{2\pi}{l}r\right\} = \sum_{\rho=-\infty}^{\infty} \mathbf{P}\{\xi = r + l\rho\}.$$

Lévy [11] obtained the definition of Poisson distribution on $\mathbb{S}$ by wrapping the Poisson distribution

$$\mathbf{P}\{\xi = m\} = \frac{\lambda^m \mathrm{e}^{-\lambda}}{m!}, \quad m = 0, 1, \dots;\ \lambda > 0,$$

on the circumference of the unit circle $\mathbb{S}$ with the centre at the origin. One can find a wrapped t-distribution in [10], a wrapped classic exponential distribution and the Laplace probability distribution in [8]. Thus, the wrapping relates the probability distributions on $\mathbb{S}$ to probability distributions on $\mathbb{R}$.

The main results of this paper are related to the wrapped lattice and quasi-lattice probability distributions.

3 Main results

Suppose, we have a random variable ξ and a random angle $\Theta_w = \xi \pmod{2\pi}$ given on probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$. First of all, we are going to make an important notice, which enables us to write the wrapped distribution function in a useful form. So, we can use the set

$$B(\theta) = \bigcup_{k=-\infty}^{\infty} [2k\pi,\, \theta + 2k\pi), \quad 0 \leqslant \theta < 2\pi, \tag{5}$$

in equality (4) and to write the wrapped distribution function as follows:

$$F_w(\theta) = \sum_{k=-\infty}^{\infty} \mathbf{P}\bigl\{\xi \in [2k\pi,\, \theta + 2k\pi)\bigr\} = \mathbf{P}\bigl\{\xi \in B(\theta)\bigr\} \tag{6}$$

for all $\theta \in [0; 2\pi)$ if

$$\sum_{k=-\infty}^{\infty} \mathbf{P}\{\xi = 2k\pi\} = 0.$$

Remark 1. In equality (5), the defined Borel set $B(\theta)$ is a union of disjoint intervals $[2k\pi,\, \theta + 2k\pi)$, $k \in \mathbb{Z}$, as $0 \leqslant \theta < 2\pi$.

Thus, it follows from equality (6) that

$$\mathbf{P}\{0 < \Theta_w \leqslant \theta\} = \mathbf{P}\bigl\{\xi \in B(\theta)\bigr\}, \quad \theta \in [0; 2\pi),$$

if

$$\sum_{k=-\infty}^{\infty} \mathbf{P}\{\xi = 2k\pi\} = 0.$$

A natural question arises whether we get the probability distribution of a lattice random angle by wrapping the probability distribution of a quasi-lattice random variable. A hypothesis would be that we obtain the Haar probability distribution.

Theorem 1. *Assume that ξ is a random variable taking values in*

$$\{\beta_1\nu_1 + \beta_2\nu_2\colon\ \nu_1, \nu_2 = 0, \pm 1, \pm 2, \dots\},$$

where (β_1, β_2) is the integer basis. After wrapping the probability function of the random variable ξ on the unit circle $\mathbb{S}$, we obtain the probability distribution of random angle Θ_ξ, which takes its values with probabilities

$$\mathbf{P}\left\{\Theta_\xi = \frac{2\pi}{l} r\right\} = \sum_{\rho=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = \beta_1(r + l\rho) + \beta_2\nu\bigr\} \tag{7}$$

for all $r = 0, 1, \dots, l-1$ and any integer $l \geqslant 1$.

We use formula (7) to obtain the Bergström identity on the unit circle $\mathbb{S}$.

Proof of Theorem 1. It is obvious that

$$\sum_{r=0}^{l-1}\sum_{\rho=-\infty}^{\infty}\sum_{\nu=-\infty}^{\infty}\mathbf{P}\bigl\{\xi=\beta_1(r+l\rho)+\beta_2\nu\bigr\}=1.$$

Let us refer to the characteristic function of a quasi-lattice random variable on $\mathbb{R}$, i.e. the function

$$\mathbf{E}\mathrm{e}^{\mathrm{i}t\xi}=\sum_{\nu_1=-\infty}^{\infty}\sum_{\nu_2=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}t(\beta_1\nu_1+\beta_2\nu_2)}\mathbf{P}\{\xi=\beta_1\nu_1+\beta_2\nu_2\}.$$

We can rewrite it in another form using equality (12). Let

$$\nu_1=r_1+l\rho_1 \quad \text{and} \quad \nu_2=r_2+l\rho_2,$$

where $r_1, r_2=0,1,\dots,l-1$, $l\geqslant 1$, $\rho_1,\rho_2=0,\pm1,\pm2,\dots$. Then

$$\begin{aligned}&\sum_{\nu_1=-\infty}^{\infty}\sum_{\nu_2=-\infty}^{\infty}\{\beta_1\nu_1+\beta_2\nu_2\}\\&\quad=\sum_{r_1=0}^{l-1}\sum_{\rho_1=-\infty}^{\infty}\sum_{r_2=0}^{l-1}\sum_{\rho_2=-\infty}^{\infty}\bigl\{\beta_1(r_1+l\rho_1)+\beta_2(r_2+l\rho_2)\bigr\}.\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{E}\mathrm{e}^{\mathrm{i}t\xi}&=\sum_{r_1=0}^{l-1}\sum_{\rho_1=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}t\beta_1(r_1+\rho_1 l)}\\&\quad\times\sum_{r_2=0}^{l-1}\sum_{\rho_2=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}t\beta_2(r_2+\rho_2 l)}\mathbf{P}\bigl\{\xi=\beta_1(r_1+l\rho_1)+\beta_2(r_2+l\rho_2)\bigr\}\end{aligned}$$

for all $t\in\mathbb{R}$.

Let us take

$$t=\frac{2\pi}{l\beta_2}t', \quad t'\in\mathbb{R}.$$

Then

$$\begin{aligned}\mathbf{E}\mathrm{e}^{\mathrm{i}\frac{2\pi}{l\beta_2}t'\xi}&=\sum_{r_1=0}^{l-1}\sum_{\rho_1=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{2\pi}{l\beta_2}t'\beta_1(r_1+\rho_1 l)}\sum_{r_2=0}^{l-1}\mathrm{e}^{\mathrm{i}\frac{2\pi}{l\beta_2}t'r_2\beta_2}\\&\quad\times\sum_{\rho_2=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{2\pi}{l\beta_2}t'\rho_2 l\beta_2}\mathbf{P}\bigl\{\xi=\beta_1(r_1+l\rho_1)+\beta_2(r_2+l\rho_2)\bigr\}.\end{aligned}$$

Let us substitute $p = 0, \pm 1, \pm 2, \ldots$ for t' in the previous equality. Therefore

$$\begin{aligned}\mathbf{E}\mathrm{e}^{\mathrm{i}\frac{2\pi}{l\beta_2}p\xi} = {}& \sum_{r_1=0}^{l-1} \mathrm{e}^{\mathrm{i}\frac{2\pi}{l}\frac{\beta_1}{\beta_2}r_1 p} \sum_{\rho_1=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi\frac{\beta_1}{\beta_2}\rho_1 p} \sum_{r_2=0}^{l-1} \mathrm{e}^{\mathrm{i}\frac{2\pi}{l}pr_2} \\ & \times \sum_{\rho_2=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = \beta_1(r_1 + l\rho_1) + \beta_2(r_2 + l\rho_2)\bigr\}. \qquad (8)\end{aligned}$$

Note that β_1 and β_2 are rationally independent, i.e.

$$\beta_1\nu_1 + \beta_2\nu_2 = 0$$

if all $\nu_1 = \nu_2 = 0$. Consequently, β_1/β_2 is not a rational number and

$$\mathrm{e}^{\mathrm{i}2\pi\frac{\beta_1}{\beta_2}\rho_1 p} \neq 1$$

as ρ_1 or $p \neq 0$.

Let us substitute $t \in \mathbb{R}$ for p in equality (8). We obtain

$$\begin{aligned}\mathbf{E}\mathrm{e}^{\mathrm{i}\frac{2\pi}{l\beta_2}t\xi} = {}& \sum_{r_1=0}^{l-1} \mathrm{e}^{\mathrm{i}\frac{2\pi}{l}\frac{\beta_1}{\beta_2}r_1 t} \sum_{\rho_1=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi\frac{\beta_1}{\beta_2}\rho_1 t} \sum_{r_2=0}^{l-1} \mathrm{e}^{\mathrm{i}\frac{2\pi}{l}tr_2} \\ & \times \sum_{\rho_2=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = \beta_1(r_1 + l\rho_1) + \beta_2(r_2 + l\rho_2)\bigr\}. \qquad (9)\end{aligned}$$

In the previous equality, the number series absolutely converges for all $t \in \mathbb{R}$. Consequently, we take

$$t = lt', \quad t' \in \mathbb{R},$$

in equality (9). Then we substitute $p = 0, \pm 1, \pm 2, \ldots$ for t'. Hence

$$\begin{aligned}\mathbf{E}\mathrm{e}^{\mathrm{i}\frac{2\pi}{l\beta_2}lp\xi} &= \mathbf{E}\mathrm{e}^{\mathrm{i}\frac{2\pi}{\beta_2}p\xi} \\ &= \sum_{r_1=0}^{l-1} \mathrm{e}^{\mathrm{i}\frac{2\pi}{l}\frac{\beta_1}{\beta_2}r_1 lp} \sum_{\rho_1=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi\frac{\beta_1}{\beta_2}\rho_1 lp} \\ &\quad \times \sum_{r_2=0}^{l-1} \sum_{\rho_2=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = \beta_1(r_1 + l\rho_1) + \beta_2(r_2 + l\rho_2)\bigr\}. \qquad (10)\end{aligned}$$

Once again we substitute $t \in \mathbb{R}$ for p in equality (10) and obtain

$$\begin{aligned}\mathbf{E}\mathrm{e}^{\mathrm{i}\frac{2\pi}{\beta_2}t\xi} = {}& \sum_{r_1=0}^{l-1} \mathrm{e}^{\mathrm{i}2\pi\frac{\beta_1}{\beta_2}r_1 t} \sum_{\rho_1=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}2\pi\frac{\beta_1}{\beta_2}\rho_1 lt} \\ & \times \sum_{\nu_2=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = \beta_1(r_1 + l\rho_1) + \beta_2\nu_2\bigr\}. \qquad (11)\end{aligned}$$

In equality (11), it is useful to choose

$$t = \frac{\beta_2 t'}{\beta_1 l}, \quad t' \in \mathbb{R},$$

and then to substitute $p = 0, \pm 1, \pm 2, \ldots$ for t'. We get

$$\mathbf{E}\mathrm{e}^{\mathrm{i}\frac{2\pi}{\beta_1 l}p\xi} = \sum_{r_1=0}^{l-1} \mathrm{e}^{\mathrm{i}\frac{2\pi}{l}pr_1} \sum_{\rho_1=-\infty}^{\infty} \sum_{\nu_2=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = \beta_1(r_1 + l\rho_1) + \beta_2\nu_2\bigr\}.$$

On the right-hand side of the previous equality, there is the characteristic function of random angle $\Theta_\xi = \xi \pmod{2\pi}$, and

$$\mathbf{P}\biggl\{\Theta_\xi = \frac{2\pi}{l}r\biggr\} = \sum_{\rho=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = \beta_1(r + l\rho) + \beta_2\nu\bigr\}.$$

Now we must verify the conditions of the definition of a lattice random angle on $\mathbb{S}$. Hence

(i) It is obvious that

$$\sum_{\rho=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = \beta_1(r + l\rho) + \beta_2\nu\bigr\} \geqslant 0;$$

(ii) It is true that

$$\sum_{r=0}^{l-1} \sum_{\rho=-\infty}^{\infty} \sum_{\nu_2=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = \beta_1(r + l\rho) + \beta_2\nu_2\bigr\} = \sum_{\nu_1=-\infty}^{\infty} \sum_{\nu_2=-\infty}^{\infty} \mathbf{P}\{\xi = \beta_1\nu_1 + \beta_2\nu_2\} = 1.$$

This equality follows from the fact that ξ is a quasi-lattice random variable.

The proof of Theorem 1 is complete. □

Remark 2. If ξ is quasi-lattice random variable taking values in $\{\beta_1\nu_1 + \beta_2\nu_2\colon \nu_1, \nu_2 = 0, \pm 1, \pm 2, \ldots\}$, where (β_1, β_2) is the integer basis, then the values of random angle $\Theta_\xi = \xi \pmod{2\pi}$ do not depend on the parameters β_1 and β_2.

Remark 3. In the proof of Theorem 1, one can find a method how to wrap the probability distribution on $\mathbb{S}$.

Corollary 1. *Suppose that a random variable ξ takes values $h\nu$, $\nu = 0, \pm 1, \pm 2, \ldots,$ $h > 0$, and $\Theta_w = 2\pi\xi/lh \pmod{2\pi}$, $l \geqslant 1$ is integer, $h > 0$. For any $h > 0$ and any integer $l \geqslant 1$, the following equality is true:*

$$\mathbf{P}\biggl\{\Theta_w = \frac{2\pi}{l}r\biggr\} = \sum_{\rho=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = h(r + \rho l)\bigr\}, \quad r = 0, 1, \ldots, l-1.$$

The statement of Corollary 1 is easy to see, but in proving it, the point is to demonstrate a method how to wrap the probability distribution of a random variable on the circumference of the unit circle.

Proof of Corollary 1. From the definition of the random variable ξ it follows that

$$\mathbf{E}\mathrm{e}^{\mathrm{i}t\xi} = \sum_{\nu=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}th\nu}\mathbf{P}\{\xi = h\nu\}.$$

It is easy to see that

$$\bigcup_{\nu=-\infty}^{\infty} \{\nu\} = \bigcup_{r=0}^{l-1} \bigcup_{\rho=-\infty}^{\infty} \{r+\rho l\}, \tag{12}$$

where l is the integer and $l \geqslant 1$. Thus, the characteristic function of the random variable ξ is

$$\mathbf{E}\mathrm{e}^{\mathrm{i}t\xi} = \sum_{r=0}^{l-1} \mathrm{e}^{\mathrm{i}trh} \sum_{\rho=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}t\rho hl}\mathbf{P}\bigl\{\xi = h(r+\rho l)\bigr\}, \quad t \in \mathbb{R}. \tag{13}$$

In formula (13), let us choose

$$t = \frac{2\pi}{lh}t', \quad t' \in \mathbb{R},$$

and after this, let us substitute $p = 0, \pm 1, \pm 2, \ldots$ for t'. Consequently,

$$\mathbf{E}\mathrm{e}^{\mathrm{i}\frac{2\pi}{lh}p\xi} = \sum_{r=0}^{l-1} \mathrm{e}^{\mathrm{i}\frac{2\pi}{l}rp} \sum_{\rho=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = h(r+\rho l)\bigr\}$$

for all $p = 0, \pm 1, \pm 2, \ldots$.

According to (3), on the right-hand side of the previous equality we have the characteristic function of the random angle $\Theta_w = \xi \pmod{2\pi}$, which takes values $2\pi r/l$ with probabilities

$$\mathbf{P}\biggl\{\Theta_w = \frac{2\pi}{l}r\biggr\} = \sum_{\rho=-\infty}^{\infty} \mathbf{P}\bigl\{\xi = h(r+\rho l)\bigr\}$$

for all $r = 0, 1, \ldots, l-1$ and any integer $l \geqslant 1$.

The proof of Corollary 1 is complete. □

Corollary 2. *The values of lattice random angle* $\Theta_w = \xi \pmod{2\pi}$ *do not depend on the span* h *of the distribution of lattice random variable* ξ.

Let us take probability distributions of two quasi-lattice random angles ξ and η defined in the integer base

$$x_{\nu_1,\nu_2} = \beta_1\nu_1 + \beta_2\nu_2, \quad \beta_1, \beta_2 > 0,\ \nu_1, \nu_2 \in \mathbb{Z},$$

where β_1, β_2 are rationally independent.

By using formula (7) we derive

$$\mathbf{P}\left\{\Theta_\xi = \frac{2\pi}{l}r\right\} = \sum_{\rho\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}}\mathbf{P}\bigl\{\xi = \beta_1(r+l\rho)+\beta_2\nu\bigr\},$$
$$\mathbf{P}\left\{\Theta_\eta = \frac{2\pi}{l}r\right\} = \sum_{\rho\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}}\mathbf{P}\bigl\{\eta = \beta_1(r+l\rho)+\beta_2\nu\bigr\}.$$

Note that ξ and η are defined in the same base $(\beta_1;\beta_2)$.

For comparison of distributions of random angles Θ_ξ and Θ_η we can use the equality

$$\mathbf{P}\left\{\Theta_\xi = \frac{2\pi}{l}r\right\} - \mathbf{P}\left\{\Theta_\eta = \frac{2\pi}{l}r\right\} = \sum_{\rho\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}}\bigl[\mathbf{P}\bigl\{\xi = \beta_1(r+l\rho)+\beta_2\nu\bigr\} - \mathbf{P}\bigl\{\eta = \beta_1(r+l\rho)+\beta_2\nu\bigr\}\bigr]$$

for all $r = 0,1,\dots,l-1$ and for the chosen integer $l \geqslant 1$.

The Bergström identity on $\mathbb{S}$ has the following form.

Theorem 2. *Assume that $S_n = \xi_1+\xi_2+\dots+\xi_n$ and $Z_n = \eta_1+\eta_2+\dots+\eta_n$, where $\xi_1,\xi_2,\dots,\xi_n$ and $\eta_1,\eta_2,\dots,\eta_n$ are independent and identically distributed random variables on the real line $\mathbb{R}$. Then for all $r = 0,1,\dots,l-1$ and for any integer $l \geqslant 1$, it is true that*

$$\begin{aligned}\mathbf{P}\left\{\Theta_{S_n} = \frac{2\pi}{l}r\right\} - \mathbf{P}\left\{\Theta_{Z_n} = \frac{2\pi}{l}r\right\} &= \sum_{j=0}^{s-1}\left(\frac{1}{n}\right)^j \sum_{\nu=j+1}^{s}\frac{(-1)^j}{\nu!}C_\nu^{(j)}P_{\eta_1}^{*(n-\nu)} * \bigl(n(P_{\xi_1}-P_{\eta_1})\bigr)^{*\nu}\bigl(B_l(r)\bigr)\\ &\quad + \binom{n}{s+1}(P_{\xi_1}-P_{\eta_1})^{*(s+1)} * \mathbf{E}\bigl(P_{\xi_1}^{*(n-\lambda)} * P_{\eta_1}^{*(\lambda-s-1)}\bigr)\bigl(B_l(r)\bigr),\end{aligned}$$

where

$$B_l(r) = \bigcup_{\rho\in\mathbb{Z}}\bigcup_{\nu\in\mathbb{Z}}\bigl\{\beta_1(r+l\rho)+\beta_2\nu\bigr\}.$$

Proof. From Theorem 1 it follows that

$$\mathbf{P}\left\{\Theta_{S_n} = \frac{2\pi}{l}r\right\} - \mathbf{P}\left\{\Theta_{Z_n} = \frac{2\pi}{l}r\right\} = \sum_{\rho\in\mathbb{Z}}\sum_{\nu\in\mathbb{Z}}\bigl[\mathbf{P}\{S_n = \beta_1(r+l\rho)+\beta_2\nu\} - \mathbf{P}\{Z_n = \beta_1(r+l\rho)+\beta_2\nu\}\bigr].$$

Let us take the difference

$$\begin{aligned}&\mathbf{P}\bigl\{S_n = \beta_1(r+l\rho)+\beta_2\nu\bigr\} - \mathbf{P}\bigl\{Z_n = \beta_1(r+l\rho)+\beta_2\nu\bigr\}\\ &\quad = \mathbf{P}\bigl\{S_n \in \bigl\{\beta_1(r+l\rho)+\beta_2\nu\bigr\}\bigr\} - \mathbf{P}\bigl\{Z_n \in \bigl\{\beta_1(r+l\rho)+\beta_2\nu\bigr\}\bigr\}\end{aligned}$$

for all $r = 0,1,\dots,l-1$, $\rho\in\mathbb{Z}$ and $\nu\in\mathbb{Z}$.

Since

$$\bigcup_{\rho\in\mathbb{Z}}\bigcup_{\nu\in\mathbb{Z}}\bigl\{\beta_1(r+l\rho)+\beta_2\nu\bigr\}=B_l(r),\quad r=0,1,\dots,l-1,$$

and

$$B_l(r_1)\cap B_l(r_2)=\emptyset\quad\text{if } r_1\neq r_2,$$

we obtain

$$\bigcup_{r=0}^{l-1}\bigcup_{\rho\in\mathbb{Z}}\bigcup_{\nu\in\mathbb{Z}}\bigl\{\beta_1(r+l\rho)+\beta_2\nu\bigr\}=\mathbb{R}=(-\infty;\infty).$$

Accordingly,

$$\begin{aligned}&\mathbf{P}\biggl\{\Theta_{S_n}=\frac{2\pi}{l}r\biggr\}-\mathbf{P}\biggl\{\Theta_{Z_n}=\frac{2\pi}{l}r\biggr\}\\&\quad=\mathbf{P}\bigl\{S_n\in B_l(r)\bigr\}-\mathbf{P}\bigl\{Z_n\in B_l(r)\bigr\}=P_\xi^{*n}\bigl(B_l(r)\bigr)-P_\eta^{*n}\bigl(B_l(r)\bigr)\end{aligned}$$

for all $r=0,1,\dots,l-1$, where

$$P_\xi(A)=\mathbf{P}\{\xi\in A\},\qquad P_\eta(A)=\mathbf{P}\{\eta\in A\}$$

for all $A\in\mathcal{B}(\mathbb{R})$.

We use the modified Bergström identity (see [3]) to complete the proof of the theorem.

The proof of Theorem 2 is complete. □

Let us go back to lattice probability distributions constructed directly on the unit circle $\mathbb{S}$. Suppose, we have the random angle Θ given on the probability space $\{\Omega,\mathcal{F},\mathbf{P}\}$. A natural question arises whether we have an inverse formula type as (1) or (2) if the random angle Θ is lattice random angle. We have not found such a type of formulas in papers and books. Therefore we obtain an inverse formula of the lattice random angle Θ.

Theorem 3. *For any integer $l\geqslant 1$ and for any $m_0=0,1,\dots,l-1$, the following equality holds*:

$$\mathbf{P}\biggl\{\Theta=\frac{2\pi}{l}m_0\biggr\}=\frac{1}{l}\int\limits_{-l/2}^{l/2}\mathrm{e}^{-\mathrm{i}t\frac{2\pi}{l}m_0}\mathbf{E}\mathrm{e}^{\mathrm{i}t\eta}\,\mathrm{d}t,$$

where η is a lattice random variable taking values $2\pi m/l$, $m=0,\pm1,\pm2,\dots$, $l\geqslant 1$, with probabilities

$$\widetilde{\mathbf{P}}\biggl\{\eta=\frac{2\pi}{l}m\biggr\}=\begin{cases}\mathbf{P}\{\Theta=\frac{2\pi}{l}m\} & \text{if } m=0,1,\dots,l-1,\\ 0 & \text{if } m=-1,-2\dots \text{ and } m=l,l+1,\dots.\end{cases}$$

Proof. Suppose, we have a lattice random angle Θ. In expression (3), let us substitute $p=0,\pm1,\pm2,\dots$ for $t\in\mathbb{R}$. Then

$$\mathbf{E}\mathrm{e}^{\mathrm{i}t\eta}=\sum_{m=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\frac{2\pi}{l}mt}\mathbf{P}\biggl\{\eta=\frac{2\pi}{l}m\biggr\},$$

where η is a random variable, and

$$\mathbf{P}\left\{\eta = \frac{2\pi}{l}m\right\} = \begin{cases} \mathbf{P}\{\Theta = \frac{2\pi}{l}m\} & \text{if } m = 0, 1, \ldots, l-1, \\ 0 & \text{if } m = -1, -2\ldots \text{ and } m = l, l+1, \ldots. \end{cases}$$

We have already noticed that there exists an inverse formula for the lattice random variable ξ (see formula (1), where $a = 0$, $h = 2\pi/l$). Consequently,

$$\mathbf{P}\left\{\eta = \frac{2\pi}{l}m_0\right\} = \frac{1}{l}\sum_{m=0}^{l-1}\int_{-l/2}^{l/2} \mathrm{e}^{-\mathrm{i}tm_0\frac{2\pi}{l}+\mathrm{i}tm\frac{2\pi}{l}}\mathbf{P}\left\{\eta = \frac{2\pi}{l}m\right\}\mathrm{d}t. \tag{14}$$

Denote $h = 2\pi/l$. After replacing $th = u$ in the integral

$$\frac{h}{2\pi}\int_{-\pi/h}^{\pi/h} \mathrm{e}^{-\mathrm{i}th(m_0-m)}\,\mathrm{d}t,$$

we obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}u(m_0-m)}\,\mathrm{d}u = \begin{cases} 1 & \text{if } m = m_0, \\ 0 & \text{if } m \neq m_0. \end{cases}$$

From the equality above and equality (14) it follows that

$$\mathbf{P}\left\{\eta = \frac{2\pi}{l}m_0\right\} = \frac{1}{l}\int_{-l/2}^{l/2} \mathrm{e}^{-\mathrm{i}t\frac{2\pi}{l}m_0}\mathbf{E}\mathrm{e}^{\mathrm{i}t\eta}\,\mathrm{d}t = \mathbf{P}\left\{\Theta = \frac{2\pi}{l}m_0\right\}.$$

The proof of Theorem 3 is complete. □

4 Conclusions

(i) Expressions of a wrapped distribution function and a wrapped quasi-lattice distribution has been constructed. These expressions have been constructed by defining special Borel sets.

(ii) It has been proved that the values of wrapped lattice distribution do not depend on the span of a lattice distribution, which was wrapped, only the probabilities depend on the span of a lattice distribution. Also, it has been proved that the values of wrapped quasi-lattice distribution does not depend on the quasi-lattice parameters, only the probabilities depend on parameters. The method, how to wrap the distribution defined on $\mathbb{R}$ on the unitcircle has been demonstrated in this paper.

(iii) The Bergström identity has been proved for wrapped quasi-lattice distributions. This identity might be useful for asymptotic analysis of distributions of random angles.

(iv) The inverse formula for lattice distributions on $\mathbb{S}$ has been proved.

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