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Normal form of double-Hopf singularity with 1:1 resonance for delayed differential equations

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Abstract. In this manuscript, we provide a framework for the double-Hopf singularity with 1:1 resonance for general delayed differential equations (DDEs). The corresponding normal form up to the third-order terms is derived. As an application of our framework, a double-Hopf singularity with 1:1 resonance for a van der Pol oscillator with delayed feedback is investigated to illustrate the theoretical results.

Keywords: double-Hopf bifurcation with 1:1 resonance, normal form, van der Pol oscillator with delayed feedback.

1 Introduction

In this research, we study the double-Hopf singularity with 1:1 resonance for the following general delayed differential equations (DDEs):

$$\dot{X}(t) = AX(t) + BX(t-1) + F(X_t, \mu), \quad X \in \mathbb{R}^n, \tag{1}$$

where

- A and B are real constant $n \times n$ matrices;
- $F \in C^k(\mathbb{R}^{n+m}, \mathbb{R}^n)$ $(k \geqslant 4)$ satisfies $F(0, \mu) = 0$ and $DF_X(0, \mu) = 0$;
- $\mu \in \mathbb{R}^m$ is the bifurcation parameter.

Obviously, 0 is an equilibrium of system (1), and the characteristic equation of system (1) at 0 is

$$f(\lambda) = \det\bigl(\lambda I - A - B\mathrm{e}^{-\lambda}\bigr) = 0. \tag{2}$$

The dynamical behavior, especially, the bifurcation behavior around the equilibrium 0, presented by system (1), is generally determined by the distribution of the roots of Eq. (2) and has been studied extensively by many researchers. Some of the bifurcation results in literature regarding system (1) and Eq. (2) are summarized in the following.

(i) If Eq. (2) has a pair of purely imaginary roots and other roots have negative real parts, system (1) may exhibit a Hopf bifurcation; see [5, 6, 9] and references therein.

(ii) If Eq. (2) has a double or triple zero root and other roots have negative real parts, a double or triple zero singularity for system (1) may occur. The mathematical frameworks for these singularities were established in [3] and [19], where the explicit conditions were formulated and the normal forms up to order 3 were derived.

(iii) If Eq. (2) has a simple zero root and a pair of purely imaginary roots and other roots have negative real parts, a zero-Hopf singularity may occur. Wu and Wang [18] studied this case for system (1) and provided the explicit conditions for system (1) to exhibit zero-Hopf singularity and derived the normal forms up to order 3.

(iv) If Eq. (2) has two pairs of purely imaginary roots $\pm\omega_1\mathrm{i}$ and $\pm\omega_2\mathrm{i}$ and other roots have negative real parts, a double-Hopf bifurcation may occur. For the case $\omega_1/\omega_2 \notin \mathbb{Q}$, Buono and Bélair [2] computed the corresponding normal form for scalar DDE, and Qesmia and Babram [15] derived the same for systems of DDEs. Recently, Ma et al. [13] studied a similar double-Hopf bifurcation for van der Pol-Duffing oscillator with parametric delayed feedback control.

(v) For the case that Eq. (2) has two pairs of purely imaginary roots $\pm\omega_1\mathrm{i}$ and $\pm\omega_2\mathrm{i}$ with $\omega_1 = \omega_2$, Guo and Wu [7, 8] studied the following van der Pol oscillator:

$$\ddot{x} - \varepsilon\big(1 - x^2\big)\dot{x} + x = f\big(x(t-\tau)\big), \tag{3}$$

where $f(x(t-\tau))$ is the delayed feedback for the position x. They established the explicit conditions such that the corresponding characteristic equation has a pair of purely imaginary roots with multiplicity 2. Zhang and Guo [21] studied the double-Hopf bifurcation with 1:1 resonance for system (3). Using the center manifold reduction method developed in [9], they derived the corresponding normal forms up to order 2 for $f(x) = \gamma x$ and provided the bifurcation diagrams.

For general DDEs (1), to the authors' knowledge, the explicit conditions for the double-Hopf singularity have not been formulated, and the corresponding normal forms have not been given in the literature, perhaps due to the extreme complexity and difficulty. In this manuscript, we will focus on deriving the normal forms for system (1) assuming that a double-Hopf bifurcation occurs. In particular, we study the case that Eq. (2) has a pair of purely imaginary roots with algebraic multiplicity 2 and geometric multiplicity 1, namely, double-Hopf singularity with 1:1 resonance. The main contribution of this manuscript is to characterize the center manifold for this singularity and to derive the corresponding normal forms up to order 3.

The rest of this manuscript is organized as follows. In Section 2, we formulate and characterize the double-Hopf singularity for general DDEs with 1:1 resonance. In Section 3, we use the normal form theory developed by Faria and Magalhães [5,6] to compute the normal form for system (1) up to order 3. In Section 4, to illustrate our theoretical results, we study a double-Hopf singularity for the van der Pol oscillator with delayed feedback (3). The normal form up to order 3 is derived. Finally, the manuscript ends with a conclusion in Section 5.

2 The double-Hopf singularity of the general DDEs with 1:1 resonance

In this section, we characterize the double-Hopf singularity for the general DDEs with 1:1 resonance. Write system (1) in the following form:

$$\dot{X}(t) = LX_t + F(X_t, \mu), \tag{4}$$

where $LX_t = AX(t) + BX(t-1)$. Define $C := C([-1,0], \mathbb{R}^n)$ with supreme norm and $X_t \in C$ by $X_t(\theta) = X(t+\theta)$, $-1 \leqslant \theta \leqslant 0$. Here $L : C \to \mathbb{R}^n$ is a bounded linear operator and $F : C \times \mathbb{R}^m \to R^n$ a C^k ($k \geqslant 4$) function with $F(0,\mu) = 0$ and $D_X F(0,\mu) = 0$. Consider the following linear system:

$$\dot{X}(t) = LX_t. \tag{5}$$

Since L is a bounded linear operator, L can be represented by a Riemann–Stieltjes integral

$$L\varphi = \int_{-1}^{0} \mathrm{d}\eta(\theta)\varphi(\theta) \quad \forall \varphi \in C$$

by the Riesz representation theorem, where $\eta(\theta)$ ($\theta \in [-1,0]$) is an $n \times n$ matrix function of bounded variation. Let I be the $n \times n$ identity matrix, and define

$$\Delta(\lambda) = \lambda I - \int_{-1}^{0} \mathrm{e}^{\lambda\theta}\,\mathrm{d}\eta(\theta).$$

Let $\mathcal{A}_0$ be the infinitesimal generator for the solution semigroup defined by system (5) such that

$$\mathcal{A}_0\varphi = \dot{\varphi}, \quad D(\mathcal{A}_0) = \bigl\{\varphi \in C^1\bigl([-1,0], \mathbb{R}^n\bigr):\ \dot{\varphi}(0) = L\varphi\bigr\}.$$

Define the bilinear form between C and $C^* = C([0,1], \mathbb{R}^{n*})$ (where $\mathbb{R}^{n*}$ is the space of all n-dimensional row vectors by

$$\langle\psi, \varphi\rangle = \psi(0)\varphi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi - \theta)\,\mathrm{d}\eta(\theta)\varphi(\xi)\,\mathrm{d}\xi \quad \forall \psi \in C^*,\ \forall \varphi \in C.$$

The adjoint of $\mathcal{A}_0$ is defined by $\mathcal{A}_0^*$

$$\mathcal{A}_0^*\psi = -\dot{\psi}, \qquad D(\mathcal{A}_0^*) = \left\{\psi \in C^1\bigl([0,1], \mathbb{R}^{n*}\bigr):\ \dot{\psi}(0) = -\int_{-1}^{0} \psi(-\theta)\,\mathrm{d}\eta(\theta)\right\}.$$

Since $LX_t = AX(t) + BX(t-1)$, $\eta(\theta)$ and $\Delta(\lambda)$ can be expressed, respectively, as

$$\eta(\theta) = \begin{cases} A + B & \text{if } \theta = 0, \\ B & \text{if } -1 < \theta < 0, \\ 0 & \text{if } \theta = -1, \end{cases} \qquad \Delta(\lambda) = \lambda I - \bigl(A + B\mathrm{e}^{-\lambda}\bigr).$$

Using this, we can rewrite the bilinear form as

$$\langle \psi, \varphi \rangle = \psi(0)\varphi(0) + \int_{-1}^{0} \psi(\xi + 1)B\varphi(\xi)\,\mathrm{d}\xi.$$

Note that, for a function $\varphi \in C$, $L\varphi = A\varphi(0) + B\varphi(-1)$. For simplicity, we still use C and C^* to represent the vector spaces on $[0,1]$ to the corresponding complex field, namely,

$$C = C\bigl([0,1], \mathbb{C}^n\bigr), \qquad C^* = C\bigl([0,1], \mathbb{C}^{n*}\bigr).$$

Since we only study the double-Hopf singularity with 1:1 resonance for system (1), we make the following assumption.

(H) $\mathcal{A}_0$ has a simple pair of purely imaginary roots $\lambda = \pm\mathrm{i}\omega$ ($\omega > 0$) with algebraic multiplicity 2 and geometric multiplicity 1.

Note that $\mathcal{A}_0$ has an eigenspace P, which is invariant under the flow (5). Let P^* be the space adjoint to P in C^*. Then C can be decomposed as $C = P \oplus Q$ where $Q = \{\varphi \in C:\ \langle \psi, \varphi \rangle = 0\ \forall \psi \in P^*\}$.

The following theorem characterizes P and P^*.

Theorem 1. *Assume that assumption* (H) *holds. Then there exist* $\phi_1^0, \phi_2^0 \in \mathbb{C}^n \setminus \{0\}$ *and* $\psi_1^0, \psi_2^0 \in \mathbb{C}^{n*} \setminus \{0\}$ *that are constant vectors* (*see in the proof*) *such that*

$$\phi_1(\theta) = \phi_1^0\mathrm{e}^{\mathrm{i}\omega\theta}, \quad \phi_2(\theta) = \bigl(\phi_2^0 + \phi_1^0\theta\bigr)\mathrm{e}^{\mathrm{i}\omega\theta}, \quad \theta \in [-1, 0],$$

and

$$\psi_1(s) = \bigl(-s\psi_2^0 + \psi_1^0\bigr)\mathrm{e}^{\mathrm{i}\omega s}, \quad \psi_2(s) = \psi_2^0\mathrm{e}^{\mathrm{i}\omega s}, \quad s \in [0, 1],$$

are (*generalized*) *eigenvectors of* $\mathcal{A}_0$ *and* $\mathcal{A}_0^*$ *corresponding to eigenvalues* $\omega\mathrm{i}$ *and* $-\omega\mathrm{i}$, *respectively*; *namely,*

$$(\mathcal{A}_0 - \mathrm{i}\omega I)\phi_1 = 0, \qquad (\mathcal{A}_0 - \mathrm{i}\omega I)\phi_2 = \phi_1,$$
$$(\mathcal{A}_0^* + \mathrm{i}\omega I)\psi_2 = 0, \qquad (\mathcal{A}_0^* + \mathrm{i}\omega I)\psi_1 = \psi_2.$$

In addition, if

$$\big(B\mathrm{e}^{-\mathrm{i}\omega} + I\big)\phi_2^0 - \frac{1}{2}B\mathrm{e}^{-\mathrm{i}\omega}\phi_1^0 \notin \mathcal{R}\big(A + B\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{i}\omega I\big),$$

then

$$\dot{\Phi} = \Phi J, \qquad \dot{\Psi} = -J\Psi,$$

where $\Phi = (\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2)$, $\Psi = (\bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2)^{\mathrm{T}}$ *and*

$$J = \begin{pmatrix} \mathrm{i}\omega & 1 & 0 & 0 \\ 0 & \mathrm{i}\omega & 0 & 0 \\ 0 & 0 & -\mathrm{i}\omega & 1 \\ 0 & 0 & 0 & -\mathrm{i}\omega \end{pmatrix}. \tag{6}$$

Furthermore, ϕ_1^0, ϕ_2^0, ψ_1^0, ψ_2^0 *can be chosen such that* $\langle\Psi, \Phi\rangle = I$.

Proof. Since $\pm\mathrm{i}\omega$ ($\omega > 0$) is a root with algebraic multiplicity 2 and geometric multiplicity 1, there exist functions ϕ_1, ϕ_2 such that

$$(\mathcal{A}_0 - \mathrm{i}\omega I)\phi_1 = 0, \qquad (\mathcal{A}_0 - \mathrm{i}\omega I)\phi_2 = \phi_1, \tag{7}$$

and the equation

$$(\mathcal{A}_0 - \mathrm{i}\omega I)\phi = \phi_2 \tag{8}$$

has no solution. Then by the definition of $\mathcal{A}_0$, we know that (7) is equivalent to

$$\begin{aligned} L\phi_1(\theta) &= \mathrm{i}\omega\phi_1(0) \quad \text{if } \theta = 0, \\ \dot{\phi}_1(\theta) &= \mathrm{i}\omega\phi_1(\theta) \quad \text{if } -1 \leqslant \theta < 0, \end{aligned}$$

and

$$\begin{aligned} L\phi_2(\theta) &= \mathrm{i}\omega\phi_2(0) + \phi_1(0) \quad \text{if } \theta = 0, \\ \dot{\phi}_2(\theta) &= \mathrm{i}\omega\phi_2(\theta) + \phi_1(\theta) \quad \text{if } -1 \leqslant \theta < 0. \end{aligned}$$

Thus ϕ_1 and ϕ_2 can be expressed as

$$\phi_1(\theta) = \mathrm{e}^{\mathrm{i}\omega\theta}\phi_1^0, \qquad \phi_2(\theta) = \mathrm{e}^{\mathrm{i}\omega\theta}\big(\theta\phi_1^0 + \phi_2^0\big),$$

where $\phi_1^0 \in \mathbb{C}^n \setminus \{0\}$ and $\phi_2^0 \in \mathbb{C}^n$ are constant vectors satisfying

$$\big(A + B\mathrm{e}^{-\mathrm{i}\omega}\big)\phi_1^0 = \mathrm{i}\omega\phi_1^0, \qquad \big(A + B\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{i}\omega I\big)\phi_2^0 = \big(B\mathrm{e}^{-\mathrm{i}\omega} + I\big)\phi_1^0.$$

Note that (8) is equivalent to

$$\begin{aligned} L\phi(\theta) &= \mathrm{i}\omega\phi(0) + \phi_2(0) \quad \text{if } \theta = 0, \\ \dot{\phi}(\theta) &= \mathrm{i}\omega\phi(\theta) + \phi_2(\theta) \quad \text{if } -1 \leqslant \theta < 0. \end{aligned}$$

The second equation above gives us $\phi(\theta) = (\phi_3^0 + \phi_2^0\theta + \phi_1^0\theta^2/2)\mathrm{e}^{\mathrm{i}\omega\theta}$, where $\phi_3^0 \in \mathbb{C}^n$ is a constant vector. Substituting this into the first equation, we see that ϕ_3^0 satisfies

$$\big(A + B\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{i}\omega I\big)\phi_3^0 = \big(B\mathrm{e}^{-\mathrm{i}\omega} + I\big)\phi_2^0 - \frac{1}{2}B\mathrm{e}^{-\mathrm{i}\omega}\phi_1^0.$$

Thus the fact that (8) has no solutions is equivalent to

$$\big(B\mathrm{e}^{-\mathrm{i}\omega} + I\big)\phi_2^0 - \frac{1}{2}B\mathrm{e}^{-\mathrm{i}\omega}\phi_1^0 \notin \mathcal{R}\big(A + B\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{i}\omega I\big).$$

Therefore $P = \operatorname{span}\{\phi_1, \phi_2, \bar{\phi}_1, \bar{\phi}_2\}$.

Similarly, let ψ_1, ψ_2 be the (generalized) eigenvectors of $\mathcal{A}_0^*$ corresponding to the eigenvalue $-\mathrm{i}\omega$; namely,

$$(\mathcal{A}_0^* + \mathrm{i}\omega I)\psi_2 = 0, \qquad (\mathcal{A}_0^* + \mathrm{i}\omega I)\psi_1 = \psi_2.$$

Then ψ_1, ψ_2 can be expressed as

$$\psi_2(s) = \mathrm{e}^{-\mathrm{i}\omega s}\psi_2^0, \qquad \psi_1(s) = \mathrm{e}^{-\mathrm{i}\omega s}\big(-s\psi_2^0 + \psi_1^0\big),$$

where $\psi_1^0 \in \mathbb{C}^{n*} \setminus \{0\}$ and $\psi_2^0 \in \mathbb{R}^{n*} \setminus \{0\}$ are constant vectors satisfying

$$\psi_2^0\big(A + B\mathrm{e}^{\mathrm{i}\omega}\big) = -\mathrm{i}\omega\psi_2^0, \qquad \psi_1^0\big(A + B\mathrm{e}^{\mathrm{i}\omega} + \mathrm{i}\omega\big) = \psi_2^0\big(B\mathrm{e}^{\mathrm{i}\omega} + I\big).$$

Therefore $P^* = \operatorname{span}\{\bar{\psi}_1, \bar{\psi}_2, \psi_1, \psi_2\}$.

Clearly,

$$\langle\bar{\psi}_1, \bar{\phi}_1\rangle = \langle\bar{\psi}_1, \phi_2\rangle = \langle\psi_1, \phi_1\rangle = \langle\psi_1, \phi_2\rangle = \langle\psi_2, \bar{\phi}_1\rangle = \langle\psi_2, \phi_1\rangle = 0,$$

and

$$\langle\bar{\psi}_1, \phi_1\rangle \neq 0, \qquad \langle\bar{\psi}_2, \phi_2\rangle \neq 0.$$

In fact, we can choose ψ_1, ψ_2 such that

$$\langle\bar{\psi}_1, \phi_1\rangle = 1, \qquad \langle\bar{\psi}_2, \phi_2\rangle = 1.$$

This finishes the proof the theorem. □

From the proof of this theorem we can get the following equivalent conditions to assumption (H).

Corollary 1. *Assumption* (H) *is equivalent to the following conditions*:

(H1) $\operatorname{rank}(A + B\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{i}\omega I) = n - 1$;

(H2) *if* $\ker(A + B\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{i}\omega I) = \operatorname{span}\{\phi_1^0\}$*, then*

$$\big(B\mathrm{e}^{-\mathrm{i}\omega} + I\big)\phi_1^0 \in \mathcal{R}\big(A + B\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{i}\omega I\big);$$

(H3) *if* $(B\mathrm{e}^{-\mathrm{i}\omega} + I)\phi_1^0 = (A + B\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{i}\omega I)\phi_2^0$*, then*

$$\big(B\mathrm{e}^{-\mathrm{i}\omega} + I\big)\phi_2^0 - \frac{1}{2}B\mathrm{e}^{-\mathrm{i}\omega}\phi_1^0 \notin \mathcal{R}\big(A + B\mathrm{e}^{-\mathrm{i}\omega} - \mathrm{i}\omega I\big).$$

3 The Faria–Magalhães normal forms

In this section, we use the idea of Faria and Magalhães [5,6] to conduct a center manifold reduction and to compute the normal form for system (1) for the double-Hopf singularity with 1:1 resonance. We assume that assumption (H) holds. Let

$$BC = \Big\{\varphi\colon [-1,0] \to \mathbb{C}^n\colon \ \varphi \text{ is continuous on } [-1,0),\ \exists \lim_{\theta\to 0^-} \varphi(\theta) \in \mathbb{C}^n\Big\}.$$

The elements of BC can be expressed as $\psi = \varphi + X_0\nu$ with $\varphi \in C,\ \nu \in \mathbb{C}^n$, where

$$X_0(\theta) = \begin{cases} 0 & \text{if } -1 \leqslant \theta < 0, \\ I & \text{if } \theta = 0, \end{cases}$$

and I is the $n \times n$ identity matrix. Define the projection $\pi\colon BC \to P$ by

$$\pi(\varphi + X_0\alpha) = \Phi\big[(\Psi, \varphi) + \Psi(0)\nu\big],$$

where Φ and Ψ are defined in Section 2. Let $F = \sum_{j\geqslant 2} F_j/j!$ and $X = \Phi x + y$ with $x = (x_1, x_2, \bar{x}_1, \bar{x}_2)^{\mathrm{T}} \in \mathbb{C}^4$ and $y \in Q^1 := \{\varphi \in Q\colon\ \dot{\varphi} \in C\}$. Then system (4) becomes

$$\begin{aligned} \dot{x} &= Jx + \sum_{j\geqslant 2} \frac{1}{j!} f_j^1(x, y, \mu), \\ \dot{y} &= \mathcal{A}_{Q^1} y + \sum_{j\geqslant 2} \frac{1}{j!} f_j^2(x, y, \mu), \end{aligned} \tag{9}$$

where J is given in (6) and

$$\begin{aligned} f_j^1(x, y, \mu) &= \Psi(0) F_j(\Phi x + y,\ \mu), \\ f_j^2(x, y, \mu) &= (I - \pi) X_0 F_j(\Phi x + y,\ \mu). \end{aligned}$$

Note that, for each j, the first and the third, and the second and the fourth components of $f_j^1(x, 0, \mu)$ are conjugate. On the center manifold, system (9) can be transformed to the following normal form:

$$\dot{x} = Jx + \frac{1}{2} g_2^1(x, 0, \mu) + \frac{1}{3!} g_3^1(x, 0, 0) + \mathcal{O}\big(|\mu||x|^2 + |x|^4\big), \tag{10}$$

where $g_j^1(x, 0, \mu)$ are homogeneous polynomials of degree j in (x, μ). Let Y be a normed space and $j \in \mathbb{N}$. Let $V_j(Y)$ be the space of homogeneous polynomials with degree j in a linear space Y. Define M_j^1, M_j^2 to be the operators in $V_j(\mathbb{C}^4)$ and $V_j(\ker \pi)$, respectively, by

$$M_j^1 p = D_x p J x - J p, \qquad M_j^2 h = D_x h J x - A_{Q^1} h$$

with $p(x) \in V_j(\mathbb{C}^4)$, $h(x)(\theta) \in V_j(\ker \pi)$. Then $V_j(\mathbb{C}^4) = \operatorname{Im}(M_j^1) \oplus (\operatorname{Im}(M_j^1))^c$. By the above decompositions, $g_2^1(x, 0, \mu)$, $g_3^1(x, 0, \mu)$ can be expressed as

$$\begin{aligned} g_2^1(x, 0, \mu) &= \operatorname{Proj}_{(\operatorname{Im}(M_2^1))^c} f_2^1(x, 0, \mu), \\ g_3^1(x, 0, 0) &= \operatorname{Proj}_{(\operatorname{Im}(M_3^1))^c} \tilde{f}_3^1(x, 0, 0), \end{aligned}$$

where

$$\tilde{f}_3^1(x,0,0) = f_3^1(x,0,0) + \frac{3}{2}\big[\big(D_x f_2^1\big)(x,0,0)U_2^1(x,0) + \big(D_y f_2^1\big)(x,0,0)U_2^2(x,0)\big],$$

and where U_2^1 and U_2^2 are determined by

$$U_2^1(x,0) = \big(M_2^1\big)^{-1}\mathrm{Proj}_{\mathrm{Im}(M_2^1)} f_2^1(x,0,0), \qquad \big(M_2^2 U_2^2\big)(x,0) = f_2^2(x,0,0).$$

Let $x = (\mathbf{x},\bar{\mathbf{x}})$, $U_2^1(x,0) = (\mathbf{U}_2^1(\mathbf{x},\bar{\mathbf{x}},0), \bar{\mathbf{U}}_2^1(\mathbf{x},\bar{\mathbf{x}},0))^{\mathrm{T}}$, where $\mathbf{x} = (x_1,x_2)^{\mathrm{T}}$. Note, for $p = (\mathbf{p},\bar{\mathbf{p}})^{\mathrm{T}}$, where $\mathbf{p} = (p_1,p_2)^{\mathrm{T}}$,

$$\begin{aligned} M_j^1\begin{pmatrix}\mathbf{p}\\ \bar{\mathbf{p}}\end{pmatrix} &= \begin{pmatrix}\mathbf{p}_{\mathbf{x}} & \mathbf{p}_{\bar{\mathbf{x}}}\\ \bar{\mathbf{p}}_{\mathbf{x}} & \bar{\mathbf{p}}_{\bar{\mathbf{x}}}\end{pmatrix}\begin{pmatrix}J' & 0\\ 0 & \bar{J}'\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \bar{\mathbf{x}}\end{pmatrix} - \begin{pmatrix}J' & 0\\ 0 & \bar{J}'\end{pmatrix}\begin{pmatrix}\mathbf{p}\\ \bar{\mathbf{p}}\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{p}_{\mathbf{x}}J'\mathbf{x} + \mathbf{p}_{\bar{\mathbf{x}}}\bar{J}'\bar{\mathbf{x}} - J'\mathbf{p}\\ \bar{\mathbf{p}}_{\mathbf{x}}J'\mathbf{x} + \bar{\mathbf{p}}_{\bar{\mathbf{x}}}\bar{J}'\bar{\mathbf{x}} - \bar{J}'\bar{\mathbf{p}}\end{pmatrix},\\ M_j^2 h &= D_x hJx - A_{Q^1}h = (h_{\mathbf{x}}, h_{\bar{\mathbf{x}}})\begin{pmatrix}J' & 0\\ 0 & \bar{J}'\end{pmatrix}\begin{pmatrix}\mathbf{x}\\ \bar{\mathbf{x}}\end{pmatrix} - A_{Q^1}h\\ &= h_{\mathbf{x}}J'\mathbf{x} + h_{\bar{\mathbf{x}}}\bar{J}'\bar{\mathbf{x}} - A_{Q^1}h, \end{aligned}$$

where $J' = \left(\begin{smallmatrix}\mathrm{i}\omega & 1\\ 0 & \mathrm{i}\omega\end{smallmatrix}\right)$. Define $\mathbf{M}_j^1$ to be the operators in $V_j(\mathbb{C}^2)$ by

$$\mathbf{M}_j^1\mathbf{p} = D_{\mathbf{x}}\mathbf{p}J'\mathbf{x} + D_{\bar{\mathbf{x}}}\mathbf{p}J'\bar{\mathbf{x}} - J'\mathbf{p}$$

with $\mathbf{p}(\mathbf{x},\bar{\mathbf{x}},\mu) \in V_j(\mathbb{C}^2)$. Then $V_j(\mathbb{C}^2) = \mathrm{Im}(\mathbf{M}_j^1) \oplus (\mathrm{Im}(\mathbf{M}_j^1))^c$. Let

$$f_j^1(x,0,\mu) = \big(\mathbf{f}_j^1(x,y,\mu), \bar{\mathbf{f}}_j^1(x,y,\mu)\big)^{\mathrm{T}}.$$

Thus system (10) can be written as the following form:

$$\dot{\mathbf{x}} = J'\mathbf{x} + \frac{1}{2}\mathbf{g}_2^1(\mathbf{x},\bar{\mathbf{x}},0,\mu) + \frac{1}{3!}\mathbf{g}_3^1(\mathbf{x},\bar{\mathbf{x}},0,0) + \text{h.o.t.}, \tag{11}$$

where

$$\begin{aligned} \mathbf{g}_2^1(\mathbf{x},\bar{\mathbf{x}},0,\mu) &= \mathrm{Proj}_{(\mathrm{Im}(\mathbf{M}_2^1))^c}\mathbf{f}_2^1(\mathbf{x},\bar{\mathbf{x}},0,\mu),\\ \mathbf{g}_3^1(\mathbf{x},\bar{\mathbf{x}},0,\mu) &= \mathrm{Proj}_{(\mathrm{Im}(\mathbf{M}_3^1))^c}\tilde{\mathbf{f}}_3^1(\mathbf{x},\bar{\mathbf{x}},0,\mu), \end{aligned}$$

where

$$\begin{aligned} \tilde{\mathbf{f}}_3^1(\mathbf{x},\bar{\mathbf{x}},0,0) = \mathbf{f}_3^1(\mathbf{x},\bar{\mathbf{x}},0,0) + \frac{3}{2}\big[&\big(D_{\mathbf{x}}\mathbf{f}_2^1\big)(\mathbf{x},\bar{\mathbf{x}},0,0)\mathbf{U}_2^1(\mathbf{x},\bar{\mathbf{x}},0)\\ &+ \big(D_{\bar{\mathbf{x}}}\mathbf{f}_2^1\big)(\mathbf{x},\bar{\mathbf{x}},0,0)\bar{\mathbf{U}}_2^1(\mathbf{x},\bar{\mathbf{x}},0)\\ &+ \big(D_y\mathbf{f}_2^1\big)(\mathbf{x},\bar{\mathbf{x}},0,0)U_2^2(\mathbf{x},\bar{\mathbf{x}},0)\big], \end{aligned}$$

where

$$\mathbf{U}_2^1(\mathbf{x},\bar{\mathbf{x}},\mathbf{0}) = \left(\mathbf{M}_{\mathbf{2}}^{\mathbf{1}}\right)^{-\mathbf{1}}\mathbf{f}_{\mathbf{2}}^{\mathbf{1}}(\mathbf{x},\bar{\mathbf{x}},\mathbf{0},\mathbf{0}).$$

Clearly, for $\mathbf{p} = (p_1, p_2)^{\mathrm{T}}$,

$$\mathbf{M}_j^1\begin{pmatrix}p_1\\p_2\end{pmatrix} = \begin{pmatrix} x_1\frac{\partial p_1}{\partial x_1}\omega\mathrm{i} + x_2(\frac{\partial p_1}{\partial x_1} + \frac{\partial p_1}{\partial x_2}\omega\mathrm{i}) - \bar{x}_1\frac{\partial p_1}{\partial \bar{x}_1}\omega\mathrm{i} + \bar{x}_2(\frac{\partial p_1}{\partial \bar{x}_1} - \frac{\partial p_1}{\partial \bar{x}_2}\omega\mathrm{i}) - p_1\omega\mathrm{i} - p_2\\ x_1\frac{\partial p_2}{\partial x_1}\omega\mathrm{i} + x_2(\frac{\partial p_2}{\partial x_1} + \frac{\partial p_2}{\partial x_2}\omega\mathrm{i}) - \bar{x}_1\frac{\partial p_2}{\partial \bar{x}_1}\omega\mathrm{i} + \bar{x}_2(\frac{\partial p_2}{\partial \bar{x}_1} - \frac{\partial p_2}{\partial \bar{x}_2}\omega\mathrm{i}) - p_2\omega\mathrm{i}\end{pmatrix}.$$

Define

$$\Psi = \begin{pmatrix}\bar{\psi}_{11}(0) & \cdots & \bar{\psi}_{1n}(0)\\ \bar{\psi}_{21}(0) & \cdots & \bar{\psi}_{2n}(0)\end{pmatrix}$$

and write

$$\begin{aligned}\frac{1}{2}F_2(\Phi x + y,\, \mu) &= \Gamma(\mu)(x_1, x_2, \bar{x}_1, \bar{x}_2)^{\mathrm{T}} + \sum_{i+j+k+l=2} A_{ijkl}x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l\\ &\quad + \sum_{j=1}^{2}\sum_{k=1}^{n}\big((B_{jk}x_j + \bar{B}_{jk}\bar{x}_j)y_k[0] + (C_{jk}x_j + \bar{C}_{jk}\bar{x}_j)y_k[-1]\big)\\ &\quad + \sum_{j=1}^{n}\big(D_j y_j^2[0] + E_j y_j^2[-1]\big) + \sum_{j,k=1}^{n} G_{jk}y_j[0]y_k[-1]\\ &\quad + \mathcal{O}\big(|\mu|(|y| + |x|) + |\mu|^2|\mathbf{x}|\big), \qquad (12)\\ \frac{1}{6}F_3(\Phi x, 0) &= \sum_{i+j+k+l=3} A_{ijk}x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l + \mathcal{O}\big(|\mu||\mathbf{x}|^2\big),\end{aligned}$$

where $\Gamma(\mu) \in \mathbb{C}^{n\times 4}$, $A_{ijkl} = (A_{ijkl}^1, \dots, A_{ijkl}^n)^{\mathrm{T}}$, $B_{jk} = (B_{jk}^1, \dots, B_{jk}^n)^{\mathrm{T}}$, $C_{jk} = (C_{jk}^1, \dots, C_{jk}^n)^{\mathrm{T}}$, $D_j = (D_j^1, \dots, D_j^n)^{\mathrm{T}}$, $E_j = (E_j^1, \dots, E_j^n)^{\mathrm{T}}$, $G_{jk} = (G_{jk}^1, \dots, G_{jk}^n)^{\mathrm{T}} \in \mathbb{C}^n$.

Now we want to obtain the explicit expressions of $\alpha, \beta, \gamma, a, b$ and c in terms of the coefficients of $\mathbf{f}_1^1(\mathbf{x}, \bar{\mathbf{x}}, 0, \mu)$ and $\tilde{\mathbf{f}}_3^1(\mathbf{x}, \bar{\mathbf{x}}, 0, \mu)$. Let

$$\begin{aligned}\mathbf{f}_2^1(\mathbf{x},\bar{\mathbf{x}},0,\mu) &= \Psi\Gamma(\mu)(x_1, x_2, \bar{x}_1, \bar{x}_2)^{\mathrm{T}} + \Psi\sum_{i+j+k+l=2} A_{ijkl}x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l\\ &= \sum_{i+j+k+l=1}\mathbf{A}_{ijkl}(\mu)x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l + \sum_{i+j+k+l=2}\mathbf{A}_{ijkl}x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l,\end{aligned}$$

$$\mathbf{f}_3^1(\mathbf{x},\bar{\mathbf{x}},0,0) = \sum_{i+j+k+l=3}\mathbf{A}_{ijkl}x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l, \quad \text{where } \mathbf{A}_{ijkl} = \begin{pmatrix}\mathbf{A}_{ijkl}^{(1)}\\ \mathbf{A}_{ijkl}^{(2)}\end{pmatrix}.$$

Note that, according to [4, 8], (10) can be written as the following normal form:

$$\dot{\mathbf{x}} = J'\mathbf{x} + x_1\begin{pmatrix}0\\P\end{pmatrix} + x_2\begin{pmatrix}0\\Q\end{pmatrix},$$

where

$$P = \alpha + \beta|x_1|^2 + \gamma(x_1\bar{x}_2 - \bar{x}_1x_2), \qquad Q = a + b|x_1|^2 + c(x_1\bar{x}_2 - \bar{x}_1x_2),$$

and α, β, γ, a, b and c are complex constants.

Note that $\mathbf{A}_{1000}, \mathbf{A}_{0100}, \mathbf{A}_{0010}$, and $\mathbf{A}_{0001}$ are linear function of μ and $\Psi = \binom{\bar{\psi}_1}{\bar{\psi}_2}$. We calculate g_1^1 first.

Lemma 1. *In fact,*

$$\mathrm{Proj}_{(\mathrm{Im}(\mathbf{M}_1^1))^c} \sum_{i+j+k+l=1} \mathbf{A}_{ijkl}(\mu)x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l = \begin{pmatrix}0\\ \alpha x_1 + a x_2\end{pmatrix},$$

$$\alpha = \mathbf{A}_{1000}^{(2)}, \qquad a = \mathbf{A}_{1000}^{(1)} + \mathbf{A}_{0100}^{(2)}.$$

Proof. Note that

$$\begin{aligned}
&\mathbf{M}_1^1\begin{pmatrix}x_1\\0\end{pmatrix} = \begin{pmatrix}-x_2\\0\end{pmatrix}, && \mathbf{M}_1^1\begin{pmatrix}x_2\\0\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix},\\
&\mathbf{M}_1^1\begin{pmatrix}\bar{x}_1\\0\end{pmatrix} = \begin{pmatrix}2\mathrm{i}\omega\bar{x}_1 - \bar{x}_2\\0\end{pmatrix}, && \mathbf{M}_1^1\begin{pmatrix}\bar{x}_2\\0\end{pmatrix} = \begin{pmatrix}2\mathrm{i}\omega\bar{x}_2\\0\end{pmatrix},\\
&\mathbf{M}_1^1\begin{pmatrix}0\\x_1\end{pmatrix} = \begin{pmatrix}x_1\\-x_2\end{pmatrix}, && \mathbf{M}_1^1\begin{pmatrix}0\\x_2\end{pmatrix} = \begin{pmatrix}x_2\\0\end{pmatrix},\\
&\mathbf{M}_1^1\begin{pmatrix}0\\\bar{x}_1\end{pmatrix} = \begin{pmatrix}\bar{x}_1\\2\mathrm{i}\omega\bar{x}_1 - \bar{x}_2\end{pmatrix}, && \mathbf{M}_1^1\begin{pmatrix}0\\\bar{x}_2\end{pmatrix} = \begin{pmatrix}\bar{x}_2\\2\mathrm{i}\omega\bar{x}_2\end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{i+j+k+l=1} \mathbf{A}_{ijkl}(\mu)x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l\\
&= \mathbf{A}_{1000}^{(1)}\begin{pmatrix}z_1\\-z_2\end{pmatrix} + \big(\mathbf{A}_{1000}^{(1)} + \mathbf{A}_{0100}^{(2)}\big)\begin{pmatrix}0\\z_2\end{pmatrix} - \mathbf{A}_{0100}^{(1)}\begin{pmatrix}-z_2\\0\end{pmatrix}\\
&\quad + \frac{1}{2\mathrm{i}\omega}\mathbf{A}_{0010}^{(1)}\begin{pmatrix}2\mathrm{i}\omega\bar{z}_1 - \bar{z}_2\\0\end{pmatrix}\\
&\quad + \frac{1}{(2\mathrm{i}\omega)^2}\left(\mathbf{A}_{0010}^{(1)} + 2\mathrm{i}\omega\mathbf{A}_{0001}^{(1)} - \frac{1}{2\mathrm{i}\omega}\mathbf{A}_{0010}^{(2)} - \mathbf{A}_{0001}^{(2)}\right)\begin{pmatrix}2\mathrm{i}\omega\bar{z}_2\\0\end{pmatrix}\\
&\quad + \mathbf{A}_{1000}^{(2)}\begin{pmatrix}0\\z_1\end{pmatrix} + \frac{1}{2\mathrm{i}\omega}\mathbf{A}_{0010}^{(2)}\begin{pmatrix}\bar{z}_1\\2\mathrm{i}\omega\bar{z}_1 - \bar{z}_2\end{pmatrix} - \frac{1}{2\mathrm{i}\omega}\mathbf{A}_{0010}^{(2)}\begin{pmatrix}\bar{z}_1\\0\end{pmatrix}\\
&\quad + \frac{1}{2\mathrm{i}\omega}\left(\frac{1}{2\mathrm{i}\omega}\mathbf{A}_{0010}^{(2)} + \mathbf{A}_{0001}^{(2)}\right)\begin{pmatrix}\bar{z}_2\\2\mathrm{i}\omega\bar{z}_2\end{pmatrix}.
\end{aligned}$$

We can see that only $\mathbf{A}_{1000}^{(2)}\binom{0}{z_1}$ and $(\mathbf{A}_{1000}^{(1)} + \mathbf{A}_{0100}^{(2)})\binom{0}{z_2}$ are not in $\mathrm{Im}(\mathbf{M}_1^1)$, and hence we obtain the result in the lemma. □

Lemma 2. *Let* $\mathbf{A} \in V_2(\mathbb{C}^2)$. *Then there exists a unique* $\mathbf{u} \in V_2(\mathbb{C}^2)$ *such that* $\mathbf{M}_2^1 u = \mathbf{A}$. *Namely,* $V_2(\mathbb{C}^2) = \mathbf{Im}(\mathbf{M}_2^1)$ *and hence* $(\mathbf{Im}(\mathbf{M}_2^1))^c \mathbf{u} = \{0\}$, *and we have*

$$\mathrm{Proj}_{(\mathbf{Im}(\mathbf{M_2^1}))^c} \sum_{i+j+k+l=2} \mathbf{A}_{ijkl} x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l = 0.$$

Proof. In fact, let

$$\mathbf{u}(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{i+j+k+l=2} u_{ijkl} x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l, \quad \text{where } u_{ijkl} = \begin{pmatrix} u_{ijkl}^{(1)} \\ u_{ijkl}^{(2)} \end{pmatrix},$$

$$\mathbf{A}(\mathbf{x}, \bar{\mathbf{x}}) = \sum_{i+j+k+l=2} \mathbf{A}_{ijkl} x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l, \quad \text{where } \mathbf{A}_{ijkl} = \begin{pmatrix} \mathbf{A}_{ijkl}^{(1)} \\ \mathbf{A}_{ijkl}^{(2)} \end{pmatrix}.$$

Using $\mathbf{M_2^1 u} = \mathbf{A}$ and after long but basic computation by Mathematica, we obtain the expressions of u_{ijkl}, $i + j + k + l = 2$:

$$\begin{aligned}
u_{2000}^{(1)} &= \frac{1}{\omega^2}\left[\mathrm{i}\omega A_{2000}^{(1)} + \mathbf{A}_{2000}^{(2)}\right], \qquad u_{2000}^{(2)} = \frac{1}{\omega^2}\mathrm{i}\mathbf{A}_{2000}^{(2)},\\
u_{1100}^{(1)} &= \frac{1}{\omega^3}\left[4\mathrm{i}\mathbf{A}_{2000}^{(2)} + \omega\left(-2\mathbf{A}_{2000}^{(1)} + \mathrm{i}\omega\mathbf{A}_{1100}^{(1)} + \mathbf{A}_{1100}^{(2)}\right)\right],\\
u_{1100}^{(2)} &= -\frac{1}{\omega^2}\left[2\mathbf{A}_{2000}^{(2)} - \mathrm{i}\omega\mathbf{A}_{1100}^{(2)}\right],\\
u_{0200}^{(1)} &= -\frac{1}{\omega^4}\left[6\mathbf{A}_{2000}^{(2)} + \omega\left(2\mathrm{i}\mathbf{A}_{2000}^{(1)} - 2\mathrm{i}\mathbf{A}_{1100}^{(2)} + \omega\left(\mathbf{A}_{1100}^{(1)} - \mathbf{A}_{0200}^{(2)} - \mathrm{i}\omega\mathbf{A}_{0200}^{(1)}\right)\right)\right],\\
u_{0200}^{(2)} &= -\frac{1}{\omega^3}\left[2\mathrm{i}\mathbf{A}_{2000}^{(2)} + \omega\left(\mathbf{A}_{1100}^{(2)} - \mathrm{i}\omega\mathbf{A}_{0200}^{(2)}\right)\right],\\
u_{0020}^{(1)} &= \frac{1}{9\omega^2}\left(\mathbf{A}_{0020}^{(2)} - 3\mathrm{i}\omega\mathbf{A}_{0020}^{(1)}\right), \qquad u_{0020}^{(2)} = -\frac{\mathrm{i}\mathbf{A}_{0020}^{(2)}}{3\omega},\\
u_{0011}^{(1)} &= \frac{1}{27\omega^3}\left[4\mathrm{i}\mathbf{A}_{0020}^{(2)} + 3\omega\left(2\mathbf{A}_{0020}^{(1)} - \mathbf{A}_{0011}^{(2)} + 3\mathrm{i}\omega\mathbf{A}_{0011}^{(1)}\right)\right],\\
u_{0011}^{(2)} &= \frac{1}{9\omega^2}\left[2\mathbf{A}_{0020}^{(2)} + 3\mathrm{i}\omega\mathbf{A}_{0011}^{(2)}\right],\\
u_{0002}^{(1)} &= \frac{1}{27\omega^4}\left[2\mathbf{A}_{0020}^{(2)} + \omega\left(-2\mathrm{i}\mathbf{A}_{0020}^{(1)} + 3\omega\left(\mathbf{A}_{0011}^{(1)} - \mathbf{A}_{0002}^{(2)} + 3\mathrm{i}\omega\mathbf{A}_{0002}^{(1)}\right)\right)\right],\\
u_{0002}^{(2)} &= \frac{1}{27\omega^3}\left[-2\mathrm{i}\mathbf{A}_{0020}^{(2)} + 3\omega\left(\mathbf{A}_{0011}^{(2)} + 3\mathrm{i}\omega\mathbf{A}_{0002}^{(2)}\right)\right],\\
u_{1010}^{(1)} &= -\frac{1}{\omega^2}\left(\mathbf{A}_{1010}^{(2)} - \mathrm{i}\omega\mathbf{A}_{1010}^{(1)}\right), \qquad u_{1010}^{(2)} = \frac{\mathrm{i}\mathbf{A}^{(2)}}{\omega},\\
u_{0101}^{(1)} &= \frac{1}{\omega^4}\left[6\mathbf{A}_{1010}^{(2)} + 2\mathrm{i}\omega\left(-\mathbf{A}_{1010}^{(1)} + \mathbf{A}_{1001}^{(2)} + \mathbf{A}_{0110}^{(2)}\right) + \omega^2\left(-\mathbf{A}_{0101}^{(2)} + \mathbf{A}_{1001}^{(1)}\right.\right.\\
&\quad \left.\left. + \mathbf{A}_{0110}^{(1)} + \mathrm{i}\omega\mathbf{A}_{0101}^{(1)}\right)\right)\Big],\\
u_{0101}^{(2)} &= \frac{1}{\omega^3}\left[-2\mathrm{i}\mathbf{A}_{1010}^{(2)} + \omega\left(\mathbf{A}_{1001}^{(2)} + \mathrm{i}\omega\mathbf{A}_{0101}^{(2)}\right)\right],
\end{aligned}$$

$$
\begin{aligned}
u_{1001}^{(1)} &= \frac{1}{\omega^3}\big[2\mathrm{i}\mathbf{A}_{1010}^{(2)} + \omega\big(\mathbf{A}_{1010}^{(1)} - \mathbf{A}_{1001}^{(2)} + \mathrm{i}\omega\mathbf{A}_{1001}^{(1)}\big)\big],\\
u_{1001}^{(2)} &= \frac{1}{\omega^2}\big(\mathbf{A}_{1010}^{(2)} + \mathrm{i}\omega\mathbf{A}_{1001}^{(2)}\big),\\
u_{0110}^{(1)} &= \frac{1}{\omega^3}\big[2\mathrm{i}\mathbf{A}_{1010}^{(2)} + \omega\big(\mathbf{A}_{1010}^{(1)} - \mathbf{A}_{0110}^{(2)} + \mathrm{i}\omega\mathbf{A}_{0110}^{(1)}\big)\big],\\
u_{0110}^{(2)} &= \frac{1}{\omega^2}\big(\mathbf{A}_{1010}^{(2)} + \mathrm{i}\omega\mathbf{A}_{0110}^{(2)}\big),
\end{aligned}
$$

and hence the proof of the lemma is complete. □

From these two lemmas we can see that

$$
\mathbf{g}_2^1(\mathbf{x}, \bar{\mathbf{x}}, 0, \mu) = \begin{pmatrix} 0 \\ \alpha x_1 + a x_2 \end{pmatrix}.
$$

Next, we compute $\mathbf{g}_3^1(\mathbf{x}, \bar{\mathbf{x}}, 0, \mu)$. Using the definition of $\mathbf{M}_3^1$ and Mathematica, we have the following result.

Lemma 3.

$$
\mathrm{Proj}_{(\mathrm{Im}(\mathbf{M}_3^1))^c} f = \begin{pmatrix} 0 \\ x_1\big[\beta|x_1|^2 + \gamma(x_1\bar{x}_2 - \bar{x}_1x_2)\big] + x_2[b|x_1|^2 + c(x_1\bar{x}_2 - \bar{x}_1x_2)\big] \end{pmatrix},
$$

where

$$
f = \sum_{i+j+k+l=3} \begin{pmatrix} a_{ijkl} \\ b_{ijkl} \end{pmatrix} x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l,
$$

$$
\beta = b_{2010}, \qquad \gamma = a_{2010} + b_{2001}, \qquad b = 3a_{2010} + b_{2001} + b_{1110},
$$

$$
c = \frac{1}{3}(2a_{2001} - 2b_{0210} - a_{1110} + b_{1101}).
$$

It is easy to get this, and hence we omit the detail. Now we have

$$
\begin{aligned}
&\frac{1}{3!}\mathrm{Proj}_{(\mathrm{Im}(\mathbf{M}_3^1))^c}\mathbf{f}_3^1(x, 0, 0)\\
&\quad = \frac{1}{3!}\mathrm{Proj}_{(\mathrm{Im}(\mathbf{M}_3^1))^c}\Psi F_3(\Phi x, 0) = \frac{1}{3!}\mathrm{Proj}_{(\mathrm{Im}(\mathbf{M}_3^1))^c}\Psi A_{ijkl}x_1^i x_2^j \bar{x}_1^k \bar{x}_2^l\\
&\quad = \begin{pmatrix} 0 \\ x_1[\beta_1|x_1|^2 + \gamma_1(x_1\bar{x}_2 - \bar{x}_1x_2)] + x_2[b_1|x_1|^2 + c_1(x_1\bar{x}_2 - \bar{x}_1x_2)] \end{pmatrix},
\end{aligned}
$$

where

$$
\begin{aligned}
\beta_1 &= \bar{\psi}_2(0)A_{2010}, \qquad \gamma_1 = \bar{\psi}_1(0)A_{2010} + \bar{\psi}_2(0)A_{2001},\\
b_1 &= 3\bar{\psi}_1(0)A_{2010} + \bar{\psi}_2(0)A_{2001} + \bar{\psi}_2(0)A_{1110},\\
c_1 &= \frac{1}{3}\big(2\bar{\psi}_1(0)A_{2001} - 2\bar{\psi}_2(0)A_{0210} - \bar{\psi}_1(0)A_{1110} + \bar{\psi}_2(0)A_{1101}\big).
\end{aligned}
$$

Next, we calculate $\mathbf{U}_2^1$ and $\mathbf{U}_2^2$, which are determined by

$$\mathbf{U}_2^1(\mathbf{x},\bar{\mathbf{x}},0) = \left(\mathbf{M}_2^1\right)^{-1}\mathrm{Proj}_{\mathrm{Im}(\mathbf{M}_2^1)}\mathbf{f}_2^1(\mathbf{x},\bar{\mathbf{x}},0,0),$$
$$\left(M_2^2 U_2^2\right)(\mathbf{x},\bar{\mathbf{x}},0) = \mathbf{f}_2^2(\mathbf{x},\bar{\mathbf{x}},0,0).$$

The expression of $\mathbf{U}_2^1(\mathbf{x},\bar{\mathbf{x}},0)$ can be attained from the proof of Lemma 1 for $\mathbf{A} = \mathbf{f}_2^1(\mathbf{x},\bar{\mathbf{x}},0,0)$. Now we work on $U_2^2(\mathbf{x},\bar{\mathbf{x}},0)$. This is the most difficult part since its computation involves solving linear systems with singular coefficient matrices.

Define $h = h(\mathbf{x},\bar{\mathbf{x}})(\theta) = U_2^2(\mathbf{x},\bar{\mathbf{x}},0)$ and write

$$\begin{aligned} h(\theta) &= \begin{pmatrix} h^{(1)}(\theta) \\ \cdots \\ h^{(n)}(\theta) \end{pmatrix} \\ &= h_{2000}x_1^2 + h_{0200}x_2^2 + h_{0020}\bar{x}_1^2 + h_{0002}\bar{x}_2^2 + h_{1010}|x_1|^2 + h_{0101}|x_2|^2 \\ &\quad + h_{1100}x_1x_2 + h_{1001}x_1\bar{x}_2 + h_{0110}x_2\bar{x}_1 + h_{0011}\bar{x}_1\bar{x}_2, \end{aligned}$$

where $h_{ijkl} \in Q^1$. Applying the definition of A_{Q^1} and π, we obtain

$$\dot{h} - (D_{\mathbf{x}}hJ'\mathbf{x} + D_{\bar{\mathbf{x}}}h\bar{J}'\bar{\mathbf{x}}) = \Phi(\theta)\Psi(0)F_2(\Phi x,0), \qquad \dot{h}(0) - Lh = F_2(\Phi x,0),$$

where $\dot{h}$ denotes the derivative of $h(\theta)$ relative to θ. Comparing the coefficients of x_1^2, x_2^2, $\bar{x}_1^2$, $\bar{x}_2^2$, $|x_1|^2$, $|x_2|^2$, x_1x_2, $x_1\bar{x}_2$, $x_2\bar{x}_1$, $\bar{x}_1\bar{x}_2$, we have that $\bar{h}_{0020} = h_{2000}$, $\bar{h}_{0002} = h_{0200}$, $\bar{h}_{0011} = h_{1100}$ and that h_{ijkl}, $i+j+k+l = 2$ satisfy the following differential equations, respectively:

$$\begin{aligned} &\dot{h}_{2000} - 2\mathrm{i}\omega h_{2000} = 2\Phi(\theta)\Psi(0)A_{2000}, \qquad \dot{h}_{2000}(0) - L(h_{2000}) = 2A_{2000}, \qquad (13)\\ &\dot{h}_{0200} - 2\mathrm{i}\omega h_{0200} - h_{1100} = 2\Phi(\theta)\Psi(0)A_{0200}, \qquad \dot{h}_{0200}(0) - L(h_{0200}) = 2A_{0200},\\ &\dot{h}_{1010} = 2\Phi(\theta)\Psi(0)A_{1010}, \qquad \dot{h}_{1010}(0) - L(h_{1010}) = 2A_{1010},\\ &\dot{h}_{0101} = h_{0110} + h_{1001} + 2\Phi(\theta)\Psi(0)A_{0101}, \qquad \dot{h}_{0101}(0) - L(h_{0101}) = 2A_{0101},\\ &\dot{h}_{1100} - 2\mathrm{i}\omega h_{1100} = 2h_{2000} + 2\Phi(\theta)\Psi(0)A_{1100}, \qquad \dot{h}_{1100}(0) - L(h_{1100}) = 2A_{1100},\\ &\dot{h}_{1001} = h_{1010} + 2\Phi(\theta)\Psi(0)A_{1010}, \qquad \dot{h}_{1001}(0) - L(h_{1001}) = 2A_{1001},\\ &\dot{h}_{0110} = h_{1010} + 2\Phi(\theta)\Psi(0)A_{0110}, \qquad \dot{h}_{0110}(0) - L(h_{0110}) = 2A_{0110}. \end{aligned}$$

By (12), we have

$$\begin{aligned} &D_y\mathbf{f}_2^1\big|_{y=0,\mu=0}U_2^2 \\ &\quad = 2\sum_{j=1}^{2}\sum_{k=1}^{n}\left[(B_{jk}x_j + \bar{B}_{jk}\bar{x}_j)h^{(k)}(0) + (C_{jk}x_j + \bar{C}_{jk}\bar{x}_j)h^{(k)}(-1)\right] \end{aligned}$$

and hence

$$\begin{aligned} &\frac{1}{4}\mathrm{Proj}_{(\mathrm{Im}(\mathbf{M}_3^1))^c}D_y\mathbf{f}_2^1\big|_{y=0,\mu=0}U_2^2 \\ &\quad = \begin{pmatrix} 0 \\ x_1\left[\beta_3|x_1|^2 + \gamma_3(x_1\bar{x}_2 - \bar{x}_1x_2)\right] + x_2\left[b_3|x_1|^2 + c_3(x_1\bar{x}_2 - \bar{x}_1x_2)\right] \end{pmatrix}, \end{aligned}$$

where

$$\beta_3 = \sum_{k=1}^{n} \bigl[B_{1k}^{(2)} h_{1010}^{(k)}(0) + \bar{B}_{1k}^{(2)} h_{2000}^{(k)}(0) + C_{1k}^{(2)} h_{1010}^{(k)}(-1) + \bar{C}_{1k}^{(2)} h_{2000}^{(k)}(-1)\bigr],$$

$$\gamma_3 = \sum_{k=1}^{n} \bigl[B_{1k}^{(1)} h_{1010}^{(k)}(0) + \bar{B}_{1k}^{(1)} h_{2000}^{(k)}(0) + B_{1k}^{(2)} h_{1001}^{(k)}(0) + \bar{B}_{2k}^{(2)} h_{2000}^{(k)}(0) + C_{1k}^{(2)} h_{1001}^{(k)}(-1) + \bar{C}_{1k}^{(1)} h_{2000}^{(k)}(-1) + C_{1k}^{(1)} h_{1010}^{(k)}(-1) + \bar{C}_{2k}^{(2)} h_{2000}^{(k)}(-1)\bigr],$$

$$b_3 = \sum_{k=1}^{n} \bigl[B_{1k}^{(2)} \bigl(h_{0110}^{(k)}(0) + h_{1001}^{(k)}(0)\bigr) + \bigl(3B_{1k}^{(1)} + B_{2k}^{(2)}\bigr) h_{1010}^{(k)}(0) + C_{1k}^{(1)} \bigl(h_{0101}^{(k)}(-1) + h_{1001}^{(k)}(-1)\bigr) + \bigl(3C_{1k}^{(1)} + C_{2k}^{(2)}\bigr) h_{1010}^{(k)}(-1) + \bar{B}_{1k}^{(2)} h_{1100}^{(k)}(0) + \bar{C}_{1k}^{(2)} h_{1100}^{(k)}(-1) + \bigl(3\bar{B}_{1k}^{(1)} + \bar{B}_{2k}^{(2)}\bigr) h_{2000}^{(k)}(0) + \bigl(3\bar{C}_{1k}^{(1)} + \bar{C}_{2k}^{(2)}\bigr) h_{2000}^{(k)}(-1)\bigr],$$

$$c_3 = \frac{1}{3} \sum_{k=1}^{n} \bigl[-\bigl(B_{1k}^{(2)} + 2B_{2k}^{(2)}\bigr) h_{0110}^{(k)}(0) + \bigl(2B_{1k}^{(1)} + B_{1k}^{(2)}\bigr) h_{1001}^{(k)}(0) + B_{1k}^{(2)} h_{0101}^{(k)}(0) - B_{2k}^{(2)} h_{1010}^{(k)}(0) - \bigl(C_{1k}^{(2)} + 2C_{2k}^{(2)}\bigr) h_{0110}^{(k)}(-1) + \bigl(2C_{1k}^{(1)} + C_{1k}^{(2)}\bigr) h_{1001}^{(k)}(-1) + C_{1k}^{(2)} h_{0101}^{(k)}(-1) - C_{2k}^{(2)} h_{1010}^{(k)}(-1) - \bar{B}_{1k}^{(2)} \bigl(2h_{0002}^{(k)}(0) + h_{1100}^{(1)}(0)\bigr) + \bar{B}_{2k}^{(2)} h_{1100}^{(k)}(0) + 2\bar{B}_{2k}^{(1)} h_{2000}^{(k)}(0) - 2\bar{C}_{1k}^{(2)} \bigl(2h_{0002}^{(k)}(-1) + h_{1100}^{(1)}(-1)\bigr) + \bar{C}_{2k}^{(2)} h_{1100}^{(k)}(-1) + 2\bar{C}_{2k}^{(1)} h_{2000}^{(k)}(-1)\bigr].$$

Next, we compute h_{2000}.

Lemma 4. *We have*

$$h_{2000}(\theta) = 2\mathrm{e}^{2\mathrm{i}\omega\theta} \int_0^\theta \mathrm{e}^{-2\mathrm{i}\omega t} \varPhi(t) \varPsi(0) A_{2000}\,\mathrm{d}t + c\mathrm{e}^{2\mathrm{i}\omega\theta},$$

where c is a constant vector, which can be determined later. See it in the proof of the lemma.

Proof. From the first equation of (13), we have

$$h_{2000}(\theta) = 2\mathrm{e}^{2\mathrm{i}\omega\theta} \int_0^\theta \mathrm{e}^{-2\mathrm{i}\omega t} \varPhi(t) \varPsi(0) A_{2000}\,\mathrm{d}t + c\,\mathrm{e}^{2\mathrm{i}\omega\theta},$$

where $c \in \mathbb{C}^2$ is a constant, and hence

$$\dot{h}_{2000}(0) = 2\varPhi(0)\varPsi(0)A_{2000} + 2\mathrm{i}\omega c$$

and

$$L(h_{2000}) = 2B \int_0^{-1} \mathrm{e}^{-2\mathrm{i}\omega(t+1)} \varPhi(t) \varPsi(0) A_{2000}\,\mathrm{d}t + L(\mathrm{e}^{2\mathrm{i}\omega\theta})c.$$

From the second equation of (13) we have

$$\begin{aligned}&\bigl(2\mathrm{i}\omega I - L\bigl(\mathrm{e}^{2\mathrm{i}\omega\theta}\bigr)\bigr)c\\&\quad= 2\bigl(I - \Phi(0)\Psi(0)\bigr)A_{2000} - 2B\int_{-1}^{0}\mathrm{e}^{-2\mathrm{i}\omega(t+1)}\Phi(t)\Psi(0)A_{2000}\,\mathrm{d}t.\end{aligned}$$

Since $2\mathrm{i}\omega$ is not an eigenvalue of L, the matrix $(2\mathrm{i}\omega I - L(\mathrm{e}^{2\mathrm{i}\omega\theta}))$ is invertible. So we have

$$\begin{aligned}c &= 2\bigl(2\mathrm{i}\omega I - L\bigl(\mathrm{e}^{2\mathrm{i}\omega\theta}\bigr)\bigr)^{-1}\\&\quad\times\left[\bigl(I - \Phi(0)\Psi(0)\bigr)A_{2000} - B\int_{-1}^{0}\mathrm{e}^{-2\mathrm{i}\omega(t+1)}\Phi(t)\Psi(0)A_{2000}\,\mathrm{d}t\right].\end{aligned}$$ □

We can solve for other h_{ijkl} similarly, and for simplicity, we omit the detail. Putting those results together, we obtain

$$\begin{aligned}\frac{1}{6}g_3^1(x,0,\mu) &= \bigl(0, x_1\bigl[(\beta_1+\beta_2+\beta_3)|x_1|^2 + (\gamma_1+\gamma_2+\gamma_3)(x_1\bar{x}_2 - \bar{x}_1x_2)\bigr]\\&\quad + x_2\bigl[(b_1+b_2+b_3)|x_1|^2 + (c_1+c_2+c_3)(x_1\bar{x}_2 - \bar{x}_1x_2)\bigr]\bigr)^{\mathrm{T}}\\&\quad + \mathcal{O}\bigl(|\mu|^2|x| + |\mu||x|^2\bigr).\end{aligned}$$

Let $\beta = \beta_1+\beta_2+\beta_3$, $\gamma = \gamma_1+\gamma_2+\gamma_3$, $b = b_1+b_2+b_3$, $c = c_1+c_2+c_3$. So, after truncating higher-order terms, we can express system (11) as

$$\begin{aligned}\dot{x}_1 &= \omega\mathrm{i}x_1 + x_2,\\\dot{x}_2 &= \omega\mathrm{i}x_2 + \alpha x_1 + ax_2 + x_1\bigl[\beta|x_1|^2 + \gamma(x_1\bar{x}_2 - \bar{x}_1x_2)\bigr]\\&\quad + x_2\bigl[b|x_1|^2 + c(x_1\bar{x}_2 - \bar{x}_1x_2)\bigr].\end{aligned}\tag{14}$$

Let $x_1 = r_1\mathrm{e}^{\mathrm{i}\omega\theta_1}$, $x_2 = r_2\mathrm{e}^{\mathrm{i}\omega\theta_2}$, $\phi = \theta_1 - \theta_2$, $a = a_R + \mathrm{i}a_I$, $b = b_R + \mathrm{i}b_I$, $c = c_R + \mathrm{i}c_I$, $\alpha = \alpha_R + \mathrm{i}\alpha_I$, $\beta = \beta_R + \mathrm{i}\beta_I$, $\gamma = \gamma_R + \mathrm{i}\gamma_I$. Then system (14) becomes

$$\begin{aligned}\dot{r}_1 &= r_2\cos(\omega\phi),\\\dot{r}_2 &= \alpha_R r_2 + \alpha_R r_1\cos(\omega\phi) - \alpha_I r_1\sin(\omega\phi) + b_R r_1^2 r_2 - 2r_1r_2^2\sin(\omega\phi)\\&\quad - \beta_I r_1^3\sin(\omega\phi) + \beta_R r_1^3\cos(\omega\phi) - \gamma_I r_1^2 r_2\sin(2\omega\phi) - \gamma_R r_1^2 r_2\\&\quad + \gamma_R r_1^2 r_2\cos(2\omega\phi),\\\dot{\phi} &= 1 - \frac{r_2\sin(\omega\phi)}{r_1} - \frac{1}{r_2}\bigl(2c_R r_1 r_2^2\sin(\omega\phi)\\&\quad + r_1\bigl(\alpha_I\cos(\omega\phi) + \alpha_R\sin(\omega\phi) + r_1^2\bigl(\beta_I\cos(\omega\phi) + \beta_R\sin(\omega\phi)\bigr)\\&\quad + r_2\bigl(\omega + a_I + b_I r_1^2 - \gamma_I r_1^2 + \gamma_I r_1^2\cos(\omega\phi) + \gamma_R r_1^2\sin(2\omega\phi)\bigr)\bigr)\bigr).\end{aligned}\tag{15}$$

This is the normal form for the double-Hopf singularity with 1:1 resonance for system (1). The number of the positive equilibrium points of system (15) corresponds to the number of limit cycles of system (14) and hence the number of limit cycles of (1).

4 Example: the van der Pol oscillator with delayed feedback

In this section, we apply the framework developed in Sections 2 and 3 to study a double-Hopf bifurcation with 1:1 resonance for Eq. (3) in Section 1. For studies of van der Pol equations, please see [1, 10–12, 14, 16, 20]. For simplicity, we assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a C^4 function such that

$$f(x) = \gamma x + \eta x^2 + \delta x^3 + \mathcal{O}\big(x^4\big).$$

Then the corresponding characteristic equation at 0 is

$$\lambda^2 - \varepsilon\lambda + 1 - \gamma \mathrm{e}^{-\lambda\tau} = 0. \tag{16}$$

Atay [1] showed that, for small $\mathrm{e} > 0$, Eq. (3) possesses both stable and unstable periodic orbits. Wei and Jiang [16] showed that Eq. (3) undergoes a Hopf bifurcation at the origin when τ passes through a sequence of critical values and then determined the direction of the Hopf bifurcation and the stability of the periodic solutions by using the normal form theory. Wu and Wang [17] showed that Eq. (3) undergoes a zero-Hopf bifurcation at the origin and gave the corresponding bifurcation diagram near critical values of τ and γ. Let $\{\xi_n\}_{n=1}^{\infty}$ be the monotonic sequence of positive solutions of the equation

$$x = \tan x.$$

Guo and Wu [8] showed that, for each $n \in \mathbb{N}$, when

$$\varepsilon = \varepsilon_n = \frac{2}{\sqrt{2 + \xi_n^2}}, \qquad \tau = \tau_n = \sqrt{2 + \xi_n^2},$$
$$\gamma = \gamma_n = \frac{2}{(2 + \xi_n^2)\cos\xi_n},$$

Eq. (16) has a pair of purely imaginary roots $\pm\omega_n\mathrm{i}$ with multiplicity 2, where

$$\omega_n = \frac{\xi_n}{\sqrt{2 + \xi_n^2}}.$$

This means that Eq. (3) exhibits double-Hopf bifurcation with 1:1 resonance. They also gave the criteria for the existence of bifurcating periodic solutions and the description of the bifurcation direction. Now we use the framework developed in Sections 2 and 3 to obtain the normal form of Eq. (3) for this singularity for $n = 1$. When $n = 1$, we have

$$\xi_1 = 4.493409457909064,$$

and calculations give

$$\varepsilon_1 = 0.42456502526584, \qquad \tau_1 = 4.710703615854709,$$
$$\gamma_1 = -0.4148884825521903, \qquad \omega_1 = 0.9538722501635843.$$

Now we use $(\varepsilon,\tau,\gamma)$ near $(\varepsilon_1,\tau_1,\gamma_1)$ as bifurcation parameter to study double-Hopf bifurcation with 1:1 resonance for Eq. (3). Let $\varepsilon=\varepsilon_1+\mu_1$, $\tau=\tau_1+\mu_2$, $\gamma=\gamma_1+\mu_3$, $x_1=x, x_2=\dot{x}$, $t\to t/\tau$ and rewrite Eq. (3) as

$$\begin{aligned}\dot{x_1}(t)&=(\tau_1+\mu_2)x_2(t),\\ \dot{x_2}(t)&=(\tau_1+\mu_2)\bigl[-x_1(t)+(\varepsilon_1+\mu_1)\bigl(1-x_1^2(t)\bigr)x_2(t)\\ &\quad+(\gamma_1+\mu_3)x_1(t-1)+\eta x_1^2(t-1)+\delta x_1^3(t-1)\bigr]\\ &\quad+\mathcal{O}\bigl(\|x\|^4\bigr).\end{aligned}\tag{17}$$

Let $X=\binom{x_1}{x_2}$. Then system (17) can be written as

$$\dot{X}(t)=LX_t+F(X_t,\mu),$$

where

$$LX_t=AX(0)+BX(-1),\qquad A=\begin{pmatrix}0&\tau_1\\-\tau_1&\tau_1\varepsilon_1\end{pmatrix},\qquad B=\begin{pmatrix}0&0\\\tau_1\gamma_1&0\end{pmatrix},$$

$$\begin{aligned}F(X_t,\mu)=\bigl(&0,-\mu_2x_1(0)+(\tau_1\mu_1+\varepsilon_1\mu_2)x_2(0)+(\tau_1\mu_3+\gamma_1\mu_2)x_1(-1)\\&-\tau_1\varepsilon_1x_1^2(0)x_2(0)+\tau_1\eta x_1^2(t-1)+\tau_1\delta x_1^3(-1)\bigr)^{\mathrm{T}}\\&+\mathcal{O}\bigl(\|x\|^4+\|\mu\|^2\|x\|\bigr).\end{aligned}$$

Now for the above L, we apply Theorem 1 to obtain

$$\begin{aligned}\varphi_1(\theta)&=\mathrm{e}^{\mathrm{i}\omega_1\tau_1\theta}\phi_1^0, &\varphi_2(\theta)&=\mathrm{e}^{\mathrm{i}\omega_1\tau_1\theta}\bigl(\theta\varphi_1^0+\phi_2^0\bigr),\\ \psi_1(s)&=\mathrm{e}^{\mathrm{i}\omega_1\tau_1 s}\bigl(-s\psi_2^0+\psi_1^0\bigr), &\psi_2(s)&=\mathrm{e}^{\mathrm{i}\omega_1\tau_1 s}\psi_2^0,\end{aligned}$$

where

$$\begin{aligned}\phi_1^0&=(1,\omega_1\mathrm{i})^{\mathrm{T}}, &\phi_2^0&=\left(1,\,\frac{1}{\tau_1}+\omega_1\mathrm{i}\right)^{\mathrm{T}},\\ \psi_1^0&=\left(c_2,\,-\frac{c_1+c_2\varepsilon_1\tau_1+\mathrm{i}c_2\tau_1\omega_1}{\tau_1(\varepsilon_1+\omega_1\mathrm{i})^2}\right), &\psi_2^0&=\left(c_1,\,-\frac{c_1}{\varepsilon_1+\omega_1\mathrm{i}}\right),\end{aligned}$$

and

$$\begin{aligned}c_1&=1.0000000000000002-0.44509631689118\mathrm{i},\\ c_2&=-0.6996851218850385+0.44509631689117\mathrm{i}.\end{aligned}$$

It is easy to check that

$$\begin{aligned}&\langle\bar{\psi}_1,\bar{\varphi}_1\rangle=\langle\bar{\psi}_1,\varphi_2\rangle=\langle\psi_1,\varphi_1\rangle=\langle\psi_1,\varphi_2\rangle=\langle\psi_2,\bar{\varphi}_1\rangle=\langle\psi_2,\varphi_1\rangle=0,\\ &\langle\bar{\psi}_1,\varphi_1\rangle=\langle\bar{\psi}_2,\varphi_2\rangle=1.\end{aligned}$$

Let

$$\varPhi=(\varphi_1,\varphi_2,\bar{\varphi}_1,\bar{\varphi}_2),\qquad \varPsi=(\bar{\psi}_1,\bar{\psi}_2,\psi_1,\psi_2)^{\mathrm{T}}.$$

Use the algorithm developed in the previous section, we are able to get

$$\alpha = -4.7107\mu_1 - 0.953872\mathrm{i}\mu_2 - (4.82057 + 1.07281\mathrm{i})\mu_3,$$
$$\beta = 2 - (0.951697 - 0.293811\mathrm{i})\eta^2 - (14.4617 + 3.21843\mathrm{i})\delta,$$
$$\gamma = (0.666667 - 0.593462\mathrm{i}) + (0.608343 + 0.0925879\mathrm{i})\eta^2 + (9.40239 + 3.21843\mathrm{i})\delta,$$
$$a = (-1.57023 + 1.39781\mathrm{i})\mu_1 - (0.495326 + 0.317957\mathrm{i})\mu_2 + (3.13413 + 1.07281\mathrm{i})\mu_3,$$
$$b = (2 - 1.78039\mathrm{i}) + (1.2711 + 0.369845\mathrm{i})\eta^2 + (28.2072 + 9.65529\mathrm{i})\delta,$$
$$c = (-0.232722 + 0.0354662\mathrm{i})\eta^2.$$

Let $x_1 = r_1 \mathrm{e}^{\mathrm{i}\omega\theta_1}$, $x_2 = r_2 \mathrm{e}^{\mathrm{i}\omega\theta_2}$, $\theta = \theta_1 - \theta_2$. Then we have

$$\dot r_1 = r_2 \cos(4.49341\phi), \tag{18$_1$}$$

$$\begin{aligned}\dot r_2 = {} & \bigl(2\cos(4.49341\theta) + \bigl(-0.951697\cos(4.49341\theta) - 0.293811\sin(4.49341\theta)\bigr)\eta^2 \\ & + \bigl(-14.4617\cos(4.49341\theta) + 3.21843\sin(4.49341\theta)\bigr)\delta\bigr) r_1^3 \\ & + \bigl(1.3 + 0.666667\cos(8.986682\theta) + 0.593462\sin(8.986682\theta) \\ & + \bigl(0.662757 + 0.608343\cos(8.986682\theta) - 0.0925879\sin(8.986682\theta)\bigr)\eta^2 \\ & + \bigl(18.8048 + 9.40239\cos(8.986682\theta) - 3.21843\sin(8.986682\theta)\bigr)\delta\bigr) r_1^2 r_2 \\ & + r_2(-1.57023\mu_1 - 0.495326\mu_2 + 3.13413\mu_3) \\ & + r_1\bigl(-0.0709324\sin(4.49341\theta)\eta^2 r_2^2 - 4.7107\cos(4.49341\theta)\mu_1 \\ & + 0.953872\sin(4.49341\theta)\mu_2 - 4.82057\cos(4.49341\theta)\mu_3 \\ & + 1.07281\sin(4.49341\theta)\mu_3\bigr),\end{aligned} \tag{18$_2$}$$

$$\begin{aligned}\dot\theta = {} & \frac{1}{r_1 r_2}\bigl(\bigl(-0.445096\sin(4.49341\theta) + \bigl(-0.065387\cos(4.49341\theta) \\ & + 0.211798\sin(4.49341\theta)\bigr)\eta^2 \\ & + \bigl(0.716255\cos(4.49341\theta) + 3.21843\sin(4.49341\theta)\bigr)\delta\bigr) r_1^4 \\ & + \bigl(0.264148 + 0.132074\cos(8.986682\theta) - 0.148365\sin(8.986682\theta) \\ & + \bigl(-0.0617031 - 0.0206053\cos(8.986682\theta) - 0.135386\sin(8.986682\theta)\bigr)\eta^2 \\ & + \bigl(-1.43251 - 0.716255\cos(8.986682\theta) - 2.09249\sin(8.986682\theta)\bigr)\delta\bigr) r_1^3 r_2 \\ & - 0.222548\sin(4.49341\theta) r_2^2 + r_1 r_2(-0.31108\mu_1 + 0.0707608\mu_2 - 0.238752\mu_3) \\ & + r_1^2\bigl(0.10358\sin(4.49341\theta)\eta^2 r_2^2 + 1.04836\sin(4.49341\theta)\mu_1 \\ & + 0.212283\cos(4.49341\theta)\mu_2 + 0.238752\cos(4.49341\theta)\mu_3 \\ & + 1.07281\sin(4.49341\theta)\mu_3\bigr)\bigr).\end{aligned} \tag{18$_3$}$$

Note that the number of positive equilibrium points corresponds to the number of limit cycles of the original system (17).

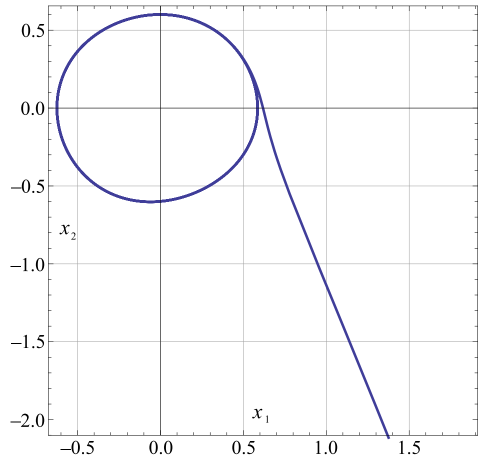

Figure 1. The VDP oscillator has one limit cycle when $\mu_1 = 0.0001$, $\mu_2 = -0.0001$, $\mu_3 = -0.0001$, $\eta = -0.1$, $\delta = 0.1$ with the initial values $x_1(t) = -0.001$, $y(t) = 0.01$ for $t \leqslant 0$.

Choose $\mu_1 = 0.0001$, $\mu_2 = -0.0001$, $\mu_3 = -0.0001$, $\eta = -0.1$, $\delta = 0.1$. Then system (18) has a positive equilibrium point

$$E(14.474,\ 32803.3,\ 0.349578),$$

and hence system (17) (or Eq. (3)) has a limit cycle as shown in Fig. 1. It is not hard to check that the eigenvalues of the Jacobian matrix at E are

$$335.994 + 3205.86\mathrm{i}, \quad 335.994 - 3205.86\mathrm{i}, \quad -333.0,$$

which implies that the limit cycle is unstable.

5 Conclusion

In this manuscript, we studied the double-Hopf singularity with 1:1 resonance for general DDEs. We characterized this complicated singularity and derived some equivalent conditions that will guarantee this singularity to occur. The corresponding normal form up to the third order was derived by using the idea of Faria and Magalhães. The unimaginable complexity and difficulty of some symbolic manipulation during the derivation were made possible by using the powerful symbolic software Mathematica. Our results were applied to a Van der Pol's oscillator with delayed feedback. The existence of a periodic solution and its stability were established.

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