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A nonlinear control system with a Hilfer derivative and its optimization

Rafał Kamocki

Faculty of Mathematics and Computer Science, University of Lodz,
Banacha str. 22, 90-238 Lodz, Poland
rafal.kamocki@wmii.uni.lodz.pl

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Abstract. In this work, we consider a fractional optimal control problem (FOCP) containing a nonlinear control system, described by a differential equation involving a Hilfer derivative, and an integral cost functional. We study the existence and uniqueness of a solution of the control system as well as the necessary and sufficient optimality conditions of FOCP.

Keywords: fractional integrals, Hilfer, Riemann–Lioville, Caputo fractional derivatives, maximum principle, existence of optimal solutions.

1 Introduction

Fractional calculus (FC) is nowadays a field of mathematics which generates a lot of interest of many scientists and engineers. It is a useful tool for describing of many phenomena in various fields of science. FC has been successfully applied, for example, in physics (classic and quantum mechanics, thermodynamics, optics, etc.) (cf. [23, 26, 47]), mechanics (nonconservative systems, mechanical systems including fractional oscillators, viscoelastic plane bodies and plates) (cf. [14, 35, 44]), viscoelasticity (fractional models describing behaviour of viscoelastic materials: polymers, gelatin phantoms, etc.) (cf. [10, 24, 41]), electrochemistry (ultracapacitors modelling, heat transfer models) (cf. [15, 16, 48]), medicine (fractional epidemic models) (cf. [5, 8, 9, 21]) and fractional calculus of variations (the Euler–Lagrange equations of fractional order, a fractional version of the Du Bois–Reymond lemma, fractional Noether-type theorems, etc.) (cf. [12, 13, 30, 40] and references therein). Over the last years, FC has been applied increasingly in fractional optimal control problems (FOCPs) that contain a control system described by a fractional differential equation and a performance index. The most popular fractional derivatives used in FOCPs are the Riemann–Liouville and Caputo derivatives (cf. [1, 3, 17, 22, 29, 31, 32, 42, 43]). In [1, 3], using the Lagrange multiplier technique, the authors obtained the necessary optimality conditions for FOCPs with the Riemann–Liouville derivative. Using the same technique, results of such a type were obtained also in [31] and [22]. In [31], the control system contains a fractional derivative in the Riemann–Liouville sense as well as the classical derivative of order 1. In [22], the control system involving the Caputo derivative is considered with the Bolza cost functional. Frederico and Torres [17] formulated a Noether-type theorem in the general context of the fractional optimal control in the sense of Caputo and studied fractional conservation laws in (FOCPs). In [42, 43], linear control systems with the Caputo and Riemann–Liouville derivative, respectively, were studied. Sufficient condition for controllability (controllability with memory) were derived. Some numerical methods for solving of FOCPs (including a combination of the perturbation homotopy and parameterization methods, variational iteration method, the Bezier curves method, a finite difference method) are proposed in [4, 6, 20, 36]. In [11, 38, 39], the numerical method is based on the operational matrix of the Riemann–Liouville fractional integration with the help of the Legendre orthonormal polynomial basis.

In our paper, we consider FOCP involving a different type of a fractional derivative. More precisely, we study the following fractional optimal control problem:

$$\big(D^{\alpha,\beta}_{a+}x\big)(t) = f\big(t, x(t), u(t)\big), \quad t \in [a, b] \text{ a.e.}, \tag{1}$$

$$\big(I^{(1-\alpha)(1-\beta)}_{a+}x\big)(a) = x_0, \tag{2}$$

$$u(t) \in M \subset \mathbb{R}^m, \quad t \in [a, b], \tag{3}$$

$$J(x, u) = \int_a^b f_0\big(t, x(t), u(t)\big)\,\mathrm{d}t \to \min, \tag{4}$$

where $f : [a, b] \times \mathbb{R}^n \times M \to \mathbb{R}^n$, $f_0 : [a, b] \times \mathbb{R}^n \times M \to \mathbb{R}$, $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$, $D^{\alpha,\beta}_{a+}x$ denotes a fractional differential derivative of order α and a type β given by

$$\begin{aligned}&\big(D^{\alpha,\beta}_{a+}x\big)(t)\\&\quad= D^{\alpha}_{a+}\biggl(x(\cdot) - \frac{(I^{(1-\alpha)(1-\beta)}_{a+}x)(a)}{\Gamma(\alpha+\beta(1-\alpha))(\cdot-a)^{(1-\alpha)(1-\beta)}}\biggr)(t), \quad t \in [a, b] \text{ a.e.}\end{aligned} \tag{5}$$

We see that this two parameter fractional derivative introduced in [33] is expressed through the Riemann–Liouville derivative of order α (similarly as the Caputo derivative; cf. formula (8)). Moreover, under a suitable assumption on x, it is equivalent to the left-sided Hilfer derivative introduced in [26] (cf. also [33, Thm. 10]) given by

$$\big(D^{\alpha,\beta}_{a+}x\big)(t) := \biggl(I^{\beta(1-\alpha)}_{a+}\frac{\mathrm{d}}{\mathrm{d}t}I^{(1-\alpha)(1-\beta)}_{a+}x\biggr)(t), \quad t \in [a, b] \text{ a.e.} \tag{6}$$

So, it seems natural to call also derivative (5) as the Hilfer derivative. Some properties and applications of the derivative described by (6) are given in [2, 19, 26, 27]. Also differential equations involving such a derivative are studied in [18, 34]. Based on the Banach contraction principle, theorems on the existence of a unique solution of such equations in the spaces of weighted continuous functions and "γ-absolutely continuous functions", respectively, were proved. Let us note that $D^{\alpha,\beta}_{a+}$ (in both cases) interpolates between the Riemann–Liouville ($\beta = 0$) and the Caputo derivatives ($\beta = 1$). So, if $\beta = 0$ then FOCP (1)–(4) contains the control system (1) described by the Riemann–Liouville derivative with the initial condition

$$\left(I_{a+}^{1-\alpha}x\right)(a) = x_0. \tag{7}$$

In the case of $\beta = 1$, the control system (1) is described by the Caputo derivative with the initial condition

$$x(a) = x_0.$$

It is worth to point the initial conditions (2) out. They do not have a clear physical meaning unless $\beta = 1$ and $\beta = 0$. Nevertheless, in view of numerous applications of the Hilfer derivative (for example, in modelling of anomaluos diffusion process; cf. [19, 46]), it seems that some interpretations of (2) exist. In the case of $\beta = 0$, condition (7) is interpreted as an initial memory (cf. [43]) in the context of fractional optimal control. The first to introduce this condition in the study of fractional problems of the calculus of variations were Almeida and Torres (cf. [7, Thm. 6.4]). Several physical interpretations (from the field of viscoelasticity) of this condition are demonstrated in [25].

Problem (1)–(4) (with the Riemann–Liouville derivative) was investigated in [29] and [32]. Based on the implicit function theorem for multivalued mappings, Idczak and Kamocki [29] formulated and proved a theorem on the existence of optimal solutions for such a problem. In [32], using a smooth-convex extremum principle, the necessary optimality conditions were derived. Both results were obtained under a general assumption of a convexity of the so called extended velocities set.

In this work, we study the existence and uniqueness of a solution of system (1)–(3) as well as the necessary and sufficient optimality conditions for problem (1)–(4). To the best knowledge of the author, results of such a type for such a problem have not been obtained yet. In our investigations, the Hilfer derivative given by (5) plays a key role. Due to this definition and a choice of a suitable space of functions (solutions), we can replace the main problem (1)–(4) with an equivalent problem (11)–(14) involving the Riemann–Liouville derivative. Next, using results proved in [29] and [31] for problem (11)–(14), we immediately obtain results of such a type for the main problem (1)–(4). So, we see that, due to such an approach, we need not prove all results of this paper directly (in the case of using formula (6), it would be necessary, and then all proofs would be long and complicated).

The paper is organized as follows. Section 2 contains basic notions and necessary facts concerning fractional calculus. In Section 3, we formulate and prove the main results of this work, mentioned necessary and sufficient optimality conditions for FOCP (1)–(4) and a theorem on the existence of a unique solution to the control system (1)–(3). A theoretical illustrative example is presented in Section 4. We end with Section 5 of conclusions.

2 Preliminaries

In this section, we present necessary notations, definitions and some properties concerning fractional derivatives and integrals that will be used throughout this paper (cf. [18,37,45]).

We shall assume that $[a,b] \subset \mathbb{R}$ is any bounded interval.

Let $\alpha > 0$ and $f \in L^1([a,b],\mathbb{R}^n)$. The functions

$$\big(I^\alpha_{a+} f\big)(t) := \frac{1}{\Gamma(\alpha)} \int\limits_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau, \quad t \in [a,b] \text{ a.e.},$$

$$\big(I^\alpha_{b-} f\big)(t) := \frac{1}{\Gamma(\alpha)} \int\limits_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}}\,\mathrm{d}\tau, \quad t \in [a,b] \text{ a.e.}$$

are called the left- and the right-sided Riemann–Liouville integral of the function f of order α, respectively.

In [45, (2.21)] the following semigroup properties for fractional integrals are given.

Lemma 1. *If $\alpha_1 > 0$, $\alpha_2 > 0$ and $f \in L^1([a,b],\mathbb{R}^n)$ then*

$$\big(I^{\alpha_1}_{a+} I^{\alpha_2}_{a+} f\big)(t) = \big(I^{\alpha_1+\alpha_2}_{a+} f\big)(t), \quad t \in [a,b] \textit{ a.e.},$$
$$\big(I^{\alpha_1}_{b-} I^{\alpha_2}_{b-} f\big)(t) = \big(I^{\alpha_1+\alpha_2}_{b-} f\big)(t), \quad t \in [a,b] \textit{ a.e.}$$

Let $1 \leqslant p < \infty$. By $I^\alpha_{a+}(L^p)$ ($I^\alpha_{b-}(L^p)$) we denote the space

$$I^\alpha_{a+}(L^p) := \big\{ f : [a,b] \to \mathbb{R}^n\colon\ f = I^\alpha_{a+} g \text{ a.e. on } [a,b],\ g \in L^p\big([a,b],\mathbb{R}^n\big)\big\}$$
$$\big(I^\alpha_{b-}(L^p) := \big\{ f : [a,b] \to \mathbb{R}^n\colon\ f = I^\alpha_{b-} g \text{ a.e. on } [a,b],\ g \in L^p\big([a,b],\mathbb{R}^n\big)\big\}\big).$$

We identify functions belonging to the spaces $I^\alpha_{a+}(L^p)$, $I^\alpha_{b-}(L^p)$ and equal almost everywhere on $[a,b]$.

We shall use the following characterization of the space $I^\alpha_{a+}(L^p)$ (cf. [32, Prop. 2]).

Proposition 1. *Let $f \in L^1([a,b],\mathbb{R}^n)$. Then*

$$f \in I^\alpha_{a+}\big(L^p\big) \quad \Longleftrightarrow \quad I^{1-\alpha}_{a+} f \in AC^p\big([a,b],\mathbb{R}^n\big) \quad \textit{and} \quad \big(I^{1-\alpha}_{a+} f\big)(a) = 0,$$

where

$$AC^p\big([a,b],\mathbb{R}^n\big) = \big\{ f \in AC\big([a,b],\mathbb{R}^n\big)\colon\ f' \in L^p\big([a,b],\mathbb{R}^n\big)\big\}.$$

In the rest of this article, we shall assume that $\alpha \in (0,1)$.

Let $f \in L^1([a,b],\mathbb{R}^n)$. The *left-sided Riemann–Liouville derivative $D^\alpha_{a+} f$ (right-sided Riemann–Liouville derivative $D^\alpha_{b-} f$) of order α of f* is defined by

$$\big(D^\alpha_{a+} f\big)(t) := \frac{\mathrm{d}}{\mathrm{d}t}\big(I^{1-\alpha}_{a+} f\big)(t), \quad t \in [a,b] \text{ a.e.}$$
$$\bigg(\big(D^\alpha_{b-} f\big)(t) := -\frac{\mathrm{d}}{\mathrm{d}t}\big(I^{1-\alpha}_{b-} f\big)(t), \quad t \in [a,b] \text{ a.e.}\bigg),$$

provided that the function $I^{1-\alpha}_{a+} f$ is absolutely continuous on $[a,b]$ (the function $I^{1-\alpha}_{b-} f \in AC([a,b],\mathbb{R}^n)$).

By the left-sided (right-sided) Caputo derivative of order α of the function f on the interval $[a,b]$ we mean a function ${}^{C}D^{\alpha}_{a+}f$ (${}^{C}D^{\alpha}_{b-}f$) given by

$$\begin{aligned} \left({}^{C}D^{\alpha}_{a+}f\right)(t) &:= D^{\alpha}_{a+}\left(f(\cdot)-f(a)\right)(t), \quad t\in[a,b] \text{ a.e.} \\ \left(\left({}^{C}D^{\alpha}_{b-}f\right)(t) \right.&:= \left. D^{\alpha}_{b-}\left(f(\cdot)-f(b)\right)(t), \quad t\in[a,b] \text{ a.e.}\right), \end{aligned} \tag{8}$$

provided that derivatives in the Riemann–Liouville sense on the right side exist. Let $\beta\in[0,1]$ and $f\in L^1([a,b],\mathbb{R}^n)$. We say that the function f possesses the *left-sided generalized Riemann–Liouville derivative* (*so called Hilfer derivative*) $D^{\alpha,\beta}_{a+}f$ *of order* α *and a type* β if the function $I^{(1-\alpha)(1-\beta)}_{a+}f$ is absolutely continuous on $[a,b]$ and then

$$\left(D^{\alpha,\beta}_{a+}f\right)(t) := \left(I^{\beta(1-\alpha)}_{a+}\frac{\mathrm{d}}{\mathrm{d}t}I^{(1-\alpha)(1-\beta)}_{a+}f\right)(t), \quad t\in[a,b] \text{ a.e.} \tag{9}$$

The operator $D^{\alpha,\beta}_{a+}f$ given by (9) was indroduced by Hilfer in [26].

Remark 1. Similarly, we can define the right-sided Hilfer derivative $D^{\alpha,\beta}_{b-}f$, i.e.

$$\left(D^{\alpha,\beta}_{b-}f\right)(t) := -\left(I^{\beta(1-\alpha)}_{b-}\frac{\mathrm{d}}{\mathrm{d}t}I^{(1-\alpha)(1-\beta)}_{b-}f\right)(t), \quad t\in[a,b] \text{ a.e.},$$

provided that the function $I^{(1-\alpha)(1-\beta)}_{b-}f$ is absolutely continuous on $[a,b]$.

Let $\eta\in(0,1)$. By $AC^{\eta}_{a+}([a,b],\mathbb{R}^n)$ (briefly AC^{η}_{a+}) we denote the set of all functions $f:[a,b]\to\mathbb{R}^n$ that have the following representation:

$$f(t)=\frac{c}{\Gamma(\eta)}(t-a)^{\eta-1}+\left(I^{\eta}_{a+}\varphi\right)(t), \quad t\in[a,b] \text{ a.e.},$$

for some $c\in\mathbb{R}^n$ and $\varphi\in L^1([a,b],\mathbb{R}^n)$.

If $f\in AC^{\alpha+\beta(1-\alpha)}_{a+}$ then the left-sided Hilfer derivative can be equivalently defined as (cf. [33, Thm. 10])

$$\begin{aligned} &\left(D^{\alpha,\beta}_{a+}f\right)(t) \\ &\quad = D^{\alpha}_{a+}\left(f(\cdot)-\frac{(I^{(1-\alpha)(1-\beta)}_{a+}f)(a)}{\Gamma(\alpha+\beta(1-\alpha))(\cdot-a)^{(1-\alpha)(1-\beta)}}\right)(t), \quad t\in[a,b] \text{ a.e.} \end{aligned} \tag{10}$$

Let us note that the above derivative is well defined if there exists the Riemann–Liouville derivative on the right side. So, definition (10) is correct for the function f which needn't belong to the space $AC^{\alpha+\beta(1-\alpha)}_{a+}$ (f can be a "less regular function" – more precisely, f can be a non-$(\alpha+\beta(1-\alpha))$-differentiable in the Riemann–Liouville sense function) (we shall see that this fact will be very helpful in the study of our problem). Of course, then both definitions ((9) and (10)) are not equivalent. Newertheless, in my opinion, we can still name the derivative given by (10) as the Hilfer derivative (similarly as in the case of a definition of the Caputo derivative; see formulae (2.4.4) and (2.4.17) in [37]).

In the rest of this paper, we shall use the Hilfer derivative operator given by formula (10) and defined on the appropriate set of functions.

3 Main results

In this section, we study existence and uniqueness of a solution of (1)–(3) as well as the necessary and sufficient optimality conditions for problem (1)–(4) are formulated.

3.1 Homogeneous problem

We start with the study of problem (1)–(4) with zero initial condition. So, let us consider the following:

$$\big(D^{\alpha,\beta}_{a^+}x\big)(t) = f\big(t, x(t), u(t)\big), \quad t \in [a,b] \text{ a.e.,} \tag{11}$$

$$\big(I^{(1-\alpha)(1-\beta)}_{a^+}x\big)(a) = 0, \tag{12}$$

$$u(t) \in M \subset \mathbb{R}^m, \quad t \in [a,b], \tag{13}$$

$$J(x,u) = \int_a^b f_0\big(t, x(t), u(t)\big)\,\mathrm{d}t \to \min, \tag{14}$$

where $f : [a,b] \times \mathbb{R}^n \times M \to \mathbb{R}^n$, $f_0 : [a,b] \times \mathbb{R}^n \times M \to \mathbb{R}$, $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$.

Let $1 \leqslant p < \infty$. In a whole paper, we consider the following set of controls:

$$\mathcal{U}_M := \big\{u \in L^p\big([a,b], \mathbb{R}^m\big)\colon\ u(t) \in M,\ t \in [a,b]\big\}.$$

By a solution of the control system (11)–(13) corresponding to any fixed control $u \in \mathcal{U}_M$ we mean a function

$$x \in K^{\alpha,\beta,p}_{a+,0} := \big\{z : [a,b] \to \mathbb{R}^n\colon\ z \in I^{\alpha}_{a+}\big(L^p\big) \text{ and } I^{(1-\alpha)(1-\beta)}_{a+}z \in C_0\big([a,b], \mathbb{R}^n\big)\big\}$$

satisfying equation (11) a.e. on $[a,b]$ (here $C_0([a,b], \mathbb{R}^n)$ denotes the set of all continuous functions $v : [a,b] \to \mathbb{R}^n$ such that $v(a) = 0$).

We say that a pair $(x_*, u_*) \in K^{\alpha,\beta,p}_{a+,0} \times \mathcal{U}_M$ is a locally optimal solution of problem (11)–(14) if x_* is a solution of system (11)–(13) corresponding to the control u_* and there exists a neighborhood V of the point x_* in $K^{\alpha,\beta,p}_{a+,0}$ such that

$$J(x_*, u_*) \leqslant J(x,u)$$

for all pairs $(x,u) \in V \times \mathcal{U}_M$ satisfying (11)–(14).

If $V = K^{\alpha,\beta,p}_{a+,0}$ then the pair (x_*, u_*) is called a globally optimal solution of problem (11)–(14).

Now, we assume that $p > 1/(1-\beta(1-\alpha))$. Let us note that since $(I^{(1-\alpha)(1-\beta)}_{a+}x)(a) = 0$, therefore $D^{\alpha,\beta}_{a+}x = D^{\alpha}_{a+}x$. Moreover, if $x \in I^{\alpha}_{a+}(L^p)$ then there exists a function $\varphi \in L^p([a,b], \mathbb{R}^n)$ such that $x = I^{\alpha}_{a+}\varphi$ and, consequently, using Lemma 1, we obtain

$$I^{(1-\alpha)(1-\beta)}_{a+}x = I^{(1-\alpha)(1-\beta)}_{a+}I^{\alpha}_{a+}\varphi = I^{1-\beta(1-\alpha)}_{a+}\varphi.$$

The condition $p>1/(1-\beta(1-\alpha))$ and [12, Prop. 4] lead to $I_{a+}^{(1-\alpha)(1-\beta)}x\in C_0([a,b],\mathbb{R}^n)$. So, we proved the inclusion $I_{a+}^{\alpha}(L^p)\subset K_{a+,0}^{\alpha,\beta,p}$. The relation $K_{a+,0}^{\alpha,\beta,p}\subset I_{a+}^{\alpha}(L^p)$ is obvious. Hence $K_{a+,0}^{\alpha,\beta,p}=I_{a+}^{\alpha}(L^p)$.

From the above observation and Proposition 1 we immediately obtain

Theorem 1. *Let $p>1/(1-\beta(1-\alpha))$. Then a pair $(x_*,u_*)\in K_{a+,0}^{\alpha,\beta,p}\times\mathcal{U}_M$ is a locally* (*globally*) *optimal solution of problem* (11)–(14) *if and only if it is a locally* (*globally*) *optimal solution of the following problem*:

$$\big(D_{a^+}^{\alpha}x\big)(t)=f\big(t,x(t),u(t)\big),\quad t\in[a,b]\ \textit{a.e.},\tag{15}$$

$$\big(I_{a^+}^{1-\alpha}x\big)(a)=0,\tag{16}$$

$$u(t)\in M\subset\mathbb{R}^m,\quad t\in[a,b],\tag{17}$$

$$J(x,u)=\int_a^b f_0\big(t,x(t),u(t)\big)\,\mathrm{d}t\to\min.\tag{18}$$

Remark 2. From the above theorem it follows in particular that a function $x_*\in K_{a+,0}^{\alpha,\beta,p}$ is a solution of the control system (11)–(13) corresponding to a control $u\in\mathcal{U}_M$ if and only if it is a solution of system (15)–(17) corresponding to u.

Now, we formulate the main results of this paper (in the case $(I_{a^+}^{(1-\alpha)(1-\beta)}x)(a)=0$), mentioned at the beginning of this section. From Theorem 1 and Remark 2 it follows that they can be obtained by using analogous results for problem (15)–(18).

So, the first result is the following:

Theorem 2 [Existence and uniqueness of a solution]. *Let $p>1/(1-\beta(1-\alpha))$. If*

- (1_f) *$f(\cdot,x,u)$ is measurable on $[a,b]$ for all $x\in\mathbb{R}^n$, $u\in M$, $f(t,x,\cdot)$ is continuous on M for $t\in[a,b]$ a.e. and all $x\in\mathbb{R}^n$;*
- (2_f) *there exists $N>0$ such that*

$$\big|f(t,x_1,u)-f(t,x_2,u)\big|\leqslant N|x_1-x_2|$$

for $t\in[a,b]$ a.e. and all $x_1,x_2\in\mathbb{R}^n$, $u\in M$;

- (3_f) *there exist $w\in L^p([a,b],\mathbb{R})$ and $\gamma\geqslant 0$ such that*

$$\big|f(t,0,u)\big|\leqslant w(t)+\gamma|u|$$

for $t\in[a,b]$ a.e. and all $u\in M$,

then, for any fixed $u\in\mathcal{U}_M$, there exists a unique solution $x_u\in K_{a+,0}^{\alpha,\beta,p}$ to (11)–(13).

Proof. This result follows from [32, Thm. 6]. □

Now, we give the necessary and sufficient optimality conditions for problem (11)–(14).

Theorem 3 [Necessary optimality conditions]. *Let $p > 1/(1-\beta(1-\alpha))$. We assume that M is compact and*

- (A_f) $f \in C^1$ *with respect to* $x \in \mathbb{R}^n$ *and satisfies assumptions* (1_f)–(3_f) *of Theorem* 2;
- (B_f) $f_0(\cdot, x, u)$ *is measurable on* $[a, b]$ *for all* $x \in \mathbb{R}^n$, $u \in M$ *and* $f_0(t, x, \cdot)$ *is continuous on* M *for* $t \in [a, b]$ *a.e. and all* $x \in \mathbb{R}^n$;
- (C_f) $f_0 \in C^1$ *with respect to* $x \in \mathbb{R}^n$ *and*

$$\left|f_0(t, x, u)\right| \leqslant a_1(t) + C_1|x|^p,$$
$$\left|\left(f_0\right)_x(t, x, u)\right| \leqslant a_2(t) + C_2|x|^{p-1}$$

for a.e. $t \in [a, b]$ *and all* $x \in \mathbb{R}^n$, $u \in M$, *where* $a_2 \in L^{p'}([a, b], \mathbb{R}_0^+)$ $(1/p + 1/p' = 1)$, $a_1 \in L^1([a, b], \mathbb{R}_0^+)$, $C_1, C_2 \geqslant 0$;

- (D_f) $f_x(\cdot, x, u)$, $(f_0)_x(\cdot, x, u)$ *are measurable on* $[a, b]$ *for all* $x \in \mathbb{R}^n$, $u \in M$;
- (E_f) $f_x(t, x, \cdot)$, $(f_0)_x(t, x, \cdot)$ *are continuous on* M *for* $t \in [a, b]$ *a.e. and all* $x \in \mathbb{R}^n$;
- (F_f) *for a.e.* $t \in [a, b]$ *and all* $x \in \mathbb{R}^n$, *the set of extended velocities*

$$Z := \left\{\left(f_0(t, x, u), f(t, x, u)\right) \in \mathbb{R}^{n+1},\ u \in M\right\}$$

is convex.

If the pair $(x_*, u_*) \in K_{a+,0}^{\alpha,\beta,p} \times \mathcal{U}_M$ *is a locally optimal solution of problem* (11)–(14) *then there exists a function* $\lambda(\cdot) \in I_{b-}^{\alpha}(L^{p'})$ $(1/p + 1/p' = 1)$ *such that*

$$\left(D_{b-}^{\alpha}\lambda\right)(t) = f_x^{\mathrm{T}}\left(t, x_*(t), u_*(t)\right)\lambda(t) - (f_0)_x\left(t, x_*(t), u_*(t)\right)$$

for $t \in [a, b]$ *a.e. and*

$$\left(I_{b-}^{1-\alpha}\lambda\right)(b) = 0.$$

Moreover,

$$f_0\left(t, x_*(t), u_*(t)\right) - \lambda(t) f\left(t, x_*(t), u_*(t)\right)$$
$$= \min_{u \in M}\left\{f_0\left(t, x_*(t), u\right) - \lambda(t) f\left(t, x_*(t), u\right)\right\}, \quad t \in [a, b] \text{ a.e.}$$

Proof. This theorem follows from [32, Thm. 7]. □

Theorem 4 [Sufficient optimality conditions]. *Let $p > 1/(1-\beta(1-\alpha))$. Moreover, let us assume that*

- (a) *the set* M *is compact*;
- (b) f *satisfies assumptions* (1_f) *and* (2_f) *of Theorem* 2, *and there exist constants* $c_1 \geqslant 0$, $c_2 \geqslant 0$ *such that*

$$\left|f(t, 0, u)\right| \leqslant c_1 + c_2(t-a)^{\lambda}$$

for $t \in [a, b]$ *a.e. and all* $u \in M$ *with* $\lambda > -1/p$;

(c) $f_0(\cdot, x, u)$ *is measurable on* $[a, b]$ *for all* $x \in \mathbb{R}^n$ *and* $u \in M$;
(d) $f_0(t, \cdot, \cdot)$ *is continuous on* $\mathbb{R}^n \times M$ *for a.e.* $t \in [a, b]$;
(e) *assumption* (F_f) *from Theorem* 3 *is satisfied*;
(f) *for any function* $\kappa \in L^p([a, b], \mathbb{R}^+)$, *there exists a function* $\psi \in L^1([a, b], \mathbb{R}_0^+)$ *such that*

$$\left|f_0(t, x, u)\right| \leqslant \psi(t)$$

for a.e. $t \in [a, b]$, $|x| \leqslant \kappa(t)$ *and* $u \in M$.

Then problem (11)–(14) *possesses a globally optimal solution* $(x_*, u_*) \in K_{a+,0}^{\alpha,\beta,p} \times \mathcal{U}_M$.

Proof. This fact follows from [29, Thm. 3.5]. □

3.2 Nonhomogeneous problem

Now, we consider the following fractional optimal control problem with nonzero initial condition:

$$\left(D_{a+}^{\alpha,\beta} y\right)(t) = g\left(t, y(t), u(t)\right), \quad t \in [a, b] \text{ a.e.}, \tag{19}$$

$$\left(I_{a+}^{(1-\alpha)(1-\beta)} y\right)(a) = y_0, \tag{20}$$

$$u(t) \in M \subset \mathbb{R}^m, \quad t \in [a, b], \tag{21}$$

$$H(y, u) = \int_a^b g_0\left(t, y(t), u(t)\right) \mathrm{d}t \to \min, \tag{22}$$

where $g : [a, b] \times \mathbb{R}^n \times M \to \mathbb{R}^n$, $g_0 : [a, b] \times \mathbb{R}^n \times M \to \mathbb{R}$, $y_0 \in \mathbb{R}^n$, $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$.

By a solution of the control system (19)–(21) corresponding to any fixed control $u \in \mathcal{U}_M$ we mean a function

$$y \in K_{a+}^{\alpha,\beta,p} := K_{a+,0}^{\alpha,\beta,p} + \left\{\frac{d}{(t-a)^{(1-\alpha)(1-\beta)}} \colon d \in \mathbb{R}^n\right\}$$

satisfying equation (19) a.e. on $[a, b]$ and the initial condition (21). (Functions belonging to the set $\{d/(t-a)^{(1-\alpha)(1-\beta)} \colon d \in \mathbb{R}^n\}$ and equal a.e. on [a,b] are identified.)

We say that a pair $(y_*, u_*) \in K_{a+}^{\alpha,\beta,p} \times \mathcal{U}_M$ is a locally (globally) optimal solution of problem (19)–(22) if y_* is the solution of the control system (19)–(21) corresponding to the control u_* and there exists a neighborhood W (in the case of a globally optimal solution $W = K_{a+}^{\alpha,\beta,p}$) of the point y_* in $K_{a+}^{\alpha,\beta,p}$ such that

$$H(y_*, u_*) \leqslant H(y, u)$$

for any pair $(y, u) \in K_{a+}^{\alpha,\beta,p} \times \mathcal{U}_M$ satisfying (19)–(21).

In order to prove all results of this section, we use a technique presented in [29, 32]. Namely, we obtain the existence and uniqueness of a solution to problem (19)–(21) as well as the necessary and sufficient optimality conditions for problem (19)–(22) combining obtained results of such a type for the homogeneous problem (Theorems 2–4) with some substitutions. So, let us put

$$f(t,x,u) = g\biggl(t,\, x + \frac{y_0}{\Gamma(\alpha+\beta(1-\alpha))}\frac{1}{(t-a)^{(1-\alpha)(1-\beta)}},\, u\biggr), \tag{23}$$

$$f_0(t,x,u) = g_0\biggl(t,\, x + \frac{y_0}{\Gamma(\alpha+\beta(1-\alpha))}\frac{1}{(t-a)^{(1-\alpha)(1-\beta)}},\, u\biggr). \tag{24}$$

It is easy to show that if a pair $(x_*(\cdot), u_*(\cdot)) \in K^{\alpha,\beta,p}_{a+,0} \times \mathcal{U}_M$ is a locally (globally) optimal solution of problem (11)–(14) with functions f and f_0 given by (23), (24) then the pair $(y_*(\cdot), u_*(\cdot)) \in K^{\alpha,\beta,p}_{a+} \times \mathcal{U}_M$ of the form

$$\bigl(y_*(\cdot), u_*(\cdot)\bigr) = \biggl(x_*(\cdot) + \frac{y_0}{\Gamma(\alpha+\beta(1-\alpha))}\frac{1}{(\cdot-a)^{(1-\alpha)(1-\beta)}},\, u_*(\cdot)\biggr)$$

is a locally (globally) optimal solution to problem (19)–(22). Conversely, if a pair $(y_*(\cdot), u_*(\cdot)) \in K^{\alpha,\beta,p}_{a+} \times \mathcal{U}_M$ is a locally (globally) optimal solution of problem (19)–(22) with functions g and g_0 given by

$$g(t,y,u) = f\biggl(t,\, y - \frac{y_0}{\Gamma(\alpha+\beta(1-\alpha))}\frac{1}{(t-a)^{(1-\alpha)(1-\beta)}},\, u\biggr),$$

$$g_0(t,y,u) = f_0\bigl(t,\, y - \frac{y_0}{\Gamma(\alpha+\beta(1-\alpha))}\frac{1}{(t-a)^{(1-\alpha)(1-\beta)}},\, u\bigr),$$

then the pair $(x_*(\cdot), u_*(\cdot)) \in K^{\alpha,\beta,p}_{a+,0} \times \mathcal{U}_M$ of the form

$$\bigl(x_*(\cdot), u_*(\cdot)\bigr) = \biggl(y_*(\cdot) - \frac{y_0}{\Gamma(\alpha+\beta(1-\alpha))}\frac{1}{(\cdot-a)^{(1-\alpha)(1-\beta)}},\, u_*(\cdot)\biggr)$$

is a locally (globally) optimal solution to problem (11)–(14).

We have

Theorem 5 [Existence and uniqueness]. *Let* $1/(1-\beta(1-\alpha)) < p < 1/((1-\alpha)(1-\beta))$. *If*

- (1_g) $g(\cdot, y, u)$ *is measurable on* $[a,b]$ *for all* $y \in \mathbb{R}^n$, $u \in M$, $g(t,y,\cdot)$ *is continuous on* M *for* $t \in [a,b]$ *a.e. and all* $y \in \mathbb{R}^n$;
- (2_g) *there exists* $L > 0$ *such that*

$$\bigl|g(t,y_1,u) - g(t,y_2,u)\bigr| \leqslant L|y_1 - y_2|$$

for $t \in [a,b]$ *a.e. and all* $y_1, y_2 \in \mathbb{R}^n$, $u \in M$;

- (3_g) *there exist* $v \in L^p([a,b],\mathbb{R})$ *and* $\theta \geqslant 0$ *such that*

$$\bigl|g(t,0,u)\bigr| \leqslant v(t) + \theta|u|$$

for $t \in [a,b]$ *a.e. and all* $u \in M$, *then, for any fixed* $u \in \mathcal{U}_M$*, there exists a unique solution* $y_u \in K_{a+}^{\alpha,\beta,p}$ *of the control system* (19)–(21).

Proof. Existence. For the proof of the existence part, it is sufficient to check that if g satisfies assumptions (1_g)–(3_g) then the function f given by (23) satisfies conditions (1_f)–(3_f). Indeed, the fact that f satisfies (1_f) and (2_f) follows immediately from (1_g) and (2_g). Moreover, using (2_g) and (3_g), we obtain

$$\begin{aligned}
&\bigl|f(t,0,u)\bigr| \\
&\quad\leqslant \left|g\Bigl(t,\ \frac{y_0}{\Gamma(\alpha+\beta(1-\alpha))}\frac{1}{(t-a)^{(1-\alpha)(1-\beta)}},\ u\Bigr) - g(t,0,u)\right| + \bigl|g(t,0,u)\bigr| \\
&\quad\leqslant \frac{L|y_0|}{\Gamma(\alpha+\beta(1-\alpha))}\frac{1}{(t-a)^{(1-\alpha)(1-\beta)}} + v(t) + \theta|u|, \quad t \in [a,b] \text{ a.e. } u \in M.
\end{aligned}$$

Since $p < 1/((1-\alpha)(1-\beta))$, therefore

$$z(\cdot) = \frac{1}{(\cdot - a)^{(1-\alpha)(1-\beta)}} \in L^p\bigl([a,b],\mathbb{R}^n\bigr).$$

Consequently, putting

$$w(\cdot) = \frac{L|y_0|}{\Gamma(\alpha+\beta(1-\alpha))}\frac{1}{(\cdot - a)^{(1-\alpha)(1-\beta)}} + v(\cdot),$$

we assert that condition (3_f) holds.

Uniqueness. The proof of this fact is analogous to the second part of the proof of [28, Thm. 3.2]. The proof is completed. □

Theorem 6 [Necessary optimality conditions]. *Let* $1/(1-\beta(1-\alpha)) < p < 1/((1-\alpha)\times(1-\beta))$. *We assume that* M *is compact and*

- (A_g) $g \in C^1$ *with respect to* $y \in \mathbb{R}^n$ *and satisfies assumptions* (1_g)–(3_g) *of Theorem* 5;
- (B_g) $g_0(\cdot,y,u)$ *is measurable on* $[a,b]$ *for all* $y \in \mathbb{R}^n$, $u \in M$, *and* $g_0(t,y,\cdot)$ *is continuous on* M *for a.e.* $t \in [a,b]$ *and all* $y \in \mathbb{R}^n$;
- (C_g) $g_0 \in C^1$ *with respect to* $y \in \mathbb{R}^n$ *and*

  $$\begin{aligned}\bigl|g_0(t,y,u)\bigr| &\leqslant \overline{a}_1(t) + \overline{C}_1|y|^p, \\ \bigl|\bigl(g_0\bigr)_x(t,y,u)\bigr| &\leqslant \overline{a}_2(t) + \overline{C}_2|y|^{p-1}\end{aligned}$$

  for a.e. $t \in [a,b]$ *and all* $y \in \mathbb{R}^n$, $u \in M$, *where* $\overline{a}_2 \in L^{p'}([a,b],\mathbb{R}_0^+)$ $(1/p + 1/p' = 1)$, $\overline{a}_1 \in L^1([a,b],\mathbb{R}_0^+)$, $\overline{C}_1, \overline{C}_2 \geqslant 0$;
- (D_g) $g_y(\cdot,y,u)$, $(g_0)_y(\cdot,y,u)$ *are measurable on* $[a,b]$ *for all* $y \in \mathbb{R}^n$, $u \in M$;
- (E_g) $g_y(t,y,\cdot)$, $(g_0)_y(t,y,\cdot)$ *are continuous on* M *for a.e.* $t \in [a,b]$ *and all* $y \in \mathbb{R}^n$;

(F_g) *for a.e.* $t \in [a, b]$ *and all* $y \in \mathbb{R}^n$ *the set*

$$\tilde{Z} := \{(g_0(t, y, u), g(t, y, u)) \in \mathbb{R}^{n+1},\ u \in M\}$$

is convex.

If the pair $(y_*, u_*) \in K_{a+}^{\alpha,\beta,p} \times \mathcal{U}_M$ *is a locally optimal solution of problem* (19)–(22) *then there exists a function* $\lambda \in I_{b-}^{\alpha}(L^{p'})$ *such that*

$$(D_{b-}^{\alpha}\lambda)(t) = g_y^{\mathrm{T}}(t, y_*(t), u_*(t))\lambda(t) - (g_0)_y(t, y_*(t), u_*(t))$$

for a.e. $t \in [a, b]$ *and*

$$(I_{b-}^{1-\alpha}\lambda)(b) = 0.$$

Moreover,

$$\begin{aligned} &g_0(t, y_*(t), u_*(t)) - \lambda(t)g(t, y_*(t), u_*(t)) \\ &\quad = \min_{u \in M}\{g_0(t, y_*(t), u) - \lambda(t)g(t, y_*(t), u)\}, \quad t \in [a, b] \text{ a.e.} \end{aligned}$$

Proof. The proof is analogous to the proof of [32, Thm. 9]. □

Theorem 7 [Sufficient optimality conditions]. *Let* $p > 1/(1 - \beta(1 - \alpha))$. *If*

(A) *the set* M *is compact*;

(B) g *satisfies assumptions* (1_g)–(2_g) *of Theorem* 5;

(C) *the function* g *is such that*

$$\left|g\left(t, \frac{1}{\Gamma(\alpha + \beta(1 - \alpha))}\frac{y_0}{(t - a)^{(1-\alpha)(1-\beta)}}, u\right)\right| \leqslant c_1 + c_2(t - a)^{\lambda}$$

for a.e. $t \in [a, b]$ *and all* $u \in M$, *where* $c_1 \geqslant 0$, $c_2 \geqslant 0$, $\lambda > -1/p$;

(D) $g_0(\cdot, y, u)$ *is measurable on* $[a, b]$ *for all* $y \in \mathbb{R}^n$ *and* $u \in M$;

(E) $g_0(t, \cdot, \cdot)$ *is continuous on* $\mathbb{R}^n \times M$ *for* $t \in [a, b]$ *a.e.*;

(F) *assumption* (F_g) *from Theorem* 6 *holds*;

(G) *for any function* $\kappa \in L^p([a, b], \mathbb{R}^+)$, *there exists a function* $\tilde{\psi} \in L^1([a, b], \mathbb{R}_0^+)$ *such that*

$$\left|g_0\left(t,\ y + \frac{1}{\Gamma(\alpha + \beta(1 - \alpha))}\frac{y_0}{(t - a)^{(1-\alpha)(1-\beta)}},\ u\right)\right| \leqslant \tilde{\psi}(t)$$

for $t \in [a, b]$ *a.e.*, $|y| \leqslant \kappa(t)$ *and all* $u \in M$,

then problem (19)–(22) *possesses a globally optimal solution* $(y_*, u_*) \in K_{a+}^{\alpha,\beta,p} \times \mathcal{U}_M$.

Proof. The proof is analogous to the proof of [29, Thm. 3.7]. □

Remark 3. If $1/(1 - \beta(1 - \alpha)) < p < 1/((1 - \alpha)(1 - \beta))$ then the following condition implies (C) with $\lambda \in (-1/p, (\alpha - 1)(1 - \beta))$: there exist constants $c_3 > 0$, $c_4 \geqslant 0$ such that

$$|g(t, 0, u)| \leqslant c_3 + c_4(t - a)^{\lambda}$$

for a.e. $t \in [a, b]$ and all $u \in M$.

Proof. The proof of this fact is analogous to the proof of [29, Remark 6]. □

4 Example

In this part of our paper, we present a simple theoretical example which illustrates obtained results.

First, we give a formula for a solution of the following Cauchy problem:

$$\begin{aligned}&\big(D^{\alpha,\beta}_{a+}x\big)(t) = Cx(t) + v(t), \quad t \in [a,b] \text{ a.e.},\\ &\big(I^{(1-\alpha)(1-\beta)}_{a+}x\big)(a) = x_0,\end{aligned} \tag{25}$$

where $C \in \mathbb{R}^{n\times n}$, $v : [a,b] \to \mathbb{R}^n$.

Let $x_0 = 0$. If $p > 1/(1-\beta(1-\alpha))$ and $v \in L^p([a,b],\mathbb{R}^n)$ then, using analogous arguments as in Section 3.1, we assert that a function $x \in K^{\alpha,\beta,p}_{a+,0}$ is a solution of system (25) if and only if it is a solution of the system

$$\begin{aligned}&\big(D^{\alpha}_{a+}x\big)(t) = Cx(t) + v(t), \quad t \in [a,b] \text{ a.e.},\\ &\big(I^{1-\alpha}_{a+}x\big)(a) = 0.\end{aligned}$$

Consequently, from [32, Thm. 10] it follows that the solution of (25) is given by

$$x(t) = \int_a^t \Phi_\alpha(t-s)v(s)\,\mathrm{d}s, \quad t \in [a,b] \text{ a.e.},$$

where

$$\Phi_\alpha(t) = \sum_{k=0}^{\infty} \frac{C^k t^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))}, \quad t \in [a,b] \text{ a.e.}$$

If $x_0 \neq 0$ then, using similar arguments as in the proof of [28, Thm. 4.2], we obtain the following formula for a solution of (25):

$$x(t) = \Psi_{\alpha,\beta}(t-a)x_0 + \int_a^t \Phi_\alpha(t-s)v(s)\,\mathrm{d}s, \quad t \in [a,b] \text{ a.e.}, \tag{26}$$

where

$$\Psi_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{C^k t^{\alpha(k+1)+\beta(1-\alpha)-1}}{\Gamma(\alpha(k+1)+\beta(1-\alpha))}, \quad t \in [a,b] \text{ a.e.}$$

Formula (26) will be used later on.

Now, let us consider the following fractional optimal control problem:

$$\big(D^{\alpha,\beta}_{0+}y\big)(t) = Ay(t) + Bu^3(t), \quad t \in [0,1] \text{ a.e.}, \tag{27}$$

$$\big(I^{(1-\alpha)(1-\beta)}_{0+}y\big)(0) = y_0, \tag{28}$$

$$u(t) \in [-2,2], \quad t \in [0,1], \tag{29}$$

$$H(y,u) = \int_0^1 \left(-y_1(t) + y_2(t) + \frac{1}{2}u^3(t)\right)\mathrm{d}t \to \min, \tag{30}$$

where $y = (y_1, y_2) \in \mathbb{R}^2$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad y_0 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

$\alpha = 1/2$, $\beta = 1/3$, $p = 2$. In this case,

$$g(t, y, u) = Ay + Bu^3, \qquad g_0(t, y, u) = \langle(-1, 1), y\rangle + \frac{1}{2}u^3,$$
$$A^k = \left(A^{\mathrm{T}}\right)^k = 0, \quad k \geqslant 2,$$
$$g_y(t, y, u) = A, \qquad (g_0)_y(t, y, u) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

It is easy to verify that all assumptions of Theorems 6 and 7 are satisfied. In particular, the set $\tilde{Z}$ from assumption (F_g) is convex, although both functions g and g_0 are not convex with respect to the variable u. Consequently, from Theorem 6 it follows that if $(y_*, u_*) \in K_{0+}^{1/2,\,1/3,\,2} \times \mathcal{U}_M$ is a locally optimal solution of problem (27)–(30) then there exists $\lambda \in I_{1-}^{1/2}(L^2)$ such that

$$\left(D_{1-}^{1/2}\lambda\right)(t) = A^{\mathrm{T}}\lambda(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad t \in [0, 1] \text{ a.e.,} \tag{31}$$

$$\left(I_{1-}^{1/2}\lambda\right)(1) = 0. \tag{32}$$

Moreover,

$$\frac{1}{2}u_*^3(t) - \lambda(t)Bu_*^3(t) = \min_{u\in[-2,2]} \left\{\frac{1}{2}u^3 - \lambda(t)Bu^3\right\}, \quad t \in [0, 1] \text{ a.e.} \tag{33}$$

From [32, Thm. 11] it follows that a solution of problem (31)–(32) is given by

$$\begin{bmatrix} \lambda_1(t) \\ \lambda_2(t) \end{bmatrix} = \begin{bmatrix} \frac{(1-t)^{1/2}}{\Gamma(3/2)} \\ \frac{(1-t)}{\Gamma(2)} - \frac{(1-t)^{1/2}}{\Gamma(3/2)} \end{bmatrix}, \quad t \in [0, 1].$$

Consequently, condition (33) is equivalent to the following one:

$$\begin{aligned} \frac{1}{2}u_*^3(t) - (1-t)u_*^3(t) &= \min_{u\in[-2,2]} \left\{\frac{1}{2}u^3 - (1-t)u^3\right\} \\ &= \begin{cases} 8(t - \frac{1}{2}), & t \in [0, \frac{1}{2}], \\ -8(t - \frac{1}{2}), & t \in (\frac{1}{2}, 1], \end{cases} \quad t \in [0, 1] \text{ a.e.} \end{aligned}$$

Hence

$$u_*(t) = \begin{cases} 2, & t \in [0, \frac{1}{2}] \text{ a.e.,} \\ -2, & t \in (\frac{1}{2}, 1] \text{ a.e.} \end{cases}$$

Using formula (26), we conclude that the solution of system (27)–(28) corresponding to u_* is given by

$$y_*(t) = \Psi_{1/2,\,1/3}(t)y_0 + \int_0^t \Phi_{1/2}(t-s)Bu_*^3(s)\,\mathrm{d}s = \Psi_{1/2,\,1/3}(t)\begin{bmatrix}1\\2\end{bmatrix}$$
$$+\begin{cases} 8\int_0^t \Phi_{1/2}(t-s)B\,\mathrm{d}s, & t\in[0,\frac{1}{2}]\ \text{a.e.},\\ 8(\int_0^{1/2}\Phi_{1/2}(t-s)B\,\mathrm{d}s - \int_{1/2}^t \Phi_{1/2}(t-s)B\,\mathrm{d}s), & t\in[\frac{1}{2},1]\ \text{a.e.} \end{cases}$$

So,

$$y_*(t) = \begin{cases} \begin{bmatrix} \frac{t^{-1/3}}{\Gamma(2/3)} + \frac{2t^{1/6}}{\Gamma(7/6)} + \frac{8t^{1/2}}{\Gamma(3/2)} + 8t \\ \frac{2t^{-1/3}}{\Gamma(2/3)} + \frac{8t^{1/2}}{\Gamma(3/2)} \end{bmatrix}, & t\in[0,\frac{1}{2}]\ \text{a.e.},\\ \begin{bmatrix} \frac{t^{-1/3}}{\Gamma(2/3)} + \frac{2t^{1/6}}{\Gamma(7/6)} + \frac{8t^{1/2}-16(t-1/2)^{1/2}}{\Gamma(3/2)} - 8(t-1) \\ \frac{2t^{-1/3}}{\Gamma(2/3)} + \frac{8t^{1/2}-16(t-1/2)^{1/2}}{\Gamma(3/2)} \end{bmatrix}, & t\in[\frac{1}{2},1]\ \text{a.e.} \end{cases}$$

It means that the pair (y_*, u_*) is the only pair which can be a locally optimal solution of problem (27)–(30). From Theorem 7 it follows that this pair is a globally optimal solution.

5 Conclusion

In this paper, fractional optimal control problems involving the Hilfer derivative have been studied. The necessary and sufficient optimality conditions for such problems were established. Existence and uniqueness of a solution to control systems were also obtained. In our study, we applied a different definition of the Hilfer derivative (formula (10)). By a suitable choice of the set of functions (solutions) all results can be immediately obtained due to well-known results of such a type for fractional optimal control problems involving the Riemann–Liouville derivative. Obtained optimality conditions were illustrated with a theoretical example.

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