Nonlinear Analysis: Modelling and Control, Vol. 24, No. 3, 447–461
https://doi.org/10.15388/NA.2019.3.8

ISSN: 1392-5113
eISSN: 2335-8963

Positive solutions for ϕ-Laplace equations with discontinuous state-dependent forcing terms

Radu Precup[a], **Jorge Rodríguez-López**[b,1]

[a]Department of Mathematics, Babeş-Bolyai University,
400084 Cluj-Napoca, Romania
r.precup@math.ubbcluj.ro

[b]Instituto de Matemáticas, Universidade de Santiago de Compostela,
15782, Facultade de Matemáticas, Campus Vida, Santiago, Spain
jorgerodriguez.lopez@usc.es

Received: October 24, 2018 / **Revised:** November 21, 2018 / **Published online:** April 19, 2019

Abstract. This paper concerns the existence, localization and multiplicity of positive solutions for a ϕ-Laplacian problem with a perturbed term that may have discontinuities in the state variable. First, the initial discontinuous differential equation is replaced by a differential inclusion with an upper semicontinuous term. Next, the existence and localization of a positive solution of the inclusion is obtained via a compression-expansion fixed point theorem for a composition of two multivalued maps, and finally, a suitable control of discontinuities allows to prove that any solution of the inclusion is a solution in the sense of Carathéodory of the initial discontinuous equation. No monotonicity assumptions on the nonlinearity are required.

Keywords: discontinuous differential equation, ϕ-Laplacian problem, positive solution, fixed point, multivalued map, infinitely many solutions.

1 Introduction

In this paper, we establish new existence, localization and multiplicity results of positive solutions for the problem

$$\begin{aligned} &-\big(\phi\big(u'\big)\big)' = f(t,u) \quad \text{a.e. in } I := [0,1], \\ &u(0) - \alpha u'(0) = u'(1) = 0, \end{aligned} \tag{1}$$

where $\alpha \geqslant 0$, $\phi : (-a,a) \to (-b,b)$ is an increasing homeomorphism such that $\phi(0) = 0$, $0 < a, b \leqslant \infty$, and the function $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ may have discontinuities even with respect to the second variable. Also, we achieve a multiplicity result concerning the existence of infinitely many solutions to problem (1).

[1]The author was financially supported by Xunta de Galicia Scholarship ED481A-2017/178.

Differential equations with discontinuous nonlinearities (discontinuous differential equations, for short) arise from mathematical modeling of real processes from physics, engineering, biology, medicine, economics etc., whose dynamics are designed by discontinuous feedback controllers. For example, the regulation of temperature in a room is achieved discontinuously by switching on and off the cooling system as operated by a thermostat. In mechanics, such a process is the movement in presence of dry friction [26], while in population dynamics, it is the harvesting whose intensity is changed depending on some population thresholds. A nonstandard example of discontinuous differential equation can be found in [14] modeling opinion dynamics. In mathematics, discontinuous differential equations appear in control theory, where, even when a system is controllable, it may fail to admit a continuous feedback, which stabilizes it [1, 16, 20].

Discontinuous differential equations have been studied by many authors, see, e.g., [4, 5, 8, 10, 15, 25, 32] and the references therein. To this aim, several techniques have been used, such as methods of fixed point theory [19, 27], lower and upper solution techniques [32], or methods of nonsmooth critical point theory [5, 7, 8]. In connection with the investigation methods, several notions of solution have been defined, most of them as solutions of some differential inclusions, see [22, 31].

Also, in the last decades, ϕ-Laplacian equations have been extensively studied by different authors and a variety of tools, see, e.g., [3, 12, 30]. However, not many results on discontinuous ϕ-Laplacian equations are known in the literature. The existing ones are based on monotonicity hypotheses for nonlinearities [11] or use solutions in the sense of set-valued analysis, mainly, as Filippov or Krasovskij solutions; see [2, 9]. Compared to these results, in our case, the solutions are in the Carathéodory sense, and no monotonicity assumptions are required.

Problem (1) with a continuous nonlinearity f was previously studied in [23, 24], using Krasnosel'skiĭ's compression-expansion fixed point theorem in cones and a Harnack-type inequality.

In the present paper, since the function f may be discontinuous, we first consider the regularized problem in the Filippov sense [20], namely the boundary value problem for a differential inclusion

$$\begin{aligned} &-\big(\phi(u')\big)' \in F(t,u) \quad \text{a.e. in } I, \\ &u(0) - \alpha u'(0) = u'(1) = 0, \end{aligned} \tag{2}$$

where the multivalued map $F : I \times \mathbb{R}_+ \to \mathcal{P}(\mathbb{R}_+)$ is defined as

$$F(t,x) = \bigcap_{\varepsilon>0} \overline{\mathrm{co}} f\big(t, \overline{B}_\varepsilon(x) \cap \mathbb{R}_+\big) \tag{3}$$

with $\overline{\mathrm{co}}$ standing for the closed convex hull and $\overline{B}_\varepsilon(x) := [x-\varepsilon,\ x+\varepsilon]$.

Unfortunately, the standard generalization of Krasnosel'skiĭ's fixed point theorem to upper semicontinuous multivalued maps with convex values, due to Fitzpatrick and Petryshyn [21], is not applicable to the integral operator associated to problem (2). The reason is that the values of the integral operator are not convex in general due to the nonlinearity of ϕ. To overcome this difficulty, we will apply a compression-expansion fixed point theorem established in [17] for the composition of two multivalued operators (see also the coincidence point theorems in [6]).

After obtaining and localizing a solution of the regularized problem, we shall concentrate on proving that any solution of the regularized problem is also a solution of the initial discontinuous equation in the sense of Carathéodory. We shall succeed in this, by using the technique from [15, 18, 19, 27, 29], under the assumption that the function f is discontinuous over the graphs of a countable number of curves satisfying some "transversality" condition.

Finally, for problems with nonlinearities having excessive oscillations towards zero or infinity, by using the localization result, we are able to emphasize the existence of infinitely many positive solutions.

2 The fixed point setting

In [17, 28], an existence theory for the operator inclusion

$$x \in \Psi\Phi x, \tag{4}$$

where Ψ and Φ are two single or multivalued operators, was developed.

Let X be a normed linear space. Let us introduce the following notations:

$$P_{fc}(X) = \{A \subset X\colon\ A \text{ is nonempty, closed and convex}\},$$

$$P_{k^wc}(X) = \{A \subset X\colon\ \text{ is nonempty, weakly compact and convex}\}.$$

Also, recall that a multivalued operator Φ from a subset D of a normed linear space to an other normed linear space is said to be

- *upper semicontinuous* (usc, for short) on D if for every closed subset C of D, the set
$$\Phi^-(C) = \{x \in D\colon\ C \cap \Phi x \neq \emptyset\}$$
is closed in D.
- *sequentially weakly upper semicontinuous* (w-usc, for short) on D if for every weakly closed subset C of D, the set $\Phi^-(C)$ is sequentially closed for the weak topology on D.

Now we state the compression-expansion fixed point theorem for inclusion (4).

Theorem 1. (*See* [17, *Thm.* 2.3].) *Let $(X, \|\cdot\|)$ and Y be normed linear spaces, and K a wedge of X. Let $\Phi : K \to P_{k^wc}(Y)$, $\Psi : C \to P_{fc}(K)$ be two bounded multivalued maps, where $C = \overline{\mathrm{co}}(\{0\} \cup \Phi(K))$. Assume that*

(i) *if $A \subset K$, $A = \overline{\mathrm{co}}(\{0\} \cup \Psi(\overline{\mathrm{co}}(\{0\} \cup \Phi(A))))$, then A is weakly compact, and Φ, Ψ are w-usc on A and $\overline{\mathrm{co}}(\{0\} \cup \Phi(A))$, respectively.*

In addition, assume that there exist $r_1, r_2 > 0$, $r_1 \neq r_2$, and $h \in K \setminus \{0\}$ such that

(ii) *$x \notin \lambda\Psi\Phi x$ for $\lambda \in (0, 1)$ and $x \in K$ with $\|x\| = r_1$; and $x \notin \Psi\Phi x + \mu h$ for $\mu > 0$ and $x \in K$ with $\|x\| = r_2$.*

Then there exists at least one $x \in K$ with $x \in \Psi\Phi x$ such that

$$\min\{r_1, r_2\} \leqslant \|x\| \leqslant \max\{r_1, r_2\}.$$

Remark 1. Since any usc map on a compact set is sequentially w-usc, Theorem 1 remains true if instead of (i) we assume condition

(iii) if $A \subset K$, $A = \overline{\mathrm{co}}(\{0\} \cup \Psi(\overline{\mathrm{co}}(\{0\} \cup \Phi(A))))$, then A is compact, and Φ, Ψ are usc on A and $\overline{\mathrm{co}}(\{0\} \cup \Phi(A))$, respectively.

3 Main result

In this section, we study the existence of *positive solutions* to problem (1), that is, a function $u \in C^1(I)$, $u \geqslant 0$, $u \not\equiv 0$, with $u(0) - \alpha u'(0) = u'(1) = 0$ such that

$$\phi \circ u' \in W^{1,1}(I) \quad \text{and} \quad -\big(\phi\big(u'(t)\big)\big)' = f\big(t, u(t)\big) \quad \text{for a.a. } t \in I.$$

Equivalently, we will look for fixed points of the integral operator $T : P \to P$ given by

$$Tu(t) = \alpha\phi^{-1}\left(\int_0^1 f\big(s, u(s)\big)\,\mathrm{d}s\right) + \int_0^t \phi^{-1}\left(\int_r^1 f\big(s, u(s)\big)\,\mathrm{d}s\right)\mathrm{d}r, \tag{5}$$

where P is the cone of nonnegative functions in the Banach space of the continuous functions with the maximum norm $(C(I), \|\cdot\|_\infty)$.

As mentioned above, since f is not necessarily continuous, the operator T may be discontinuous, and the usual compression-expansion-type results are not applicable. This is the motivation to consider inclusion (2) and to look for solutions of this problem by means of the multivalued operator $\mathbf{T} : P \to \mathcal{P}(P)$ defined as

$$\mathbf{T}u(t) = \alpha\phi^{-1}\left(\int_0^1 F\big(s, u(s)\big)\,\mathrm{d}s\right) + \int_0^t \phi^{-1}\left(\int_r^1 F\big(s, u(s)\big)\,\mathrm{d}s\right)\mathrm{d}r, \tag{6}$$

where F stands for the map obtained after "convexification" of the function f as in (3).

Notice that the operator $\mathbf{T}$ can be decomposed as

$$\mathbf{T} = \Psi\Phi,$$

where for every $v \in P$,

$$\Psi v(t) = \alpha\phi^{-1}\big(v(0)\big) + \int_0^t \phi^{-1}\big(v(s)\big)\,\mathrm{d}s, \quad t \in I,$$

and

$$\Phi v(t) = \Lambda\mathcal{N}_F v(t)$$

with

$$\Lambda w(t) = \int_t^1 w(s)\,\mathrm{d}s$$

and the Nemytskii operator

$$\mathcal{N}_F(u) = \bigl\{v \in L^1(I)\colon\ v(t) \in F\bigl(t, u(t)\bigr) \text{ for a.a. } t \in I\bigr\}. \tag{7}$$

First, let us mention the following result about the upper semicontinuity of the Nemytskii operator (for details, see [13, 29]).

Lemma 1. *Assume that the function $f : I \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions*:

(H1) *The composed function $f(\cdot, u(\cdot))$ is measurable for every $u \in P$*;

(H2) *$f(t, u) < b$ on $I \times \mathbb{R}_+$, and if $b = \infty$, there exist $c_1, c_2 \in \mathbb{R}_+$ and $p \geqslant 1$ such that $f(t, u) \leqslant c_1 u^p + c_2$ for a.a. $t \in I$ and all $u \in \mathbb{R}_+$.*

Then the Nemytskii operator $\mathcal{N}_F : P \to \mathcal{P}(L^1(I))$ defined as in (7) *is an usc map on P from the topology of $C(I)$ to that of $L^1(I)$.*

In order to apply Theorem 1, we need the following Harnack-type inequality established in [23, 24] for the case $a = \infty$. Notice that with the same proof, the result still holds true for $a < \infty$.

Lemma 2. *For each $c \in (0, 1)$ and any $u \in C^1(I)$, $u \geqslant 0$, with $u(0) - \alpha u'(0) = u'(1) = 0$ and $\phi \circ u'$ nonincreasing in I, one has*

$$u(t) \geqslant M\|u\|_\infty \quad \textit{for all } t \in [c, 1],$$

where $M = (\alpha + c)/(\alpha + 1)$.

From now on, the point $c \in (0, 1)$ is fixed. The essential properties of the operators Φ and Ψ from above are given by the following theorem involving two subcones of P, namely

$$K_1 = \bigl\{u \in P\colon\ u(t) \geqslant M\|u\|_\infty \text{ for all } t \in [c, 1]\bigr\},$$
$$K_2 = \bigl\{u \in P\colon\ u \text{ is nonincreasing},\ u(1) = 0\bigr\},$$

and, exclusively, the topology of $C(I)$.

Theorem 2. *If the function f satisfies conditions* (H1) *and* (H2)*, then the operators*

$$\Phi : K_1 \to P_{fc}(K_2) \quad \textit{and} \quad \Psi : K_2 \to K_1$$

are well defined; *Φ is usc and maps bounded sets into relatively compact sets*; *and Ψ is a single-valued continuous operator, which maps bounded sets into relatively compact sets.*

Proof. Since $\Phi = \Lambda\mathcal{N}_F$, it follows from the definition of the operator Λ that $\Phi(K_1) \subset K_2$. To show that $\Psi(K_2) \subset K_1$, take any $v \in K_2$ and let $u := \Psi(v)$. Clearly, $u \in P$. Also, $\phi \circ u' = v$, and so $\phi \circ u'$ is nonincreasing in I. Moreover, $u(0) - \alpha u'(0) = u'(1) = 0$. Consequently, by Lemma 2, $u(t) \geqslant M\|u\|_\infty$ for all $t \in [c, 1]$. Hence $u \in K_1$, as desired.

In addition, Λ, as a linear operator from $L^1(I)$ to $C(I)$, is compact, while in view of Lemma 1, $\mathcal{N}_F$ is usc from the topology of $C(I)$ to that of $L^1(I)$. Thus Φ is usc and maps bounded sets into relatively compact sets.

Clearly, Φ has convex values. To show that its values are also closed in $C(I)$, take any element $u \in K_1$ and any sequence $v_n \in \Phi u$ with $v_n \to v$ in $C(I)$. Then $v_n = \Lambda w_n$ for some $w_n \in \mathcal{N}_F(u)$. From the definition of F we have that $\mathcal{N}_F(u)(t)$ is bounded uniformly with respect to $t \in I$. As a result, the sequence w_n is bounded in $L^p(I)$ for any (fixed) $p \in (1, \infty)$. The space $L^p(I)$ (for $1 < p < \infty$) being reflexive, we may assume without less of generality that w_n is weakly convergent in $L^p(I)$ to some w. It is easy to see that $w \in \mathcal{N}_F(u)$. Then there is a sequence $\overline{w}_n$ of convex combinations of w_n, which strongly converges in $L^p(I)$, and consequently in $L^1(I)$, to w. From $\overline{v}_n = \Lambda \overline{w}_n$ we deduce that the corresponding sequence $\overline{v}_n$ of convex combinations of v_n converges in $C(I)$ to Λw. But since $v_n \to v$, the limit of $\overline{v}_n$ is v. Then $v = \Lambda w$, where $w \in \mathcal{N}_F(u)$, which proves that $v \in \Phi u$, as wished.

Finally, the continuity and the compactness of the operator Ψ are standard consequences of Lebesgue's dominated convergence and Ascoli–Arzela's theorems. □

Now we are ready to state and prove the main result about the existence and localization of positive solutions to the discontinuous problem (1).

Theorem 3. *Assume that the function f satisfies conditions* (H1), (H2) *and*

(H3) *There is a countable number of functions $\gamma_n \in C^1(I)$ $(n \in \mathbb{N})$ with $\phi \circ \gamma_n' \in W^{1,1}(I)$ and a countable number of closed subintervals I_n of I such that*

$$\bigl\{-\bigl(\phi\bigl(\gamma_n'(t)\bigr)\bigr)'\bigr\} \cap F\bigl(t, \gamma_n(t)\bigr) \subset \bigl\{f\bigl(t, \gamma_n(t)\bigr)\bigr\} \quad \text{for a.a. } t \in I_n,\ n \in \mathbb{N}, \tag{8}$$

and

$$f(t, \cdot) \text{ is continuous on } \mathbb{R}_+ \setminus \bigcup_{\{n:\, t \in I_n\}} \bigl\{\gamma_n(t)\bigr\} \quad \text{for a.a. } t \in I. \tag{9}$$

In addition, assume that there exist $0 < r_1, r_2$, $r_1 \neq r_2$, and $\varepsilon > 0$ such that

$$\alpha \phi^{-1}\left(\int_0^1 \Gamma_{r_1}^{\varepsilon}(s)\,\mathrm{d}s\right) + \int_0^1 \phi^{-1}\left(\int_r^1 \Gamma_{r_1}^{\varepsilon}(s)\,\mathrm{d}s\right)\mathrm{d}r \leqslant r_1, \tag{10}$$

$$\alpha \phi^{-1}\left(\int_c^1 \Gamma_{r_2,\varepsilon}(s)\,\mathrm{d}s\right) + \int_c^1 \phi^{-1}\left(\int_r^1 \Gamma_{r_2,\varepsilon}(s)\,\mathrm{d}s\right)\mathrm{d}r \geqslant r_2, \tag{11}$$

where

$$\Gamma_{r_1}^{\varepsilon}(s) = \max_{x \in [0,\, r_1+\varepsilon]} f(s, x) \quad \text{and} \quad \Gamma_{r_2,\varepsilon}(s) = \min_{x \in [(r_2-\varepsilon)M,\, r_2+\varepsilon]} f(s, x).$$

Then problem (1) *has at least one positive solution u such that*

$$\min\{r_1, r_2\} \leqslant \|u\|_\infty \leqslant \max\{r_1, r_2\}. \tag{12}$$

Proof. We apply Theorem 1. In virtue of Theorem 2, it only remains to prove that the operator $\mathbf{T} = \Psi\Phi$ satisfies the compression-expansion conditions as in (ii).

We first show that

$$\|v\|_\infty \leqslant r_1 \quad \text{for all } v \in \mathbf{T}u \quad \text{and} \quad u \in K_1 \quad \text{with } \|u\|_\infty = r_1,$$

which implies that

$$u \notin \lambda \mathbf{T}u \quad \text{for all } \lambda \in (0,1) \quad \text{and} \quad u \in K_1 \quad \text{with } \|u\|_\infty = r_1.$$

Assume the contrary. Then there exists $v \in \mathbf{T}u$ and $u \in K_1$ with $\|u\|_\infty = r_1$ such that $r_1 < \|v\|_\infty$. Notice that for any $\varepsilon > 0$, if $w \in \mathcal{N}_F(u)$ and $\|u\|_\infty = r_1$, then

$$w(s) \leqslant \max_{x \in [0,\, r_1+\varepsilon]} f(s,x) =: \Gamma^\varepsilon_{r_1}(s) \quad \text{for all } s \in I.$$

Hence, by the fact that $v \in \mathbf{T}u$ and (10),

$$\|v\|_\infty \leqslant \alpha\phi^{-1}\left(\int_0^1 \Gamma^\varepsilon_{r_1}(s)\,\mathrm{d}s\right) + \int_0^1 \phi^{-1}\left(\int_r^1 \Gamma^\varepsilon_{r_1}(s)\,\mathrm{d}s\right)\mathrm{d}r \leqslant r_1,$$

which yields the contradiction $r_1 < r_1$.

Next, we have to show that

$$r_2 \leqslant \|v\|_\infty \quad \text{for all } v \in \mathbf{T}u \quad \text{and} \quad u \in K_1 \quad \text{with } \|u\|_\infty = r_2,$$

which implies that

$$u \notin \mathbf{T}u + \mu \quad \text{for all } \mu > 0 \quad \text{and} \quad u \in K_1 \quad \text{with } \|u\|_\infty = r_2.$$

The proof is similar and is based on the fact that for every $\varepsilon > 0$, if $w \in \mathcal{N}_F(u)$ and $\|u\|_\infty = r_2$,

$$w(s) \geqslant \Gamma_{r_2,\varepsilon}(s) \quad \text{for all } s \in [c,1].$$

The details are left to the reader.

Therefore, Theorem 1 applies and yields the existence of a fixed point $u \in P$ for the operator $\mathbf{T}$ satisfying (12). Then

$$-\big(\phi\big(u'(t)\big)\big)' \in F\big(t,u(t)\big) \quad \text{for a.a. } t \in I. \tag{13}$$

Now we prove that u (in fact, any fixed point of $\mathbf{T}$) solves the initial discontinuous problem (1). To this aim, define

$$J_n := \big\{t \in I_n\colon\, u(t) = \gamma_n(t)\big\}, \quad n \in \mathbb{N}.$$

Clearly

$$-\big(\phi\big(u'(t)\big)\big)' = -\big(\phi\big(\gamma_n'(t)\big)\big)' \quad \text{for a.a. } t \in J_n.$$

Hence, by (13),

$$-\big(\phi\big(\gamma_n'(t)\big)\big)' \in F\big(t, u(t)\big) = F\big(t, \gamma_n(t)\big) \quad \text{for a.a. } t \in J_n.$$

This, based on condition (8), implies

$$-\big(\phi\big(\gamma_n'(t)\big)\big)' = f\big(t, \gamma_n(t)\big) \quad \text{for a.a. } t \in J_n,$$

equivalently,

$$-\big(\phi\big(u'(t)\big)\big)' = f\big(t, u(t)\big) \quad \text{for a.a. } t \in J_n.$$

Thus u satisfies the initial discontinuous differential equation a.e. in $J = \bigcup_{n\in\mathbb{N}} J_n$. Finally, from (9) one has

$$F\big(t, u(t)\big) = \big\{f\big(t, u(t)\big)\big\} \quad \text{for } t \in I \setminus J.$$

This together with (13) shows that u also satisfies the initial discontinuous differential equations a.e. in $I \setminus J$. Therefore, u solves (1) in I. □

Remark 2. If the function f is nondecreasing with respect to the second variable, then conditions (10) and (11) can be written as

$$\alpha\phi^{-1}\left(\int_0^1 f(s,\, r_1+\varepsilon)\,\mathrm{d}s\right) + \int_0^1 \phi^{-1}\left(\int_r^1 f(s,\, r_1+\varepsilon)\,\mathrm{d}s\right)\mathrm{d}r \leqslant r_1,$$

$$\alpha\phi^{-1}\left(\int_c^1 f\big(s,\, M(r_2-\varepsilon)\big)\,\mathrm{d}s\right) + \int_c^1 \phi^{-1}\left(\int_r^1 f\big(s,\, M(r_2-\varepsilon)\big)\,\mathrm{d}s\right)\mathrm{d}r \geqslant r_2,$$

and they are analogous to those considered in [23] for the case of a continuous nonlinearity.

Remark 3 [Asymptotic conditions]. In virtue of Remark 2, if the function f is nondecreasing with respect to the second variable, the existence of two numbers r_1 and r_2 satisfying (10) and (11) is guaranteed by any one of the following two conditions:

$$\text{(a)}\quad \begin{aligned} &\liminf_{x\to 0} \frac{\alpha\phi^{-1}(\int_0^1 f(s,x)\,\mathrm{d}s) + \int_0^1 \phi^{-1}(\int_r^1 f(s,x)\,\mathrm{d}s)\,\mathrm{d}r}{x} < 1,\\ &\limsup_{x\to \infty} \frac{\alpha\phi^{-1}(\int_c^1 f(s,Mx)\,\mathrm{d}s) + \int_c^1 \phi^{-1}(\int_r^1 f(s,Mx)\,\mathrm{d}s)\,\mathrm{d}r}{x} > 1; \end{aligned} \tag{14}$$

$$\text{(b)}\quad \begin{aligned} &\liminf_{x\to \infty} \frac{\alpha\phi^{-1}(\int_0^1 f(s,x)\,\mathrm{d}s) + \int_0^1 \phi^{-1}(\int_r^1 f(s,x)\,\mathrm{d}s)\,\mathrm{d}r}{x} < 1,\\ &\limsup_{x\to 0} \frac{\alpha\phi^{-1}(\int_c^1 f(s,Mx)\,\mathrm{d}s) + \int_c^1 \phi^{-1}(\int_r^1 f(s,Mx)\,\mathrm{d}s)\,\mathrm{d}r}{x} > 1. \end{aligned} \tag{15}$$

Observe that the first case is only possible if $a = +\infty$ and $b = +\infty$. Otherwise, if $a < +\infty$, then ϕ^{-1} is bounded, and if $b < +\infty$, then $f < b$, so the numerator being bounded,

$$\limsup_{x\to\infty} \frac{\alpha\phi^{-1}(\int_c^1 f(s,Mx)\,\mathrm{d}s) + \int_c^1 \phi^{-1}(\int_r^1 f(s,Mx)\,\mathrm{d}s)\,\mathrm{d}r}{x} = 0.$$

The "transversality" condition (8) was previously presented in [29] and recalls the notion of *viable* and *inviable* curves introduced in [19, 27]. It allows the function f to be discontinuous over time-dependent sets that generalize the discontinuity sets of [5, 7]. The meaning of (8) is clarified by the next remark, where some sufficient conditions are given.

Remark 4. Assumption (8) is satisfied if one of the following two conditions holds:

(i) $-(\phi(\gamma_n'(t)))' = f(t,\gamma_n(t))$ for a.a. $t \in I_n$;
(ii) $\{-(\phi(\gamma_n'(t)))'\} \notin F(t,\gamma_n(t))$ for a.a. $t \in I_n$.

In particular, alternative (ii) is satisfied if there exist $\delta, \varepsilon > 0$ such that

$$-\bigl(\phi\bigl(\gamma_n'(t)\bigr)\bigr)' + \delta \leqslant f(t,y) \quad \text{for a.a. } t \in I_n \text{ and all } y \in \bigl[\gamma_n(t)-\varepsilon,\ \gamma_n(t)+\varepsilon\bigr], \tag{16}$$

or

$$-\bigl(\phi\bigl(\gamma_n'(t)\bigr)\bigr)' - \delta \geqslant f(t,y) \quad \text{for a.a. } t \in I_n \text{ and all } y \in \bigl[\gamma_n(t)-\varepsilon,\ \gamma_n(t)+\varepsilon\bigr]. \tag{17}$$

Observe also that conditions (16) and (17) recall the notion of lower and upper solutions for the differential equation $-(\phi(u'))' = f(t,u)$.

To finish this section, we illustrate the applicability of our main result by an example.

Example 1. Consider the differential problem involving the curvature operator in Euclidean space

$$\begin{aligned} &-\biggl(\frac{u'}{\sqrt{1+u'^2}}\biggr)' = f(t,u) := \sqrt[3]{u}\mathrm{e}^{-u} + \frac{1}{2}\cos^2\biggl(\biggl[\frac{1}{u+t}\biggr]\biggr) \quad \text{a.e. in } I, \\ &u(0) = u'(1) = 0, \end{aligned} \tag{18}$$

where $[x]$ denotes the integer part of x. Here, $\phi : \mathbb{R} \to (-1,1)$ is given by

$$\phi(\tau) = \frac{\tau}{\sqrt{1+\tau^2}} \quad \text{and} \quad \phi^{-1}(\tau) = \frac{\tau}{\sqrt{1-\tau^2}}.$$

Also, notice that $f(t,u) < 1$.

For this example,

$$\gamma_n = -t + \frac{1}{n} \quad \text{and} \quad I_n = \biggl[0, \frac{1}{n}\biggr], \quad n \in \mathbb{N}.$$

Clearly, the function $u \mapsto f(t,u)$ is continuous on $\mathbb{R}_+ \setminus \bigcup_{\{n:\ t\in I_n\}}\{\gamma_n(t)\}$ for a.a. $t \in I$. Note that

$$-\big(\phi\big(\gamma_n'(t)\big)\big)' = 0 \quad \text{for a.a. } t \in I_n \text{ and } n \in \mathbb{N},$$

so condition (16) is obviously satisfied for $\delta, \varepsilon > 0$ small enough since $\cos^2 n > 0$ for all $n \in N$.

Finally, if we take $c = 1/2$, then $M = 1/2$, and it is easy to verify that conditions (10) and (11) (with $\alpha = 0$) hold for $r_1 = 1$ and $r_2 = 1/50$, respectively.

Therefore, Theorem 3 ensures the existence of a positive solution of problem (18) such that $1/50 \leqslant \|u\|_\infty \leqslant 1$.

4 Infinitely many solutions

The existence and localization result, Theorem 3, yields multiplicity results for problem (1) when several couples of numbers r_1 and r_2 satisfying conditions (10) and (11) exist such that the corresponding intervals (r, R) are disjoint.

Taking this into account and using asymptotic conditions, it is possible to derive a multiplicity result concerning the existence of infinitely many positive solutions for problem (1).

Theorem 4. *Assume that the function f satisfies conditions* (H1)–(H3). *Moreover, assume that f is a nondecreasing function with respect to the second variable. If the following asymptotic condition*

$$\text{(c)}\quad \begin{aligned} &\liminf_{x\to\infty} \frac{\alpha\phi^{-1}(\int_0^1 f(s,x)\,\mathrm{d}s) + \int_0^1 \phi^{-1}(\int_r^1 f(s,x)\,\mathrm{d}s)\,\mathrm{d}r}{x} < 1,\\ &\limsup_{x\to\infty} \frac{\alpha\phi^{-1}(\int_c^1 f(s,Mx)\,\mathrm{d}s) + \int_c^1 \phi^{-1}(\int_r^1 f(s,Mx)\,\mathrm{d}s)\,\mathrm{d}r}{x} > 1 \end{aligned}$$

holds, then problem (1) *has a sequence of positive solutions $(u_n)_{n\in\mathbb{N}}$ such that $\|u_n\|_\infty \to \infty$ as $n \to \infty$.*

If the condition

$$\text{(d)}\quad \begin{aligned} &\liminf_{x\to 0} \frac{\alpha\phi^{-1}(\int_0^1 f(s,x)\,\mathrm{d}s) + \int_0^1 \phi^{-1}(\int_r^1 f(s,x)\,\mathrm{d}s)\,\mathrm{d}r}{x} < 1,\\ &\limsup_{x\to 0} \frac{\alpha\phi^{-1}(\int_c^1 f(s,Mx)\,\mathrm{d}s) + \int_c^1 \phi^{-1}(\int_r^1 f(s,Mx)\,\mathrm{d}s)\,\mathrm{d}r}{x} > 1 \end{aligned}$$

holds, then problem (1) *has a sequence of positive solutions $(u_n)_{n\in\mathbb{N}}$ such that $\|u_n\|_\infty \to 0$ as $n \to \infty$.*

Proof. Notice that, in view of Remarks 2 and 3, condition (c) ensures that there exist two sequences $(r_{1,n})_{n\in\mathbb{N}}$ and $(r_{2,n})_{n\in\mathbb{N}}$ such that

$$0 < r_{1,n} < r_{2,n} < r_{1,n+1} \quad \text{for } n \in \mathbb{N}, \qquad r_{1,n} \to \infty \quad \text{as } n \to \infty, \tag{19}$$

and for each $n \in \mathbb{N}$, the numbers $r_{1,n}$ and $r_{2,n}$ satisfy (10) and (11). Therefore, for each $n \in \mathbb{N}$, Theorem 3 ensures the existence of a solution u_n such that

$$r_{1,n} \leqslant \|u_n\|_\infty \leqslant r_{2,n}. \tag{20}$$

Now (19) and (20) show that the solutions u_n are distinct being located in disjoint annular sets, and also that $\|u_n\|_\infty \to \infty$ as $n \to \infty$.

In a similar way, condition (d) and Theorem 3 ensure the existence of a sequence of positive solutions $(u_n)_{n\in\mathbb{N}}$ in the conditions of the statement. □

We illustrate Theorem 4 by two examples where the nonlinearities are discontinuous perturbations of those in [23, Addendum, Examples 4.2 and 4.3].

Example 2. Consider the problem

$$-u'' = f(u), \qquad u(0) = u'(1) = 0, \tag{21}$$

where

$$f(u) := \alpha u + \beta[u] + \rho u \sin\bigl(\delta \ln(u+1)\bigr),$$

α, β, ρ and δ are positive constants, and $[x]$ denotes the integer part of x.

Assume that

$$\alpha \geqslant \rho(\delta + 1). \tag{22}$$

Then, it is easy to verify that f takes nonnegative values and is nondecreasing on $\mathbb{R}_+$. Obviously, f is discontinuous at the natural numbers (the points where the integer part function is discontinuous) and so,

$$\gamma_n(t) \equiv n \quad \text{and} \quad I_n = [0,1] \quad \text{for } n \in \mathbb{N}.$$

Observe that $\inf_{u\in[1/2,\,\infty)} f(u) > 0$, and thus condition (16) in Remark 4 is clearly satisfied for the functions γ_n, $n \in N$, and $\phi(u) = u$ ($a = b = +\infty$).

Now we compute the limits in condition (c) from Theorem 4, and we find

$$\liminf_{x\to\infty} \frac{\int_0^1 \int_r^1 f(x)\,\mathrm{d}s\,\mathrm{d}r}{x} \leqslant \frac{1}{2}(\alpha + \beta - \rho),$$
$$\limsup_{x\to\infty} \frac{\int_c^1 \int_r^1 f(cx)\,\mathrm{d}s\,\mathrm{d}r}{x} \geqslant \frac{c(1-c)^2}{2}\left(\alpha + \frac{\beta}{2} + \rho\right).$$

Hence, if we choose $c = 1/2$, then the following inequalities

$$\alpha + \frac{\beta}{2} + \rho > 16 \quad \text{and} \quad \alpha + \beta - \rho < 2 \tag{23}$$

guarantee that condition (c) holds. Therefore, under conditions (22) and (23), Theorem 4 ensures that problem (21) has a sequence of positive solutions $(u_n)_{n\in\mathbb{N}}$ with $\|u_n\|_\infty \to \infty$ as $n \to \infty$.

For example, conditions (22) and (23) hold for the following values of parameters:

$$\alpha = 8.5, \qquad \beta = 0.5, \qquad \rho = 8, \qquad \delta = 0.05.$$

In the next example, ϕ is not the identity function, while the right-hand side is a slight modification of that from Example 2.

Example 3. Consider the problem

$$-\left(|u'|^{p-2}u'\right)' = f(u), \qquad u(0) = u'(1) = 0, \tag{24}$$

where $p > 1$,

$$f(u) := g(u)^{p-1},$$

$$g(u) = \alpha u + \beta[u] + \rho u \sin\left(\delta \ln \frac{1}{u}\right) \quad \text{for } u > 0, \qquad g(0) = 0,$$

and α, β, ρ, δ are positive constants.

Here, $\phi : \mathbb{R} \to \mathbb{R}$,

$$\phi(x) = |x|^{p-2}x \quad \text{and} \quad \phi^{-1}(x) = |x|^{1/(p-1)} \operatorname{sign} x.$$

As in the previous example, under assumption (22), f is nonnegative and nondecreasing on $\mathbb{R}_+$. Also, f is discontinuous at the natural numbers, which are admissible discontinuity points.

Let us compute the limits in condition (c) from Theorem 4. We obtain

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\int_0^1 \phi^{-1}(\int_r^1 f(x)\,\mathrm{d}s)\,\mathrm{d}r}{x} &= \frac{p-1}{p} \liminf_{x\to\infty} \frac{g(x)}{x} \leqslant \frac{p-1}{p}(\alpha + \beta - \rho), \\
\limsup_{x\to\infty} \frac{\int_c^1 \phi^{-1}(\int_r^1 f(cx)\,\mathrm{d}s)\,\mathrm{d}r}{x} &= \frac{p-1}{p}(1-c)^{p/(p-1)} \limsup_{x\to\infty} \frac{g(cx)}{x} \\
&\geqslant \frac{p-1}{p}(1-c)^{p/(p-1)} c\left(\alpha + \frac{\beta}{2} + \rho\right).
\end{aligned}$$

If we choose $c = 1/2$, then the following inequalities

$$\alpha + \frac{\beta}{2} + \rho > \frac{4p}{p-1} 2^{1/(p-1)} \quad \text{and} \quad \alpha + \beta - \rho < \frac{p}{p-1} \tag{25}$$

guarantee that condition (c) holds. Therefore, under conditions (22) and (25), Theorem 4 ensures that problem (24) has a sequence of positive solutions $(u_n)_{n\in\mathbb{N}}$ with $\|u_n\|_\infty \to \infty$ as $n \to \infty$.

Moreover, we have

$$\begin{aligned}
\liminf_{x\to 0} \frac{\int_0^1 \phi^{-1}(\int_r^1 f(x)\,\mathrm{d}s)\,\mathrm{d}r}{x} &= \frac{p-1}{p}(\alpha - \rho), \\
\limsup_{x\to 0} \frac{\int_c^1 \phi^{-1}(\int_r^1 f(cx)\,\mathrm{d}s)\,\mathrm{d}r}{x} &= \frac{p-1}{p}(1-c)^{p/(p-1)} c(\alpha + \rho).
\end{aligned}$$

If we choose $c = 1/2$, then the following inequalities

$$\alpha + \rho > \frac{4p}{p-1} 2^{1/(p-1)} \quad \text{and} \quad \alpha - \rho < \frac{p}{p-1} \tag{26}$$

guarantee that condition (d) holds. Hence, under conditions (22) and (26), Theorem 4 implies that problem (24) has a sequence of positive solutions $(v_n)_{n\in\mathbb{N}}$ with $\|v_n\|_\infty \to 0$ as $n \to \infty$.

Let us remark that both conditions (25) and (26) hold if

$$\alpha + \rho > \frac{4p}{p-1} 2^{1/(p-1)} \quad \text{and} \quad \alpha + \beta - \rho < \frac{p}{p-1} \tag{27}$$

when problem (24) has two sequences of positive solutions $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}}$ with $\|u_n\|_\infty \to \infty$ and $\|v_n\|_\infty \to 0$ as $n \to \infty$.

Notice that for $p \geqslant 2$ since $2^{1/(p-1)} \leqslant 2$ and $1 < p/(p-1) \leqslant 2$), in order to fulfill conditions (27), it suffices that

$$\alpha + \rho > 16 \quad \text{and} \quad \alpha + \beta - \rho \leqslant 1.$$

For example, these inequalities and (22) are satisfied for the following values of parameters:

$$\alpha = 8.5, \qquad \beta = 0.25, \qquad \rho = 8, \qquad \delta = 0.05.$$

Acknowledgment. The authors thank the referee for the careful reading and appreciation of their paper and for the suggested additional references.

References

1. F. Ancona, A. Bressan, Patchy vector fields and asymptotic stabilization, *ESAIM, Control Optim. Calc. Var.*, **4**:419–444, 1999.
2. C. Bereanu, P. Jebelean, C. Şerban, The Dirichlet problem for discontinuous perturbations of the mean curvature operator in Minkowski space, *Electron. J. Qual. Theory Differ. Equ.*, **35**:1–7, 2015.
3. C. Bereanu, J. Mawhin, Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian, *J. Differ. Equations*, **243**:536–557, 2007.
4. D.C. Biles, Existence of solutions for discontinuous differential equations, *Differ. Integral Equ.*, **8**:1525–1532, 1995.
5. G. Bonanno, G. M. Bisci, Infinitely many solutions for a boundary value problem with discontinuous nonlinearities, *Bound. Value Probl.*, **2009**:670675, 2009.
6. G. Bonanno, P. Candito, D. Motreanu, A coincidence point theorem for sequentially continuous mappings, *J. Math. Anal. Appl.*, **435**:606–615, 2016.
7. G. Bonanno, N. Giovannelli, An eigenvalue Dirichlet problem involving the p-Laplacian with discontinuous nonlinearities, *J. Math. Anal. Appl.*, **308**:596–604, 2005.

8. G. Bonanno, A. Iannizzotto, M. Marras, On ordinary differential inclusions with mixed boundary conditions, *Differ. Integral Equ.*, **30**(3–4):273–288, 2017.

9. G. Bonanno, P. Jebelean, C. Şerban, Three periodic solutions for discontinuous perturbations of the vector p-Laplacian operator, *Proc. R. Soc. Edinb., Sect. A, Math.*, **147**:673–681, 2017.

10. A. Bressan, W. Shen, On discontinuous differential equations, in J. Andres, L. Górniewicz, P. Nistri (Eds.), *Differential Inclusions and Optimal Control*, Lect. Notes Nonlinear Anal., Vol. 2, Nicolaus Copernicus University, Juliusz Schauder Center for Nonlinear Studies, Toruń, 1998, pp. 73–87.

11. A. Cabada, S. Heikkila, Implicit nonlinear discontinuous functional boundary value ϕ-Laplacian problems: Extremality results, *Appl. Math. Comput.*, **129**:537–549, 2002.

12. A. Cabada, R.L. Pouso, Extremal solutions of strongly nonlinear discontinuous second-order equations with nonlinear functional boundary conditions, *Nonlinear Anal., Theory Methods Appl.*, **42**(8):1377–1396, 2000.

13. A. Cellina, A. Fryszkowsk, T. Rzezuchowsk, Upper semicontinuity of Nemytskij operators, *Ann. Mat. Pura Appl. (4)*, **160**:321–330, 1991.

14. F. Ceragioli, P. Frasca, Continuous and discontinuous opinion dynamics with bounded confidence, *Nonlinear Anal., Real World Appl.*, **13**(3):1239–1251, 2012.

15. J. A. Cid, R. López Pouso, Ordinary differential equations and systems with time-dependent discontinuity sets, *Proc. R. Soc. Edinb., Sect. A, Math.*, **134**:617–637, 2004.

16. F. Clarke, Discontinuous feedback and nonlinear systems, in *8th IFAC Symposium on Nonlinear Control Systems, Bologna, September 1–3, 2010*, IFAC Proceedings Volumes, Vol. 43, Issue 14, Elsevier, 2010, pp. 1–29.

17. J.-F. Couchouron, R. Precup, Homotopy method for positive solutions of p-Laplace inclusions, *Topol. Methods Nonlinear Anal.*, **30**:157–169, 2007.

18. R. Figueroa, R.L. Pouso, Discontinuous first-order functional boundary value problems, *Nonlinear Anal., Theory Methods Appl.*, **69**(7):2142–2149, 2008.

19. R. Figueroa, R.L. Pouso, J. Rodríguez-López, A version of Krasnosel'skiĭ's compression-expansion fixed point theorem in cones for discontinuous operators with applications, *Topol. Methods Nonlinear Anal.*, **51**(2):493–510, 2018.

20. A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic, Dordrecht, 1988.

21. P.M. Fitzpatrick, W.V. Petryshyn, Fixed point theorems and the fixed point index for multivalued mappings in cones, *J. Lond. Math. Soc.*, **2**(11):75–85, 1975.

22. O. Hájek, Discontinuous differential equations. I, *J. Differ. Equations*, **32**(2):149–170, 1979.

23. D.-R. Herlea, Positive solutions for second-order boundary-value problems with *ϕ-Laplacian*, *Electron. J. Differ. Equ.*, **2016**:1–8, 2016.

24. D.-R. Herlea, R. Precup, Existence, localization and multiplicity of positive solutions to ϕ-Laplace equations and systems, *Taiwan. J. Math.*, **20**:77–89, 2016.

25. S. Hu, Differential equations with discontinuous right-hand sides, *J. Math. Anal. Appl.*, **154**: 377–390, 1991.

26. R. Ma, Second order periodic problems in the presence of dry friction, *Nonlinear Anal., Real World Appl.*, **12**:3306–3314, 2011.

27. R.L. Pouso, Schauder's fixed-point theorem: New applications and a new version for discontinuous operators, *Bound. Value Probl.*, **2012**:92, 2012.
28. R. Precup, Fixed point theorems for decomposable multi-valued maps and applications, *Z. Anal. Anwend.*, **22**(4):843–861, 2003.
29. R. Precup, J. Rodríguez-López, Positive solutions for discontinuous problems with applications to ϕ-Laplacian equations, *J. Fixed Point Theory Appl.*, **20**(156):1–17, 2018.
30. I. Rachůnková, M. Tvrdý, Periodic problems with ϕ-Laplacian involving non-ordered lower and upper functions, *Fixed Point Theory*, **6**:99–112, 2005.
31. J.S. Spraker, D. Biles, A comparison of the Carathéodory and Filippov solution sets, *J. Math. Anal. Appl.*, **198**:571–580, 1996.
32. K.A. Topolski, Upper and lower absolutely continuous functions with applications to discontinuous differential equations, *Electron. J. Qual. Theory Differ. Equ.*, **83**:1–12, 2017.