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On Fučík type spectrum for problem with integral nonlocal boundary condition

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Abstract. The Fučík equation $x'' = -\mu x^+ + \lambda x^-$ with two types of nonlocal boundary value conditions are considered. The Fučík type spectrum for both problems are constructed. The visualization of the spectrum for some values of parameter γ is provided.

Keywords: Fučík type problem, Fučík spectrum, integral nonlocal condition.

1 Introduction

Investigations of spectra of differential equations with nonlocal conditions are quite fast developing area currently. Eigenvalue problems for differential operators with nonlocal boundary conditions are still less investigated than classical cases of boundary conditions.

Let us consider the Fučík equation

$$x'' = -\mu x^+ + \lambda x^-, \tag{1}$$

with one classical condition

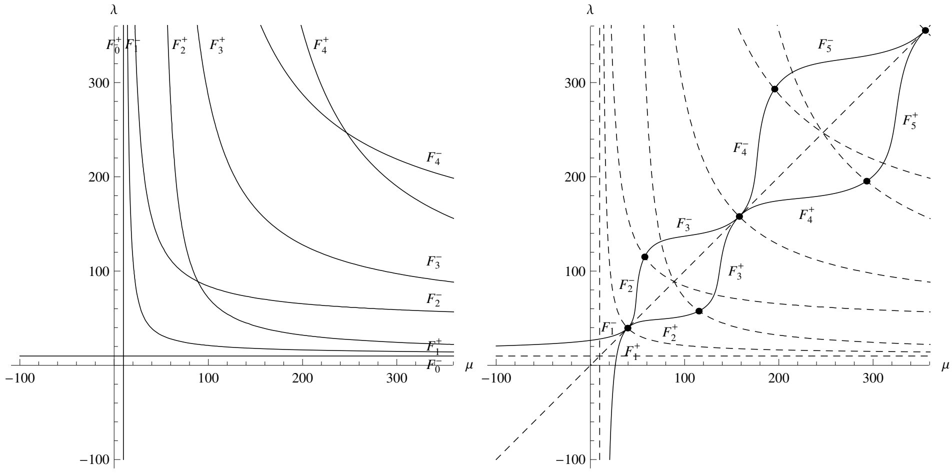
$$x(0) = 0 \tag{2}$$

and other nonlocal integral condition

$$x(1) = \gamma \int_0^{1/2} x(s)\,\mathrm{d}s, \tag{3_1}$$

$$x(1) = \gamma \int_{1/2}^{1} x(s)\,\mathrm{d}s, \tag{3_2}$$

with parameters $\mu, \lambda, \gamma \in \mathbb{R}$. Here $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$. There are analyzed two cases of nonlocal integral boundary conditions in right side of the interval.



(a) (b)

Figure 1. The spectra for the classical Fučík problem and the problem with integral condition $x(1) = \gamma \int_0^1 x(s)\mathrm{d}s$ for $\gamma \to \pm\infty$.

Let us define (1), (2), (3_1) as Problem 1, as well as (1), (2), (3_2) as Problem 2. The index in the number of a formula (e.g. in formula (3)) denotes the case. If there is no index, then the result holds on in both cases of nonlocal boundary conditions.

A set of (μ, λ) values, when the problem has nontrivial solutions, is called the Fučík spectrum. It consists of infinite set of curves (branches) F_i^+ and F_i^-, $i = 0, 1, 2, \ldots$. The lower index shows the number of zeros of the respective solution in the interval $(0; 1)$, while the upper index shows the sign of the derivative of a solution at $t = 0$. When parameter $\gamma = 0$ in (3), problem (1)–(3) reduces to the classical Fučík problem. The spectrum of such problem is well known and consists of infinite set of curves (branches), which can be obtained analytically and graphically [2]. The first five pairs of branches of the classical Fučík spectrum are presented in Fig. 1(a).

The Fučík type problem (1)–(2) with condition $x(1) = \gamma \int_0^1 x(s)\,\mathrm{d}s$ instead of condition (3) was analyzed in [10]. Graphical view of the spectrum of this problem shown in Fig. 1(b) as an example.

More general problem (1) with conditions $x(0) = \gamma \int_0^1 x(s)\,\mathrm{d}s = x(1)$ was investigated in [12].

When $\mu = \lambda$ in (1), the Fučík type problem turns into Sturm–Liouville problem

$$x'' = -\lambda x. \tag{4}$$

Sapagovas with co-authors [1,9] investigated eigenvalues for differential problem (4) with nonlocal boundary conditions $x(0) = \gamma_0 \int_0^1 \alpha_0(s)x(s)\,\mathrm{d}s$ and $x(1) = \gamma_1 \int_0^1 \alpha_1(s)x(s)\,\mathrm{d}s$ and proved that eigenvalues (constant eigenvalues) exist, which do not depend on the parameters γ_0 and γ_1 values. The spectrum of Sturm–Liouville problem (4) with various types of nonlocal boundary conditions was quite properly investigated in scientific

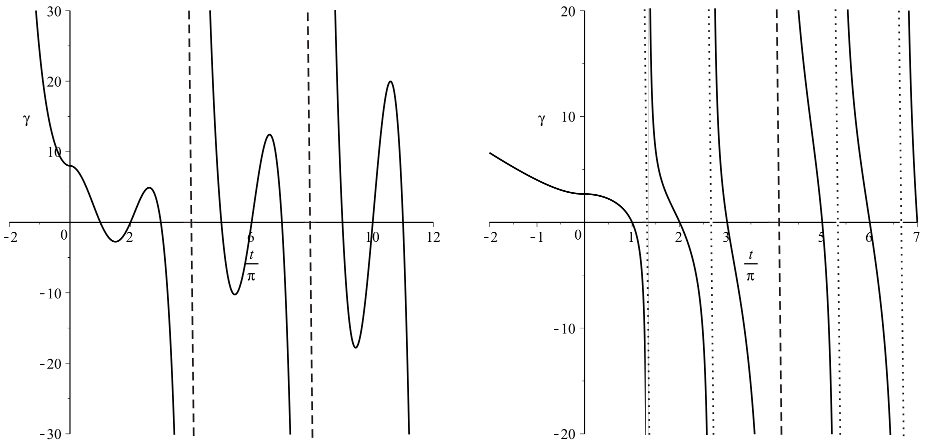


Figure 2. Characteristic functions for Sturm–Liouville problem (4) with conditions (2), (3_1) and (2), (3_2), respectively.

literature. The comprehensive analysis on results of such problems spectra is presented in [13].

There are many papers on applications of the spectrum structure of differential and difference equations with nonlocal conditions to investigation for the stability of difference schemes, for the convergence of iterative methods, for the existence of positive solutions (see, for example, [3–5, 8, 14]).

The spectrum of problems (4), (2)–(3) was deeply analysed by using the characteristic functions method in [6, 7]. Using this method, parameter γ in (3) can be expressed as a function of eigenvalue λ in (4). Let us denote $\lambda = t^2$ for $t > 0$ and $\lambda = -t^2$ for $t < 0$. Then we have formulae

$$\gamma(t) = \begin{cases} \frac{t \sinh t}{2 \sinh^2 \frac{t}{4}} & \text{when } t < 0, \\ \frac{t \sin t}{2 \sin^2 \frac{t}{4}} & \text{when } t > 0; \end{cases} \tag{5_1}$$

$$\gamma(t) = \begin{cases} \frac{t \sinh t}{2 \sinh \frac{3t}{4} \sinh \frac{t}{4}} & \text{when } t < 0, \\ \frac{t \sin t}{2 \sin \frac{3t}{4} \sin \frac{t}{4}} & \text{when } t > 0. \end{cases} \tag{5_2}$$

Graphics of characteristic functions $\gamma(t)$ are shown in Fig. 2.

In problem (4), (2), (3_1), simple, multiple and complex eigenvalues may exist for some values of parameter γ. If $\gamma > 8$, then there exists one negative eigenvalue in this case. All eigenvalues of problem (4), (2), (3_2) are real and simple. One negative eigenvalue exists if $\gamma > 8/3$ in this case. In both cases, constant eigenvalues $\lambda_k = (4\pi k)^2$, $k \in \mathbb{N}$, exist.

The purpose of this paper is to present the investigation results on the spectrum of Fučík type problem with two types of nonlocal integral boundary conditions. There are shown how spectra of these problems depend on parameter γ in nonlocal boundary conditions. In the paper, the analytical expressions and graphical visualizations of the Fučík spectra of investigated problems are provided.

2 Analytical description of the Fučík spectrum

In this section, the features of different spectral curves are presented.

2.1 On the branches $F_0^{\pm}$

Let us consider the solutions of problems (1)–(3) without zeros in the interval $(0;1)$. These solutions correspond to the branches F_0^+ and F_0^-.

Lemma 1. *The branches $F_0^{\pm}$ of the spectrum for problems* (1)–(3) *do not exist for $\gamma < 0$.*

Proof. It is clear that the solutions of problems (1)–(3) must have at least one zero in the interval $(0;1)$ in order to meet condition (3). □

The solutions of problems (1)–(3) without zeros in the interval $(0;1)$ for $\gamma > 0$ may be of three types: sine function, linear function and hyperbolic sine function. The visualization of all types of these solutions with $x'(0) > 0$ is shown in Fig. 3.

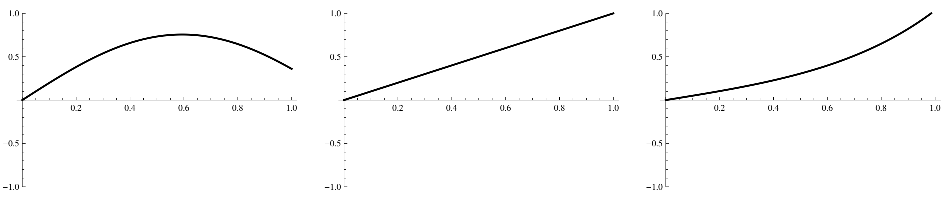
Let us analyse the solution of the considering problems without zeros in the interval $(0;1)$ when it is represented by the sine function. Without loss of generality, let $x'(0) > 0$. In this case, Problems 1 and 2 reduce to the problem $x'' = -\mu x$ with boundary conditions (2), (3_1) and (2), (3_2), accordingly.

The solution of these problems is the function $x(t) = A \sin \sqrt{\mu} t$. Taking into account that the solutions have no zeros in the interval $(0;1)$, it follows that $0 < \mu < \pi^2$. Substituting the solution into the boundary condition (3), we obtain the equations

$$\sin \sqrt{\mu} = \gamma \frac{1}{\sqrt{\mu}} \left(1 - \cos \frac{\sqrt{\mu}}{2} \right), \tag{6_1}$$

$$\sin \sqrt{\mu} = \gamma \frac{1}{\sqrt{\mu}} \left(\cos \frac{\sqrt{\mu}}{2} - \cos \sqrt{\mu} \right). \tag{6_2}$$

In order to estimate possible values of γ in both cases, consider γ as function of μ. We obtain $\gamma(\mu) = \sqrt{\mu} \sin \sqrt{\mu} / (1 - \cos \sqrt{\mu}/2)$ and $\gamma(\mu) = \sqrt{\mu} \sin \sqrt{\mu} / (\cos \sqrt{\mu}/2 - \cos \sqrt{\mu})$



(a) $0 < \mu < \pi^2$ (b) $\mu = 0$ (c) $\mu < 0$

Figure 3. The different types of solutions without zeros in the interval $(0;1)$.

from (6_1) and (6_2), respectively. Both functions decrease in the interval $\mu \in (0, \pi^2]$ and vanish at $\mu = \pi^2$. The first function tends to the maximal value $\gamma = 8$, while the second function tends to the maximal value $\gamma = 8/3$ when $\mu \to 0$.

This proves the following lemma.

Lemma 2. *The branch F_0^+ is straight line parallel to λ-axis, which is located in the first–fourth quadrants of (μ, λ)-plane for $\gamma \in [0; 8)$ in the case of Problem* 1*, for $\gamma \in [0; 8/3)$ in the case of Problem* 2*. The corresponding values of μ can be calculated from equations* (6_1) *and* (6_2)*, respectively.*

For $\mu = 0$, problems $x'' = 0$ with boundary conditions (2), (3_1) and (2), (3_2) respectively are obtained. The solution is linear function $x(t) = At$ in this case. Calculations show that this solution is possible only for $\gamma = 8$ (Problem 1) and $\gamma = 8/3$ (Problem 2).

The next result follows.

Lemma 3. *The branch F_0^+ is straight line, which coincides with λ-axis for $\gamma = 8$ in the case of Problem* 1*, for $\gamma = 8/3$ in the case of Problem* 2*.*

For negative μ, the solution of the considering problems is the hyperbolic sine function. Substituting this solution into the boundary condition (3), we obtain the equations

$$\sinh\sqrt{-\mu} = \gamma\frac{1}{\sqrt{-\mu}}\left(\cosh\frac{\sqrt{-\mu}}{2} - 1\right), \tag{7_1}$$

$$\sinh\sqrt{-\mu} = \gamma\frac{1}{\sqrt{-\mu}}\left(\cosh\sqrt{-\mu} - \cosh\frac{\sqrt{-\mu}}{2}\right). \tag{7_2}$$

Similarly as for solutions in the sine form, two functions $\gamma(\mu) = \sqrt{-\mu}\sinh\sqrt{-\mu}/(\cosh\sqrt{-\mu}/2 - 1)$ from (7_1) and $\gamma(\mu) = \sqrt{-\mu}\sinh\sqrt{-\mu}/(\cosh\sqrt{-\mu} - \cosh\sqrt{-\mu}/2)$ from (7_2) are obtained. Analysis of these functions show that the range of the first function is $(8; +\infty)$ and the range of the second function is $(8/3; +\infty)$.

This proves the following lemma.

Lemma 4. *The branch F_0^+ is straight line parallel to λ-axis, which is located in the second–third quadrants of (μ, λ)-plane for $\gamma > 8$ in the case of Problem* 1*, for $\gamma > 8/3$ in the case of Problem* 2*. The corresponding values of μ can be calculated from equations* (7_1) *and* (7_2)*, respectively.*

Remark 1. F_0^- is located symmetric to F_0^+ with respect to the bisectix of (μ, λ)-plane.

2.2 On the branches $F_1^{\pm}$

Now consider the solutions of Problem 1 (Problem 2) with one zero in the interval $(0; 1)$. Let us denote this zero by τ. These solutions correspond to the branches F_1^+ and F_1^-.

Sine function of the solution in the interval $(0; \tau)$ may be continued by sine function, linear function or hyperbolic sine function in the interval $(\tau; 1)$. The zero τ of the solution can be located before, at and behind $1/2$. So, it follows that there are nine different types

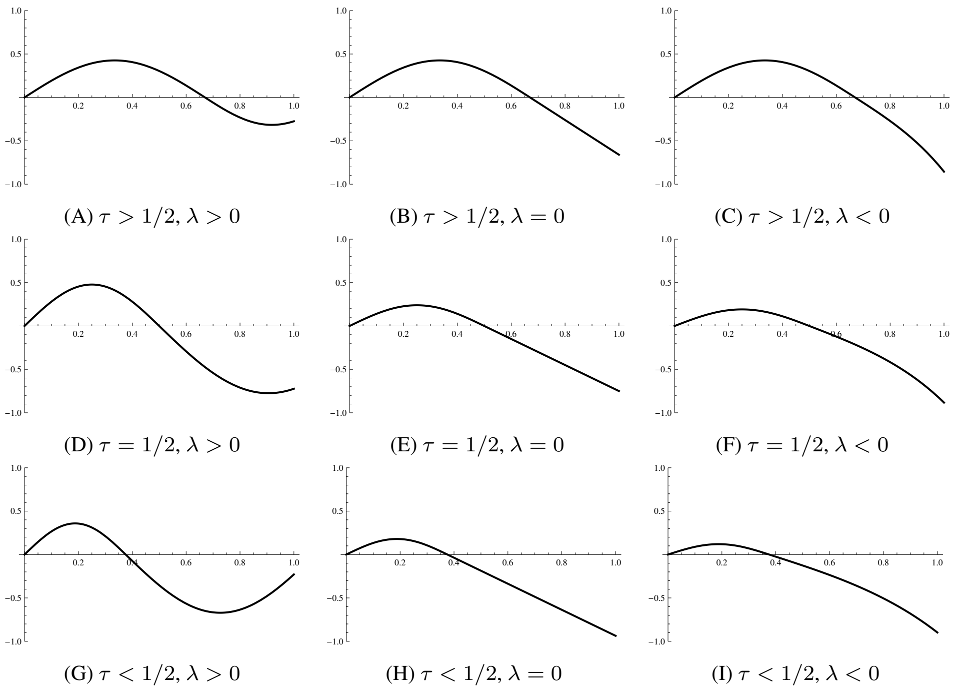


Figure 4. The different types of solutions with one zero in the interval $(0;1)$.

of solutions for Problem 1 (or Problem 2) with one zero in the interval $(0;1)$. All possible types of solutions with $x'(0)>0$ are shown in Fig. 4.

Lemma 5. *The branch $F_1^{\pm}$ of the spectrum for Problem 1 can be described with nine equations (if the respective part of branch exists for corresponding value of γ). The corresponding values of $(\mu,\lambda)\in F_1^+$ can be found from the next equations*:

$$\gamma\left(\frac{1}{\mu}-\frac{1}{\mu}\cos\frac{\sqrt{\mu}}{2}\right)+\frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}-\pi\sqrt{\frac{\lambda}{\mu}}\right)=0,$$
$$\pi^2<\mu<4\pi^2,\ \lambda>0,\ \frac{\pi}{\sqrt{\mu}}+\frac{\pi}{\sqrt{\lambda}}>1; \tag{8_A}$$

$$\gamma\left(\frac{1}{\mu}-\frac{1}{\mu}\cos\frac{\sqrt{\mu}}{2}\right)+1-\frac{\pi}{\sqrt{\mu}}=0,\quad \pi^2<\mu<4\pi^2,\ \lambda=0; \tag{8_B}$$

$$\gamma\left(\frac{1}{\mu}-\frac{1}{\mu}\cos\frac{\sqrt{\mu}}{2}\right)+\frac{1}{\sqrt{-\lambda}}\sinh\left(\sqrt{-\lambda}-\pi\sqrt{-\frac{\lambda}{\mu}}\right)=0,$$
$$\pi^2<\mu<4\pi^2,\ \lambda<0; \tag{8_C}$$

$$\gamma\frac{1}{2\pi^2}+\frac{1}{\sqrt{\lambda}}\sin\frac{\sqrt{\lambda}}{2}=0,\quad \mu=4\pi^2,\ 0<\lambda<4\pi^2; \tag{8_D}$$

$$\gamma + \pi^2 = 0, \quad \mu = 4\pi^2, \quad \lambda = 0; \tag{8_{E}}$$

$$\gamma \frac{1}{2\pi^2} + \frac{1}{\sqrt{-\lambda}} \sinh \frac{\sqrt{-\lambda}}{2} = 0, \quad \mu = 4\pi^2,\ \lambda < 0; \tag{8_{F}}$$

$$\begin{gathered} \gamma\biggl(\frac{2}{\mu} - \frac{1}{\lambda} + \frac{1}{\lambda}\cos\biggl(\frac{\sqrt{\lambda}}{2} - \pi\sqrt{\frac{\lambda}{\mu}}\biggr)\biggr) + \frac{1}{\sqrt{\lambda}}\sin\biggl(\sqrt{\lambda} - \pi\sqrt{\frac{\lambda}{\mu}}\biggr) = 0, \\ \mu > 4\pi^2,\ \lambda > 0,\ \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}} > 1; \end{gathered} \tag{8_{G}}$$

$$\gamma\biggl(\frac{2}{\mu} - \frac{1}{2}\biggl(\frac{1}{2} - \frac{\pi}{\sqrt{\mu}}\biggr)^2\biggr) + 1 - \frac{\pi}{\sqrt{\mu}} = 0, \quad \mu > 4\pi^2,\ \lambda = 0; \tag{8_{H}}$$

$$\begin{gathered} \gamma\biggl(\frac{2}{\mu} - \frac{1}{\lambda} + \frac{1}{\lambda}\cosh\biggl(\frac{\sqrt{-\lambda}}{2} - \pi\sqrt{-\frac{\lambda}{\mu}}\biggr)\biggr) \\ + \frac{1}{\sqrt{-\lambda}}\sinh\biggl(\sqrt{-\lambda} - \pi\sqrt{-\frac{\lambda}{\mu}}\biggr) = 0, \quad \mu > 4\pi^2,\ \lambda < 0. \end{gathered} \tag{8_{I}}$$

The negative spectrum branch $F_1^- = \{(\mu, \lambda) \mid (\lambda, \mu) \in F_1^+\}$.

Remark 2. Let us remark that the (μ, λ) from (8_{A}) correspond to the solutions of type A from Fig. 4, the (μ, λ) from (8_{B}) correspond to the solutions of type B and so on.

Proof of Lemma 5. The proof is similar to the previous lemmas and to the theorem, which were considered in [11]. The corresponding linear eigenvalue problems are considered on both intervals $(0; \tau)$ and $(\tau; 1)$, and the equations are obtained using required conditions.

The region, where the corresponding part of branch is located, is obtained from geometric considerations. For instance, in case A, the zero of solution $\tau = \pi/\sqrt{\mu}$ located between $1/2 < \tau < 1$. It follows that $\mu < 4\pi^2$ and $1 < \pi/\sqrt{\mu} + \pi/\sqrt{\lambda}$. Thus, the region of location (μ, λ) for (8_{A}) follows. Similar analysis is possible in cases D and G, where $\tau = 1/2$ and $\tau < 1/2$, respectively. Therefore $\mu = 4\pi^2$ and $\mu > 4\pi^2$, so the regions of location (μ, λ) for (8_{D}) and (8_{G}) follow. □

Let us analyze, which equations of (8) describe the spectrum $(\mu, \lambda) \in F_1^+$ for particular values of parameter γ. It is clear that, for negative γ values, the value of the integral $\int_0^{1/2} x(s)\,\mathrm{d}s$ in (3_1) must be positive. Thus, in all nine cases, solutions exist for negative γ values.

The result for solutions of type E (see Fig. 4(I)) follows immediately from (8_{E}), and these solutions exist only for $\gamma = -\pi^2$.

Considering the solutions of types D, F, B and H, parameter γ is expressed from (8) as a function of λ or μ, and the range of this function is calculated. Thus, the range of this function is $\gamma \in (-\pi^2; 0)$ for $0 < \lambda < 4\pi^2$ in case of type D; $\gamma \in (-\infty; -\pi^2)$ for $\lambda < 0$ in case of type F; $\gamma \in (-\pi^2; 0)$ for $\pi^2 < \mu < 4\pi^2$ in case of type B; $\gamma \in (-\infty; -\pi^2)$ for $4\pi^2 < \mu < 4(\pi+2)^2$ in case of type H (the integral $\int_0^{1/2} x(s)\,\mathrm{d}s = 0$ when $\mu = 4(\pi+2)^2$).

The value of the integral $\int_0^{1/2} x(s)\,\mathrm{d}s$ for the solutions of type A is the same as in the case of type B for the particular values of $\pi^2 < \mu < 4\pi^2$, however the value of $x(1)$ for solutions of type A is greater than the value in case of type B. Thus, it follows that the solutions of type A exist only for $\gamma \in (-\pi^2; 0)$.

The solutions of type C are bounded by the solutions of type F. It guarantees the existence of solutions for $\gamma \in (-\infty; -\pi^2)$. If μ tends to π^2, then $x(1)$ tends to 0, and γ converges to 0 also. Thus, such type of solutions exists for any negative γ value.

The solutions of type I are bounded by solutions of type F and type H. Both mentioned types of solutions exist only for $\gamma < -\pi^2$. It turns that the solutions of type I exist also only for $\gamma \in (-\infty; -\pi^2)$.

Since the solutions of type G are bounded by solutions of type H and D (with $\gamma \in (-\infty; -\pi^2)$ and $\gamma \in (-\pi^2; 0)$, accordingly), then them exist for any negative γ.

In the first six cases (from A till F, see Fig. 4), $\int_0^{1/2} x(s)\,\mathrm{d}s > 0$, $x(1) < 0$, and it turns that the solutions in these cases do not exist for $\gamma > 0$ in order to meet nonlocal condition (3_1).

It is clear that $\int_0^{1/2} x(s)\,\mathrm{d}s < 0$ for $\mu \to +\infty$. It follows that the solution of type G exists for all $\gamma > 0$.

From Lemmas 3 and 4 we obtain that solutions without zeros in the interval $(0; 1)$ in the form of a linear function and hyperbolic sine function exist only for $\gamma = 8$ and $\gamma > 8$, accordingly. The solutions of types H and I with one zero in the interval $(0; 1)$ asymptotically tend to just mentioned solutions without zeros when $\mu \to +\infty$.

Generalized results on existence of solutions in Problem 1 with particular values of parameter γ is presented in Table 1(a).

Lemma 6. *The branches $F_1^{\pm}$ of Problem 2 can be described with nine equations (if the respective part of branch exists for corresponding value of γ). The corresponding values of $(\mu, \lambda) \in F_1^{+}$ can be found from the following equations*:

$$\begin{aligned} &\gamma\left(\frac{1}{\mu} - \frac{1}{\lambda} + \frac{1}{\mu}\cos\frac{\sqrt{\mu}}{2} + \frac{1}{\lambda}\cos\left(\sqrt{\lambda} - \pi\sqrt{\frac{\lambda}{\mu}}\right)\right) \\ &\quad + \frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda} - \pi\sqrt{\frac{\lambda}{\mu}}\right) = 0, \quad \pi^2 < \mu < 4\pi^2,\ \lambda > 0,\ \frac{\pi}{\sqrt{\mu}} + \frac{\pi}{\sqrt{\lambda}} > 1; \end{aligned} \tag{9_A}$$

$$\begin{aligned} &\gamma\left(\frac{1}{\mu} + \frac{1}{\mu}\cos\frac{\sqrt{\mu}}{2} - \frac{1}{2}\left(1 - \frac{\pi}{\sqrt{\mu}}\right)^2\right) + 1 - \frac{\pi}{\sqrt{\mu}} = 0, \\ &\quad \pi^2 < \mu < 4\pi^2,\ \lambda = 0; \end{aligned} \tag{9_B}$$

$$\begin{aligned} &\gamma\left(\frac{1}{\mu} - \frac{1}{\lambda} + \frac{1}{\mu}\cos\frac{\sqrt{\mu}}{2} + \frac{1}{\lambda}\cosh\left(\sqrt{-\lambda} - \pi\sqrt{-\frac{\lambda}{\mu}}\right)\right) \\ &\quad + \frac{1}{\sqrt{-\lambda}}\sinh\left(\sqrt{-\lambda} - \pi\sqrt{-\frac{\lambda}{\mu}}\right) = 0, \quad \pi^2 < \mu < 4\pi^2,\ \lambda < 0; \end{aligned} \tag{9_C}$$

$$\gamma\left(-\frac{1}{\lambda}+\frac{1}{\lambda}\cos\frac{\sqrt{\lambda}}{2}\right)+\frac{1}{\sqrt{\lambda}}\sin\frac{\sqrt{\lambda}}{2}=0, \quad \mu=4\pi^2,\ 0<\lambda<4\pi^2; \tag{9_{D}}$$

$$\gamma-4=0, \quad \mu=4\pi^2, \quad \lambda=0; \tag{9_{E}}$$

$$\gamma\left(-\frac{1}{\lambda}+\frac{1}{\lambda}\cosh\frac{\sqrt{-\lambda}}{2}\right)+\frac{1}{\sqrt{-\lambda}}\sinh\frac{\sqrt{-\lambda}}{2}=0, \quad \mu=4\pi^2,\ \lambda<0; \tag{9_{F}}$$

$$\begin{aligned}&\gamma\left(\frac{1}{\lambda}\cos\left(\sqrt{\lambda}-\pi\sqrt{\frac{\lambda}{\mu}}\right)-\frac{1}{\lambda}\cos\left(\frac{\sqrt{\lambda}}{2}-\pi\sqrt{\frac{\lambda}{\mu}}\right)\right)\\&\quad+\frac{1}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}-\pi\sqrt{\frac{\lambda}{\mu}}\right)=0, \quad \mu>4\pi^2,\ \lambda>0,\ \frac{\pi}{\sqrt{\mu}}+\frac{\pi}{\sqrt{\lambda}}>1;\end{aligned} \tag{9_{G}}$$

$$\gamma\left(\frac{1}{2}\left(\frac{1}{2}-\frac{\pi}{\sqrt{\mu}}\right)^2-\frac{1}{2}\left(1-\frac{\pi}{\sqrt{\mu}}\right)^2\right)+1-\frac{\pi}{\sqrt{\mu}}=0, \quad \mu>4\pi^2,\ \lambda=0; \tag{9_{H}}$$

$$\begin{aligned}&\gamma\left(\frac{1}{\lambda}\cosh\left(\sqrt{-\lambda}-\pi\sqrt{-\frac{\lambda}{\mu}}\right)-\frac{1}{\lambda}\cosh\left(\frac{\sqrt{-\lambda}}{2}-\pi\sqrt{-\frac{\lambda}{\mu}}\right)\right)\\&\quad+\frac{1}{\sqrt{-\lambda}}\sinh\left(\sqrt{-\lambda}-\pi\sqrt{-\frac{\lambda}{\mu}}\right)=0, \quad \mu>4\pi^2,\ \lambda<0.\end{aligned} \tag{9_{I}}$$

The negative spectrum branch $F_1^-=\{(\mu,\lambda)\mid(\lambda,\mu)\in F_1^+\}$.

Remark 3. Let us remark that the (μ,λ) from (9_{A}) correspond to the solutions of type A from Fig. 4 and so on.

Proof of Lemma 6. The proof of this lemma is analogous to the proof of previous results. The corresponding linear eigenvalue problems are considered on both intervals $(0;\tau)$ and $(\tau;1)$, and the equations are obtained using required conditions. □

Let us analyze, which equations of (9) describe the spectrum $(\mu,\lambda)\in F_1^+$ for particular values of parameter γ. In the last six cases (from D till I, see Fig. 4), $\int_{1/2}^1 x(s)\,\mathrm{d}x<0$ and $x(1)<0$. Therefore the last six cases of solutions are not possible for $\gamma<0$ as the nonlocal condition (3_2) is not satisfied in these cases. In order to determine existence of the solutions of type A, γ is expressed from (9_{A}), and the range of this function is obtained $\gamma\in(-\infty;0)$ (in the case when the value of integral $\int_{1/2}^1 x(s)\,\mathrm{d}s$ is positive). The solutions of types B and C are bounded by solutions of type A. Thus, it follows that the solutions of these types exist for any $\gamma<0$ also.

It is clear that, for positive γ values, the value of the integral $\int_{1/2}^1 x(s)\,\mathrm{d}s$ in (3_2) must be negative. Thus, in all nine cases, solutions exist for positive γ values.

The existence of solutions of type E only for $\gamma=4$ follows from equation (9_{E}).

Considering the solutions of types D, F, B and H, parameter γ is expressed from (9) as a function of λ or μ, and the range of this function is calculated. Thus, the range of this

Table 1. Equations describing $(\mu, \lambda) \in F_1^+$ for particular γ value in: (a) Problem 1, (b) Problem 2

(a)

γ	Nos. of equations
$(-\infty; -\pi^2)$	(8_C), (8_F), (8_G), (8_H), (8_I)
$-\pi^2$	(8_C), (8_E), (8_G)
$(-\pi^2; 0)$	(8_A), (8_B), (8_C), (8_D), (8_G)
$(0; 8]$	(8_G)
$(8; +\infty)$	(8_G), (8_H), (8_I)

(b)

γ	Nos. of equations
$(-\infty; 0)$	(9_A), (9_B), (9_C)
$(0; 8/3]$	(9_A), (9_D), (9_G)
$(8/3; 4)$	(9_A), (9_D), (9_G), (9_H), (9_I)
4	(9_A), (9_E), (9_I)
$(4; +\infty)$	(9_A), (9_B), (9_C), (9_F), (9_I)

function is $\gamma \in (0; 4)$ for $0 < \lambda < 4\pi^2$ in case of type D; $\gamma \in (4; +\infty)$ for $\lambda < 0$ in case of type F; $\gamma \in (4; +\infty)$ for $\pi^2 < \mu < 4\pi^2$ (in the case when the value of integral $\int_{1/2}^1 x(s)\,\mathrm{d}s$ is negative) in case of type B; $\gamma \in (8/3; 4)$ for $4\pi^2 < \mu < +\infty$ in case of type H.

The solutions of type I are bounded by the solutions of type F and hyperbolic sine function (see Lemma 4 with $x'(0) < 0$). It follows that such type of solutions exist for $\gamma \in (8/3; +\infty)$.

The solutions of type C are bounded by solutions of type B and type F. Both mentioned types of solutions exist only for $\gamma > 4$. It proves that the solutions of type C exist also only for $\gamma > 4$.

Similarly, the solutions of type G are bounded by solutions of type D and H. It follows that the solutions of type G exist for $\gamma \in (8/3; 4)$ also.

In order to determine existence of the solutions of type A (for positive values of parameter γ), γ is expressed from (9_A), and the range of this function is obtained $\gamma \in (0; 4)$ (in the case when the value of integral $\int_{1/2}^1 x(s)\,\mathrm{d}s$ is negative). Although the solutions of type A are bounded by solutions of type B. Thus, it turns that the solutions of type A exist for any positive γ values.

Generalized results on existence of solutions in Problem 2 with particular values of parameter γ is presented in Table 1(b).

Remark 4. Equations (8_B), (8_D), (8_E), (8_F) and (8_H) (respectively (9_B), (9_D), (9_E), (9_F) and (9_H)) in Lemma 5 (respectively in Lemma 6) are degenerate equations, there are points on the μ axes or on the straight line, which is parallel to λ axes.

2.3 The features of the other branches of the spectrum

Let us consider other branches $F_i^\pm$, $i \in \mathbb{N}$, $i \neq 1$, of the spectrum for Problems 1 and 2. These branches can be grouped into $F_{4i-2}^\pm$, $F_{4i-1}^\pm$, $F_{4i}^\pm$ and $F_{4i+1}^\pm$, $i \in \mathbb{N}$.

In order to write down the equations of spectrum branches, it is important to know how many zeros are in the intervals $(0; 1/2)$ and $(1/2; 1)$. Let us denote these zeros by τ_i, $i \in \mathbb{N}$. These zeros can be calculated by using one formula out of:

$$\tau_{2i-1} = \frac{i\pi}{\sqrt{\mu}} + \frac{(i-1)\pi}{\sqrt{\lambda}}, \quad \tau_{2i} = \frac{i\pi}{\sqrt{\mu}} + \frac{i\pi}{\sqrt{\lambda}}, \quad i \in \mathbb{N}.$$

Lemma 7. *The number of zeros, which have the solutions corresponding to the branches* $F^{\pm}_{4i-2}$, $F^{\pm}_{4i-1}$, $F^{\pm}_{4i}$, $F^{\pm}_{4i+1}$ *in Problems* 1 *and* 2*, are as follow*:

Part	Branches	Number of zeroes in $(0;1/2)$	Number of zeroes in $(1/2;1)$
1	$F^{\pm}_{4i-2}$	$2i-2$	$2i$
		$2i-2$	$2i-1$ (here $\tau_{2i-1}=1/2$)
		$2i-1$	$2i-1$
2	$F^{\pm}_{4i-1}$	$2i-1$	$2i$
		$2i-1$	$2i-1$ (here $\tau_{2i}=1/2$)
3	$F^{\pm}_{4i}$	$2i$	$2i$
4	$F^{\pm}_{4i+1}$	$2i$	$2i+1$
		$2i$	$2i$ (here $\tau_{2i+1}=1/2$)
		$2i+1$	$2i$

Proof. Consider the solutions with $4i-2$ zeros in the interval $(0;1)$ (see Part 1 in the table above). It is required that $\tau_{4i-2}<1$ and $\tau_{4i-1}\geqslant 1$ in this case. Thus, it follows that such inequalities hold:

$$\frac{(2i-1)\pi}{\sqrt{\mu}}+\frac{(2i-1)\pi}{\sqrt{\lambda}}<1,\qquad \frac{2i\pi}{\sqrt{\mu}}+\frac{(2i-1)\pi}{\sqrt{\lambda}}\geqslant 1. \tag{10}$$

It is clear that the half of zeros $2i-1$ may be located in the interval $(0;1/2)$ and other half in the interval $(1/2;1)$.

Consider the case that $2i-2$ zeros are located in the interval $(0,1/2)$ and other $2i$ zeros in the interval $(1/2,1)$. It follows in this case that inequalities

$$\frac{(i-1)\pi}{\sqrt{\mu}}+\frac{(i-1)\pi}{\sqrt{\lambda}}<\frac{1}{2},\qquad \frac{i\pi}{\sqrt{\mu}}+\frac{i-1)\pi}{\sqrt{\lambda}}>\frac{1}{2} \tag{11}$$

hold. In (11), there is no contradiction with inequalities (10). Thus, it means that mentioned above location of zeros is possible.

Now consider the case that $2i$ zeros are located in the interval $(0,1/2)$ and other $2i-2$ zeros in the interval $(1/2,1)$. It follows that

$$\frac{i\pi}{\sqrt{\mu}}+\frac{i\pi}{\sqrt{\lambda}}<\frac{1}{2},\qquad \frac{(i+1)\pi}{\sqrt{\mu}}+\frac{i\pi}{\sqrt{\lambda}}>\frac{1}{2}. \tag{12}$$

The first inequality in (12) contradicts the second inequality from (10). It turns that such location of zeros is impossible. This proves Part 1 of Lemma 7.

The proofs of Parts 2, 3 and 4 of Lemma 7 are analogous to the proof of Part 1. □

Remark 5. The number of equations for each branch follows from Lemma 7. It means that the branches $F^{\pm}_{4i-2}$ can be described with three equations (Part 1); the branches $F^{\pm}_{4i-1}$ can be described with two equations (Part 2); the branches $F^{\pm}_{4i}$ – with one equation (Part 3), and the branches $F^{\pm}_{4i+1}$ – with three equations (Part 4).

In order to write the equations for spectrum branches, let us consider the solutions of Problem 1 (Problem 2) with $x'(0)>0$.

The solution of Problem 1 (Problem 2) has the form $x(t) = A\sin\sqrt{\mu}(t-\tau_{2i-2})$ in each odd interval $(\tau_{2i-2};\tau_{2i-1})$ and the form $x(t) = -A\sqrt{\mu/\lambda}\sin\sqrt{\lambda}(t-\tau_{2i-1})$ in each even interval $(\tau_{2i-1};\tau_{2i})$, where $i\in\mathbb{N}$ (let $\tau_0 = 0$).

Taking into account mentioned above, it follows that

$$x(1) = A\sin\sqrt{\mu}(1-\tau_{2i-2}) \tag{13}$$

if $t = 1$ belongs to the odd interval and

$$x(1) = -A\sqrt{\frac{\mu}{\lambda}}\sin\sqrt{\lambda}(1-\tau_{2i-1}) \tag{14}$$

if $t = 1$ belongs to the even interval.

The value of the integral in (3) in each odd interval will be equal to

$$\int\limits_{\tau_{2i-2}}^{\tau_{2i-1}} x(s)\,\mathrm{d}s = \frac{2A}{\sqrt{\mu}} \tag{15}$$

and in each even interval

$$\int\limits_{\tau_{2i-1}}^{\tau_{2i}} x(s)\,\mathrm{d}s = -\frac{2A\sqrt{\mu}}{\lambda}. \tag{16}$$

In order to write the equations for the branches of the spectrum in Problem 1, the following values of the integrals are needed:

$$\int\limits_{\tau_{2i-2}}^{1/2} x(s)\,\mathrm{d}s = -\frac{A}{\sqrt{\mu}}\left(\cos\sqrt{\mu}\left(\frac{1}{2}-\tau_{2i-2}\right)-1\right) \quad \text{if } t=\frac{1}{2}\in(\tau_{2i-2};\tau_{2i-1}); \tag{17}$$

$$\int\limits_{\tau_{2i-1}}^{1/2} x(s)\,\mathrm{d}s = \frac{A\sqrt{\mu}}{\lambda}\left(\cos\sqrt{\lambda}\left(\frac{1}{2}-\tau_{2i-1}\right)-1\right) \quad \text{if } t=\frac{1}{2}\in(\tau_{2i-1};\tau_{2i}) \tag{18}$$

and in Problem 2

$$\int\limits_{1/2}^{\tau_{2i-1}} x(s)\,\mathrm{d}s = \frac{A}{\sqrt{\mu}}\left(\cos\sqrt{\mu}(\frac{1}{2}-\tau_{2i-2})+1\right) \quad \text{if } t=\frac{1}{2}\in(\tau_{2i-2};\tau_{2i-1}); \tag{19}$$

$$\int\limits_{1/2}^{\tau_{2i}} x(s)\,\mathrm{d}s = -\frac{A\sqrt{\mu}}{\lambda}\left(\cos\sqrt{\lambda}(\frac{1}{2}-\tau_{2i-1})+1\right) \quad \text{if } t=\frac{1}{2}\in(\tau_{2i-1};\tau_{2i}). \tag{20}$$

Formulas (21) and (22) are obtained for the case τ_{2i-1}, and τ_{2i} are the last zeros of the corresponding solution of Problem 2:

$$\int\limits_{\tau_{2i-1}}^{1} x(s)\,\mathrm{d}s = \frac{A\sqrt{\mu}}{\lambda}\bigl(\cos\sqrt{\lambda}(1-\tau_{2i-1})-1\bigr); \tag{21}$$

$$\int\limits_{\tau_{2i}}^{1} x(s)\,\mathrm{d}s = -\frac{A}{\sqrt{\mu}}\bigl(\cos\sqrt{\mu}(1-\tau_{2i})-1\bigr). \tag{22}$$

Taking into account relations (13)–(22), Lemmas 8 and 9 follow.

Lemma 8. *The branches $F_i^{\pm}$ of the spectrum for Problem 1 can be described with the following equations (if the respective part of branch exists for corresponding value of γ):*

(i) *The $(\mu;\lambda) \in F_{4i-2}^{+}$ are located in the region $(2i-1)\pi/\sqrt{\mu}+(2i-1)\pi/\sqrt{\lambda} < 1$, $2i\pi/\sqrt{\mu}+(2i-1)\pi/\sqrt{\lambda} \geqslant 1$ and can be found using following equations:*

$$\gamma\left(\frac{2i-1}{\mu}-\frac{2i-2}{\lambda}-\frac{1}{\mu}\cos\sqrt{\mu}\left(\frac{1}{2}-\tau_{2i-2}\right)\right)-\frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}(1-\tau_{4i-2})=0;$$
$$\gamma\left(\frac{2i}{\mu}-\frac{2i-1}{\lambda}\right)-\frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}(1-\tau_{4i-2})=0;$$
$$\gamma\left(\frac{2i}{\mu}-\frac{2i-1}{\lambda}+\frac{1}{\lambda}\cos\sqrt{\lambda}\left(\frac{1}{2}-\tau_{2i-1}\right)\right)-\frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}(1-\tau_{4i-2})=0.$$

(ii) *The $(\mu;\lambda) \in F_{4i-1}^{+}$ are located in the region $2i\pi/\sqrt{\mu}+(2i-1)\pi/\sqrt{\lambda} < 1$, $2i\pi/\sqrt{\mu}+2i\pi/\sqrt{\lambda} \geqslant 1$ and can be found using following equations:*

$$\gamma\left(\frac{2i}{\mu}-\frac{2i-1}{\lambda}+\frac{1}{\lambda}\cos\sqrt{\lambda}\left(\frac{1}{2}-\tau_{2i-1}\right)\right)+\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(1-\tau_{4i-1})=0;$$
$$\gamma\left(\frac{2i}{\mu}-\frac{2i}{\lambda}\right)+\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(1-\tau_{4i-1})=0.$$

(iii) *The $(\mu;\lambda) \in F_{4i}^{+}$ are located in the region $2i\pi/\sqrt{\mu}+2i\pi/\sqrt{\lambda} < 1$, $(2i+1)\pi/\sqrt{\mu}+2i\pi/\sqrt{\lambda} \geqslant 1$ and can be found using the following equation:*

$$\gamma\left(\frac{2i+1}{\mu}-\frac{2i}{\lambda}-\frac{1}{\mu}\cos\sqrt{\mu}\left(\frac{1}{2}-\tau_{2i}\right)\right)-\frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}(1-\tau_{4i-1})=0.$$

(iv) *The $(\mu;\lambda) \in F_{4i+1}^{+}$ are located in the region $(2i+1)\pi/\sqrt{\mu}+2i\pi/\sqrt{\lambda} < 1$, $(2i+1)\pi/\sqrt{\mu}+(2i+1)\pi/\sqrt{\lambda} \geqslant 1$ and can be found using following equations:*

$$\gamma\left(\frac{2i+1}{\mu}-\frac{2i}{\lambda}-\frac{1}{\mu}\cos\sqrt{\mu}(\frac{1}{2}-\tau_{2i})\right)+\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(1-\tau_{4i+1})=0;$$

$$\gamma\left(\frac{2i+2}{\mu}-\frac{2i}{\lambda}\right)+\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(1-\tau_{4i+1})=0;$$
$$\gamma\left(\frac{2i+2}{\mu}-\frac{2i+1}{\lambda}+\frac{1}{\lambda}\cos\sqrt{\lambda}\left(\frac{1}{2}-\tau_{2i+1}\right)\right)+\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(1-\tau_{4i+1})=0.$$

$F_i^- = \{(\mu,\lambda) \mid (\lambda,\mu)\in F_i^+\}$.

Lemma 9. *The branches $F_i^{\pm}$ of the spectrum for Problem 2 can be described with the following equations (if the respective part of branch exists for corresponding value of γ):*

(i) *The $(\mu;\lambda)\in F_{4i-2}^+$ are located in the region $(2i-1)\pi/\sqrt{\mu}+(2i-1)\pi/\sqrt{\lambda}<1$, $2i\pi/\sqrt{\mu}+(2i-1)\pi/\sqrt{\lambda}\geqslant 1$ and can be found using following equations:*

$$\gamma\left(\frac{2i}{\mu}-\frac{2i}{\lambda}+\frac{1}{\mu}\cos\sqrt{\mu}\left(\frac{1}{2}-\tau_{2i-2}\right)-\frac{1}{\mu}\cos\sqrt{\mu}(1-\tau_{4i-2})\right)$$
$$-\frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}(1-\tau_{4i-2})=0;$$
$$\gamma\left(\frac{2i-1}{\mu}-\frac{2i}{\lambda}-\frac{1}{\mu}\cos\sqrt{\mu}(1-\tau_{4i-2})\right)-\frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}(1-\tau_{4i-2})=0;$$
$$\gamma\left(\frac{2i-1}{\mu}-\frac{2i-1}{\lambda}-\frac{1}{\lambda}\cos\sqrt{\lambda}(\frac{1}{2}-\tau_{2i-1})-\frac{1}{\mu}\cos\sqrt{\mu}(1-\tau_{4i-2})\right)$$
$$-\frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}(1-\tau_{4i-2})=0.$$

(ii) *The $(\mu;\lambda)\in F_{4i-1}^+$ are located in the region $2i\pi/\sqrt{\mu}+(2i-1)\pi/\sqrt{\lambda}<1$, $2i\pi/\sqrt{\mu}+2i\pi/\sqrt{\lambda}\geqslant 1$ and can be found using following equations:*

$$\gamma\left(\frac{2i}{\mu}-\frac{2i}{\lambda}-\frac{1}{\lambda}\cos\sqrt{\lambda}\left(\frac{1}{2}-\tau_{2i-1}\right)+\frac{1}{\lambda}\cos\sqrt{\lambda}(1-\tau_{4i-1})\right)$$
$$+\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(1-\tau_{4i-1})=0;$$
$$\gamma\left(\frac{2i}{\mu}-\frac{2i-1}{\lambda}+\frac{1}{\lambda}\cos\sqrt{\lambda}(1-\tau_{4i-1})\right)+\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(1-\tau_{4i-1})=0.$$

(iii) *The $(\mu;\lambda)\in F_{4i}^+$ are located in the region $2i\pi/\sqrt{\mu}+2i\pi/\sqrt{\lambda}<1$, $(2i+1)\pi/\sqrt{\mu}+2i\pi/\sqrt{\lambda}\geqslant 1$ and can be found using following equation:*

$$\gamma\left(\frac{2i}{\mu}-\frac{2i}{\lambda}+\frac{1}{\mu}\cos\sqrt{\mu}\left(\frac{1}{2}-\tau_{2i}\right)-\frac{1}{\mu}\cos\sqrt{\mu}(1-\tau_{4i})\right)$$
$$-\frac{1}{\sqrt{\mu}}\sin\sqrt{\mu}(1-\tau_{4i})=0.$$

(iv) *The* $(\mu;\lambda) \in F^+_{4i+1}$ *are located in the region* $(2i+1)\pi/\sqrt{\mu} + 2i\pi/\sqrt{\lambda} < 1$, $(2i+1)\pi/\sqrt{\mu} + (2i+1)\pi/\sqrt{\lambda} \geqslant 1$ *and can be found using following equations*:

$$\gamma\left(\frac{2i+1}{\mu} - \frac{2i+1}{\lambda} + \frac{1}{\mu}\cos\sqrt{\mu}\left(\frac{1}{2} - \tau_{2i}\right) + \frac{1}{\lambda}\cos\sqrt{\mu}(1-\tau_{4i+1})\right) + \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(1-\tau_{4i+1}) = 0;$$

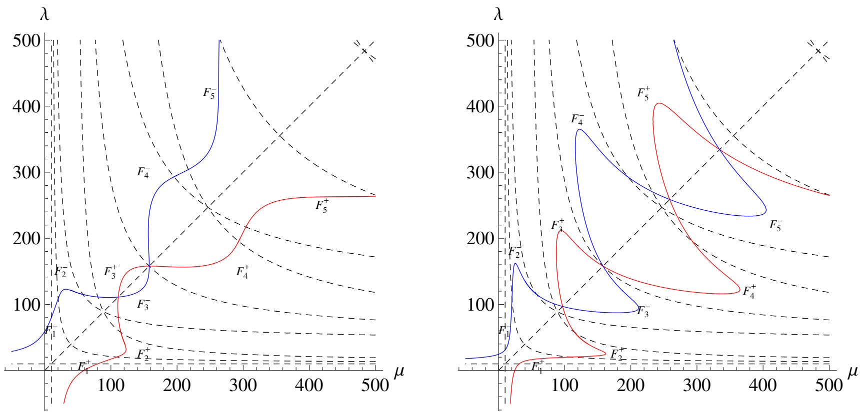
$$\gamma\left(\frac{2i}{\mu} - \frac{2i+1}{\lambda} + \frac{1}{\lambda}\cos\sqrt{\lambda}(1-\tau_{4i+1})\right) + \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(1-\tau_{4i+1}) = 0;$$

$$\gamma\left(\frac{2i}{\mu} - \frac{2i}{\lambda} - \frac{1}{\lambda}\cos\sqrt{\lambda}\left(\frac{1}{2} - \tau_{2i+1}\right) + \frac{1}{\lambda}\cos\sqrt{\lambda}(1-\tau_{4i+1})\right) + \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}(1-\tau_{4i+1}) = 0.$$

$F^-_i = \{(\mu,\lambda) \mid (\lambda,\mu) \in F^+_i\}$.

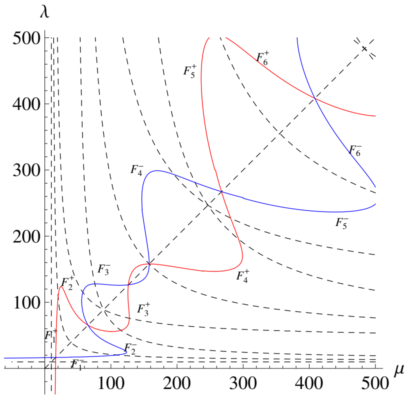
Some branches of the spectrum for Problems 1 and 2 are shown in Figs. 5, 6 and 7, 8 for some different γ values. The red curves indicate F^+_i branches and the blue ones – the F^-_i. All branches $F^\pm_i$, $i \in \mathbb{N}$, are located in the first quadrant of the coordinates plane. The branch F^+_0 is located in the first–fourth (see Figs. 7(a), 8(a), $\gamma = 2$) or in second–third (see Figs. 7(c), 8(c), $\gamma = 15$) quadrants of (μ,λ)-plane. F^-_0 is located symmetric to F^+_0. The branches of the spectra for Problems 1 and 2 are bounded by classical Fučík spectrum branches, which depicted with dashed lines in Figs. 5, 6 and 7, 8.

All real eigenvalues of the Sturm–Liouville problem (4), (2)–(3) (see Fig. 2) with particular values of parameter γ are located on the bisectrix of (μ,λ)-plane (see Figs. 5–8).

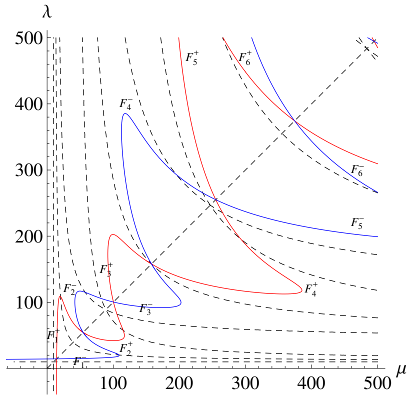


(a) $\gamma = -20$ (b) $\gamma = -5$

Figure 5. The first few spectrum curves for Problem 1 with some negative γ values.

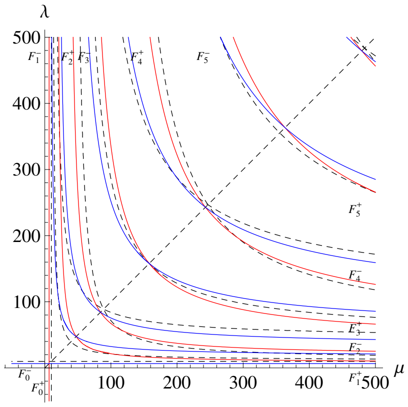


(a) $\gamma = -20$

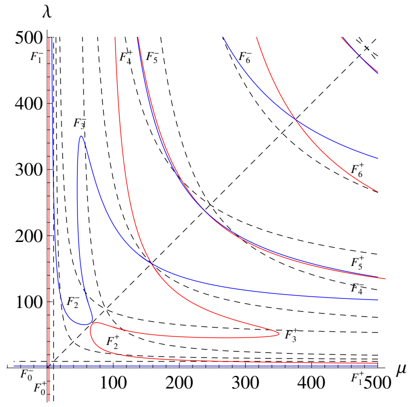


(b) $\gamma = -5$

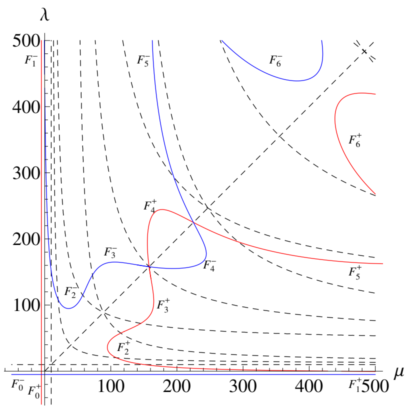
Figure 6. The first few spectrum curves for Problem 2 with some negative γ values.



(a) $\gamma = 2$

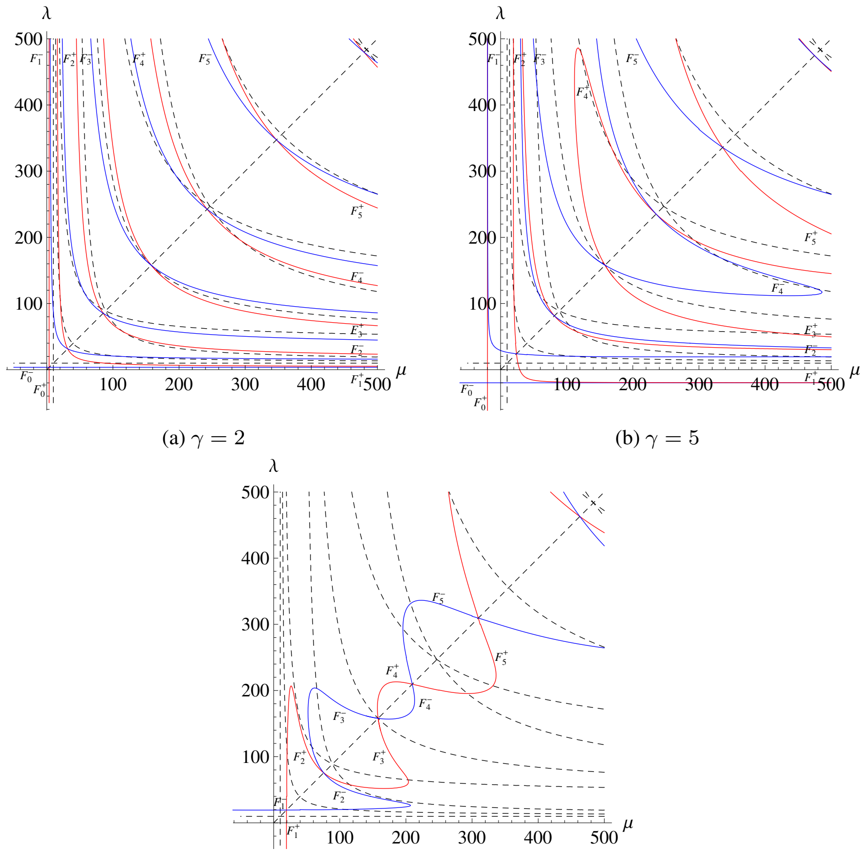


(b) $\gamma = 5$



(c) $\gamma = 15$

Figure 7. The first few spectrum curves for Problem 1 with some positive γ values.



(a) $\gamma = 2$

(b) $\gamma = 5$

(c) $\gamma = 15$

Figure 8. The first few spectrum curves for Problem 2 with some positive γ values.

3 Conclusions

1. Full analytical description of the spectrum for Problems 1 and 2 is given for all values of parameter γ in boundary conditions. The branches $F_0^{\pm}$ of the spectrum exist only for $\gamma \geqslant 0$. Analytical expressions of spectrum branches $F_i^{\pm}$, $i \in \mathbb{N}$, depend on the number of zeros of Problem 1 (Problem 2) solution located in the intervals $(0, 1/2)$ and $(1/2, 1)$.
2. The visualization of the spectrum for Problems 1 and 2 was presented for some selected values of parameter γ. The spectra differ essentially for corresponding positive and negative values of parameters.
3. The results provided in the paper generalize previously received properties of Fučík type problems with various integral conditions and Sturm–Liouville problem with various types of nonlocal boundary conditions.

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