

## The time-fractional mZK equation for gravity solitary waves and solutions using sech-tanh and radial basis function method\*

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**Abstract.** In recent years, we know that gravity solitary waves have gradually become the research spots and aroused extensive attention; on the other hand, the fractional calculus have been applied to the biology, optics and other fields, and it also has attracted more and more attention. In the paper, by employing multi-scale analysis and perturbation methods, we derive a new modified Zakharov–Kuznetsov (mZK) equation to describe the propagation features of gravity solitary waves. Furthermore, based on semi-inverse and Agrawal methods, the integer-order mZK equation is converted into the time-fractional mZK equation. In the past, fractional calculus was rarely used in ocean and atmosphere studies. Now, the study on nonlinear fluctuations of the gravity solitary waves is a hot area of research by using fractional calculus. It has potential value for deep understanding of the real ocean–atmosphere. Furthermore, by virtue of the sech-tanh method, the analytical solution of the time-fractional mZK equation is obtained. Next, using the above analytical solution, a numerical solution of the time-fractional mZK equation is given by using radial basis function method. Finally, the effect of time-fractional order on the wave propagation is explained.

**Keywords:** gravity solitary waves, time-fractional mZK equation, sech-tanh method, radial basis function method.

### 1 Introduction

Gravity solitary waves are important part of solitary waves and can cause the suddenly change of all kinds of medium and small scale local circulation. The emergence and development of solitary waves theory is a major event on the study of nonlinear partial

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differential equation and plays an important role of the study of climate change. So, gravity solitary waves can be studied by the many researchers [35, 58].

In the past, many models were used to research gravity solitary waves. Li [33, 34] obtained the KdV equation to study the formation mechanism of squall lines by using the cohesionless equation and the basic dynamic equations, respectively. By applying the multiple-scales expansion method and perturbation method [64], Munro [45] and Anjan [4] got the ZK equation. A new ZK–Burgers equation was derived by Yang [60] and could be used to describe the three-dimensional solitary waves. Using the two-dimensional motions governing equation of an incompressible perfect fluid, a BO equation was gained by Ono [47] and can be used to describe the steady progressing waves. Using the dimensionless shallow water wave equation, Luo [39] got the BO equation to study the nonlinear problems in the atmosphere. Based on Euler equation, Su [52] studied the  $(2 + 1)$ -dimensional BO equation to investigate the eigenvalue problem corresponding to the vertical structure of the solitary wave. In finite depth fluids, a forced ILW–Burgers equation on solitary waves was researched [61]. And based on the new model, we found that solitary waves in the finite deep fluid are very different from solitary waves in the infinite deep fluid.

According to above researches, we know that these models are integer-order equations [21, 59]. The researcher found that many physical phenomenas may be taken for nonconservative. In order to better understand the nature of anomalous dynamics and improve previous integer-order models, the fractional calculus should be used to describe physical phenomena [38, 65]. The biggest difference between fractional-order differential operator and integer-order operator is that its nonlocal behavior. This feature makes the fractional-order models more and more concerned by researchers. When fractional-order value  $\beta = 1$ , the fractional-order models are transformed into the integer-order models. And fractional-order models are more realistic and practical than the classical integer-order models. Thus, fractional calculus is a new research direction in scientific research. We know that fractional calculus have been applied in studies of the ocean and atmosphere [1, 18, 22, 44], colored noise [41], solid mechanics [50], nonlinear oscillations earthquake [25], modeling of physical phenomena [20], boundary value problem [7, 13, 14] and so on. Furthermore, many researchers have used the fractional calculus in the finance [57] and economics [19].

Fractional calculus is as old as integer-order calculus, and it makes use of real numbers to replaces integer numbers in differential and integration operators. Several authors have investigated the fractional equation and its special properties for two decades [6, 63]. The solution of integer-order model is research hotspot [8, 11, 16, 32, 54]. Similarly, the solution of time-fractional models is also research emphasis. There are many methods to seek the solution of time-fractional models [15, 36, 53]. For example, variational-iteration method [24, 26], sech-tanh method [23, 56], phase portraits analysis [42], multiple  $G'/G$ -expansion method [46], Lie symmetry method [29, 49], the general Adams–Bashforth–Moulton method [55], trial function method [40, 43], ansatz method [9, 10, 12] and others [28, 31] were all found in the past decades years.

In this paper, using the multi-scale analysis [27, 62] and perturbation method [5, 37], we obtain a new  $(2 + 1)$ -dimensional integer-order mZK equation. Based on the new

integer-order equation, we seek the time-fractional mZK equation by using the semi-inverse method [17] and Agrawal method [2, 3]. Further, we get the solution of time-fractional mZK equation. So, the structure of the article is as follows. In Section 2, according to two layers fluid shallow water wave equation, we get a new (2+1)-dimensional integer-order mZK equation by using the multi-scale analysis and perturbation method. Based on the new model, we acquire the time-fractional mZK equation in Section 3. In the next section, using the sech-tanh method, we get the analytical solution of time-fractional mZK equation. According to the analytical solution, the numerical solution of time-fractional mZK equation is obtained by using radial basis function method, and the effect of time-fractional order on the wave propagation is explained in Section 5. In the end, a summary of the article is presented.

## 2 The derivation of integer-order mZK equation

The two layers fluid shallow water wave equation is described as follows:

$$\begin{aligned} \frac{\partial u'}{\partial t'} + u' \frac{\partial u'}{\partial x'} + v' \frac{\partial u'}{\partial y'} - f' v' &= -\frac{\partial \phi'}{\partial x'}, \\ \frac{\partial v'}{\partial t'} + u' \frac{\partial v'}{\partial x'} + v' \frac{\partial v'}{\partial y'} + f' u' &= -\frac{\partial \phi'}{\partial y'}, \\ \frac{\partial \phi'}{\partial t'} + u' \frac{\partial \phi'}{\partial x'} + v' \frac{\partial \phi'}{\partial y'} &= -\phi' \left( \frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} \right), \end{aligned} \tag{1}$$

where  $\phi = \phi_0 + \phi'$ ,  $\phi = g^* H = g^* H_0 + g^* H'$ ,  $g^* = g(\rho - \rho')/\rho$ ,  $\rho'$  is the upper fluid density, and  $\rho$  is the lower fluid density. Importing the following characteristic quantities into Eq. (1)

$$\begin{aligned} u', v' &= U(u, v), & x', y' &= L(x, y), & f' &= f_0, & t' &= \frac{t}{f_0}, \\ \Delta \phi' &= f_0 L U, & \lambda^2 &= \frac{L^2}{L_0^2}, & L_0^2 &= \frac{C_0^2}{f_0^2}, & C_0^2 &= g^* H_0, \end{aligned}$$

we can gain the two layers of dimensionless fluid diving waves equation

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - v &= -\frac{\partial \phi}{\partial x}, \\ \frac{\partial v}{\partial t} + u' \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + u &= -\frac{\partial \phi}{\partial y}, \\ \frac{\partial \phi'}{\partial t} + u \frac{\partial \phi}{\partial x} + v \frac{\partial \phi}{\partial y} &= -\lambda^{-2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \end{aligned} \tag{2}$$

The boundary condition is in the following form:

$$v = 0 \quad \text{as } y = 0, \quad v \rightarrow 0 \quad \text{as } y = H.$$

In previous studies, researchers have obtained new models on the scale of  $1/2$  or  $1$ . However, as the disturbance increases, the nonlinearity of gravity solitary waves is also increasing. With the development of the theory and the nonlinearity increasing, the KdV equation is replaced by the mKdV equation to describe the propagation of gravity solitary waves. Furthermore, in the plane, the  $(1 + 1)$ -dimensional model cannot accurately describe the actual phenomenon. Thus, the high-order model – the ZK equation – was subsequently deduced. But, with the nonlinearity increasing continuously, what is the development trend of the new model? That is a question that cause our great attention. In this paper, we will solve this problem. In order to refine the gravity solitary waves model, we first generalize the scale of predecessors. And we research the new model on a scale of  $1/4$ .

Introducing some conversion in the following form

$$T = \epsilon^{3/2}t, \quad X = \epsilon^{1/2}(x - ct), \quad Y = \epsilon^{3/4}y, \quad (3)$$

Eq. (3) can be rewritten as

$$\frac{\partial}{\partial t} = \epsilon^{3/2} \frac{\partial}{\partial T}, \quad \frac{\partial}{\partial x} = \epsilon^{1/2} \frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y} = \epsilon^{3/4} \frac{\partial}{\partial Y} + \frac{\partial}{\partial y}. \quad (4)$$

So, we can extend  $u, v, \phi$  as follows:

$$\begin{aligned} u &= U(y) + \epsilon^{2/4}u_0 + \epsilon^{3/4}u_1 + \epsilon^{4/4}u_2 + \epsilon^{5/4}u_3 + \epsilon^{6/4}u_4 + \dots, \\ v &= \epsilon^{4/4}v_0 + \epsilon^{5/4}v_1 + \epsilon^{6/4}v_2 + \epsilon^{7/4}v_3 + \epsilon^{8/4}v_4 + \dots, \\ \phi &= \Phi(y) + \epsilon^{2/4}\phi_0 + \epsilon^{3/4}\phi_1 + \epsilon^{4/4}\phi_2 + \epsilon^{5/4}\phi_3 + \epsilon^{6/4}\phi_4 + \dots. \end{aligned} \quad (5)$$

Substituting Eqs. (4) and (5) into Eq. (2), we can get all level of approximation equations about  $\epsilon$ .

Firstly, we use the first-order approximation equation

$$\epsilon^1 \begin{cases} (U - c) \frac{\partial u_0}{\partial X} + (U_y - 1)v_0 + \frac{\partial \phi_0}{\partial X} = 0, \\ u_0 + \frac{\partial \phi_0}{\partial y} = 0, \quad \frac{\partial u_0}{\partial X} + \frac{\partial v_0}{\partial y} = 0. \end{cases} \quad (6)$$

Assume that the solution of Eq. (6) has the following form:

$$\begin{aligned} u_0 &= m(X, Y, T)\tilde{u}_0(y), \\ v_0 &= m_X(X, Y, T)\tilde{v}_0(y), \\ \phi_0 &= m(X, Y, T)\tilde{\phi}_0(y). \end{aligned} \quad (7)$$

Substituting Eq. (7) into Eq. (6), we cannot get  $m(X, Y, T)$ , where  $m(X, Y, T)$  is the amplitude of gravity solitary waves. Next, we describe the second-order approximation

equation as follows:

$$\epsilon^2 \begin{cases} (U - c) \frac{\partial u_1}{\partial X} + (U_y - 1)v_1 + \frac{\partial \phi_1}{\partial X} = 0, \\ u_1 + \frac{\partial \phi_1}{\partial y} = 0, \quad \frac{\partial u_1}{\partial X} + \frac{\partial v_1}{\partial y} = 0, \end{cases} \quad (8)$$

and we assume that the solution of Eq. (8) has the following form:

$$\begin{aligned} u_1 &= m_Y(X, Y, T) \tilde{u}_1(y), \\ v_1 &= m_{XY}(X, Y, T) \tilde{v}_1(y), \\ \phi_1 &= m_Y(X, Y, T) \tilde{\phi}_1(y). \end{aligned} \quad (9)$$

Substituting Eq. (9) into Eq. (8), we cannot gain  $m(X, Y, T)$ . Keep on using the third-order approximation equation

$$\epsilon^3 \begin{cases} (U - c) \frac{\partial u_2}{\partial X} + (U_y - 1)v_2 + \frac{\partial \phi_2}{\partial X} = - \left( u_0 \frac{\partial u_0}{\partial X} + v_0 \frac{\partial u_0}{\partial y} \right), \\ u_2 + \frac{\partial \phi_2}{\partial y} = 0, \quad \frac{\partial u_2}{\partial X} + \frac{\partial v_2}{\partial y} = 0 \end{cases}$$

and assume that it has the solution as follows:

$$\begin{aligned} u_2 &= m^2(X, Y, T) \tilde{u}_2(y), \\ v_2 &= mm_X(X, Y, T) \tilde{v}_2(y), \\ \phi_2 &= m^2(X, Y, T) \tilde{\phi}_2(y). \end{aligned}$$

Similarly,  $m(X, Y, T)$  can be not gained. Thus, we must continue to solve the high-order problem, and the fourth-order approximation equation is similar to the third-order approximation equation. So, we use the fifth-order approximation equation

$$\epsilon^5 \begin{cases} (U - c) \frac{\partial u_4}{\partial X} + (U_y - 1)v_4 + \frac{\partial \phi_4}{\partial X} \\ = - \frac{\partial u_0}{\partial T} - \left( u_0 \frac{\partial u_2}{\partial X} + u_2 \frac{\partial u_0}{\partial X} + v_0 \frac{\partial u_2}{\partial y} + v_2 \frac{\partial u_0}{\partial y} \right), \\ u_4 + \frac{\partial \phi_4}{\partial y} = - \frac{\partial \phi_1}{\partial y} - (U - c) \frac{\partial v_0}{\partial X}, \\ \frac{\partial u_2}{\partial X} + \frac{\partial v_2}{\partial y} = - \frac{\partial v_1}{\partial y} - (U - c) \frac{\partial \phi_0}{\partial X} - v_0 \Phi_y. \end{cases} \quad (10)$$

According to Eq. (10), assuming  $U - c \neq 0$  and eliminating  $u_4, \phi_4$ , we can gain the equation about  $v_4$  as follows:

$$\begin{aligned} \frac{\partial^2 v_4}{\partial y^2} - \frac{U_{yy}}{U - c} v_4 &= \tilde{u}_0 m_T + \{ (U_y - 1) [U \tilde{v}_0 - (U - c) \tilde{\phi}_0] \\ &\quad + (U - c) [-(U - c) \tilde{\phi}_{0_y} + U \tilde{v}_{0_y} + U_y \tilde{v}_0] \} m_X \end{aligned}$$

$$\begin{aligned}
& + (2\tilde{u}_{0y}\tilde{u}_2 + 2\tilde{u}_{2y}\tilde{u}_0 + \tilde{v}_{0y}\tilde{u}_{2y} + \tilde{v}_0\tilde{u}_{2yy}) \\
& + \tilde{v}_{2y}\tilde{u}_{0y} + \tilde{v}_2\tilde{u}_{0yy})m^2m_X - (U - c)\tilde{v}_0m_{XXX} \\
& + [(U_y - 1)\tilde{v}_1 + \tilde{\phi}_1]m_{XY}Y.
\end{aligned} \tag{11}$$

Multiplying Eq. (11) by  $\tilde{v}_0$ , integrating Eq. (11) with respect to  $y$  from 0 to  $H$  and using the identical equation

$$\tilde{v}_0 \frac{\partial^2 v_4}{\partial y^2} = \frac{\partial}{\partial y} \left( \tilde{v}_0 \frac{\partial v_4}{\partial y} \right) - \frac{\partial}{\partial y} \left( v_4 \frac{\partial \tilde{v}_0}{\partial y} \right) + v_4 \frac{\partial^2 \tilde{v}_0}{\partial y^2},$$

we can have the following equation:

$$m_T + a_1m_X + a_2m^2m_X + a_3m_{XXX} + a_4m_{XY}Y = 0, \tag{12}$$

where

$$\begin{aligned}
a_0 &= \int_0^H \tilde{u}_0 \, dy, \\
a_1 &= \frac{1}{a_0} \int_0^H \{ (U_y - 1)[U\tilde{v}_0 - (U - c)\tilde{\phi}_0] + (U - c)[-(U - c)\tilde{\phi}_{0y} + U\tilde{v}_{0y} \\
& \quad + U_y\tilde{v}_0] \} \, dy, \\
a_2 &= \frac{1}{a_0} \int_0^H (2\tilde{u}_{0y}\tilde{u}_2 + 2\tilde{u}_{2y}\tilde{u}_0 + \tilde{v}_{0y}\tilde{u}_{2y} + \tilde{v}_0\tilde{u}_{2yy} + \tilde{v}_{2y}\tilde{u}_{0y} + \tilde{v}_2\tilde{u}_{0yy}) \, dy, \\
a_3 &= -\frac{1}{a_0} \int_0^H (U - c)\tilde{v}_0 \, dy, \quad a_4 = -\frac{1}{a_0} \int_0^H [(U_y - 1)\tilde{v}_1 + \tilde{\phi}_1] \, dy.
\end{aligned}$$

**Remark.** Based on the above derivation, we get Eq. (12). Equation (12) is a new model. When  $a_4 = 0$ , it can be reduced to mKdV equation. When  $m^2m_X \rightarrow mm_X$ , we call it ZK equation. So, Eq. (12) is mZK equation. Generally speaking, the ZK equation governs the behaviour of nonlinearity gravity solitary waves. But the nonlinearity of ZK equation is rather weak, and it has more external disturbance. While mZK equation has the strong nonlinearity, it is a  $(2 + 1)$ -dimensional model, which is more practical. Thus, the mZK equation is more suitable than ZK equation to describe the propagation of gravity solitary waves.

### 3 The derivation of time-fractional mZK equation

In this section, we seek for the time-fractional mZK equation. We start by introducing the notion of fractional derivative. Let  $g$  is a continuous function on the interval  $[a, b]$ , and  $\beta$  is a positive real number.

**Definition 1.** (See [30, 48, 51].) The left Riemann–Liouville fractional derivative  ${}_a D_t^\beta$  of a function  $g(t)$  is defined as

$${}_a D_t^\beta g(t) = \frac{1}{\Gamma(k - \beta)} \frac{d^k}{dt^k} \left[ \int_a^t d\tau (t - \tau)^{k-\beta-1} g(\tau) \right],$$

$$k - 1 \leq \beta \leq k, t \in [a, b].$$

**Definition 2.** (See [30, 48, 51].) The right Riemann–Liouville fractional derivative  ${}_t D_b^\beta$  of a function  $g(t)$  is defined as

$${}_t D_b^\beta g(t) = \frac{(-1)^k}{\Gamma(k - \beta)} \frac{d^k}{dt^k} \left[ \int_a^t d\tau (\tau - t)^{k-\beta-1} g(\tau) \right],$$

$$k - 1 \leq \beta \leq k, t \in [a, b].$$

**Definition 3.** (See [30, 48, 51].) The well-known Riesz fractional derivative  ${}^R D_t^\beta$  of a function  $g(t)$  is defined as

$$\begin{aligned} {}^R D_t^\beta g(t) &= \frac{1}{2} \{ {}_a D_t^\beta g(t) + (-1)^k [{}_t D_b^\beta g(t)] \} \\ &= \frac{1}{2} \frac{(-1)^k}{\Gamma(k - \beta)} \frac{d^k}{dt^k} \left[ \int_a^t d\tau |\tau - t|^{k-\beta-1} g(\tau) \right], \end{aligned}$$

$$k - 1 \leq \beta \leq k, t \in [a, b].$$

According to Section 2, we get the following mZK equation:

$$m_T + a_1 m_X + a_2 m^2 m_X + a_3 m_{XXX} + a_4 m_{XYY} = 0,$$

where  $[X, Y] \subseteq \mathbb{R}^2$  is the space coordinate,  $T \times [0, T^*]$  is the time coordinate. Letting  $m = n_X$ , here  $n(X, Y, T)$  is a potential function. Then the above equation can be written as

$$n_{XT} + a_1 n_{XX} + a_2 n_X^2 n_{XX} + a_3 n_{XXX} + a_4 n_{XYY} = 0. \quad (13)$$

Next, we use the semi-inverse method to seek the Lagrangian of mZK equation. The functional of Eq. (13) has the following form:

$$\begin{aligned} \mathcal{J}(n) &= \iiint_V dV n(X, Y, T) [s_1 n_{XT} + s_2 a_1 n_{XX} + s_3 a_2 n_X^2 n_{XX} + s_4 a_3 n_{XXX} \\ &\quad + s_5 a_4 n_{XYY}], \end{aligned}$$

where  $V = \mathbb{R}^2 \times [0, T^*]$ ,  $s_i$  ( $i = 1, 2, 3, 4, 5$ ) are unknown constants, we can get it in the later calculation. Making use of integration by parts and assuming  $n_T|_R = n_X|_R = n_T|_0 = n_X|_{T^*} = 0$ , we can have

$$\mathcal{J}(n) = \iiint_V dV n(X, Y, T) \left[ -s_1 n_X n_T - s_2 a_1 n_X^2 - \frac{1}{3} s_3 a_2 n_X^4 + s_4 a_3 n_{XX}^2 - s_5 a_4 n_X n_{XY} \right].$$

Taking advantage of variation method, integration by parts and the formula

$$\begin{aligned} & \mathcal{F}(X, Y, T, n, n_T, n_X, n_{XX}, n_{XXX}, n_{XY}) \\ &= \frac{\partial \mathcal{F}}{\partial n} - \frac{\partial}{\partial T} \left( \frac{\partial \mathcal{F}}{\partial n_T} \right) - \frac{\partial}{\partial X} \left( \frac{\partial \mathcal{F}}{\partial n_X} \right) + \frac{\partial^2}{\partial X^2} \left( \frac{\partial \mathcal{F}}{\partial n_{XX}} \right) \\ & \quad - \frac{\partial^3}{\partial X^3} \left( \frac{\partial \mathcal{F}}{\partial n_{XXX}} \right) - \frac{\partial^3}{\partial X^2 \partial T} \left( \frac{\partial \mathcal{F}}{\partial n_{XY}} \right) = 0, \end{aligned}$$

the Euler equation can be obtained as follows:

$$2s_1 n_X n_T + 2s_2 a_1 n_{XX} + 4s_3 a_2 n_X^2 n_{XX} + 2s_4 a_3 n_{XXX} + 2s_5 a_4 n_X n_{XY} = 0. \quad (14)$$

Making a comparison between Eqs. (14) and (13), we can know that

$$s_1 = \frac{1}{2}, \quad s_2 = \frac{1}{2}, \quad s_3 = \frac{1}{4}, \quad s_4 = \frac{1}{2}, \quad s_5 = \frac{1}{2}.$$

Thus, we can get the integer-order Lagrangian form of mZK equation as follows:

$$\begin{aligned} & \mathcal{L}(n_T, n_X, n_{XX}, n_{XXX}, n_{XY}) \\ &= -\frac{1}{2} n_X n_T - \frac{1}{2} a_1 n_X^2 - \frac{1}{12} a_2 n_X^4 + \frac{1}{2} a_3 n_{XX}^2 - \frac{1}{2} a_4 n_X n_{XY}. \end{aligned} \quad (15)$$

At this time, we can gain the following Lagrangian form of time-fractional mZK equation by using Definition 1:

$$\begin{aligned} & \mathcal{F}(n_T, n_X, n_{XX}, n_{XXX}, n_{XY}) \\ &= -\frac{1}{2} [{}_0 D_T^\beta n(X, Y, T)] n_X - \frac{1}{2} a_1 n_X^2 - \frac{1}{12} a_2 n_X^4 + \frac{1}{2} a_3 n_{XX}^2 - \frac{1}{2} a_4 n_X n_{XY}. \end{aligned}$$

So, the functional of mZK equation is as follows:

$$\mathcal{J}(n) = \iiint_V dV \mathcal{F}({}_0 D_T^\beta n, n_X, n_{XX}, n_{XXX}, n_{XY}). \quad (16)$$

According to Agrawal method [2,3], the variation of Eq. (16) is in the following form:

$$\delta\mathcal{J}(n) = \iiint_V dV \left[ \left( \frac{\partial\mathcal{F}}{\partial\delta_0 D_T^\beta n} \right) \delta_0 D_T^\beta n + \left( \frac{\partial\mathcal{F}}{\partial n_X} \right) \delta n_X + \left( \frac{\partial\mathcal{F}}{\partial n_{XX}} \right) \delta n_{XX} \right. \\ \left. + \left( \frac{\partial\mathcal{F}}{\partial n_{XXX}} \right) \delta n_{XXX} + \left( \frac{\partial\mathcal{F}}{\partial n_{XY}} \right) \delta n_{XY} \right]. \quad (17)$$

Using Definition 2 and the transformation

$$\int_a^b dt f(t) {}_a D_t^\beta g(t) = \int_a^b dt g(t) {}_a D_t^\beta f(t), \quad t \in [a, b], \quad (18)$$

Eq. (17) can be indicated as follows:

$$\delta\mathcal{J}(n) = \iiint_V dV \left[ D_T^\beta \left( \frac{\partial\mathcal{F}}{\partial\delta_0 D_T^\beta n} \right) - \frac{\partial}{\partial X} \left( \frac{\partial\mathcal{F}}{\partial n_X} \right) + \frac{\partial^2}{\partial X^2} \left( \frac{\partial\mathcal{F}}{\partial n_{XX}} \right) \right. \\ \left. - \frac{\partial^3}{\partial X^3} \left( \frac{\partial\mathcal{F}}{\partial n_{XXX}} \right) - \frac{\partial^3}{\partial X^2 \partial T} \left( \frac{\partial\mathcal{F}}{\partial n_{XY}} \right) \right] \delta n.$$

Relying on the above assumption  $n_T|_R = n_X|_R = n_T|_0 = n_X|_{T^*} = 0$ , we can get  $\delta n|_R = \delta n|_0 = \delta n|_{T^*} = \delta n_X|_R = 0$ .

When  $\delta\mathcal{J} = 0$ , we can gain the optimization of function  $\mathcal{J}(n)$  by using variation principle. So, the Euler–Lagrange equation for the time-fractional mZK equation is as follows:

$${}_T D_{T_0}^\beta \left( \frac{\partial\mathcal{F}}{\partial\delta_0 D_T^\beta n} \right) - \frac{\partial}{\partial X} \left( \frac{\partial\mathcal{F}}{\partial n_X} \right) + \frac{\partial^2}{\partial X^2} \left( \frac{\partial\mathcal{F}}{\partial n_{XX}} \right) - \frac{\partial^3}{\partial X^3} \left( \frac{\partial\mathcal{F}}{\partial n_{XXX}} \right) \\ - \frac{\partial^3}{\partial X^2 \partial T} \left( \frac{\partial\mathcal{F}}{\partial n_{XY}} \right) = 0. \quad (19)$$

Thus, the following equation can be obtained:

$$-\frac{1}{2} {}_T D_{T_0}^\beta n_X + a_1 n_{XX} + a_2 n_X^2 n_{XX} + a_3 n_{XXX} + a_4 n_{XY} = 0. \quad (20)$$

Letting  $n_X(X, Y, T) = u(X, Y, T)$ , we obtain

$$\frac{1}{2} ({}_0 D_T^\beta m - T D_{T_0}^\beta m) + a_1 m_X + a_2 m^2 m_X + a_3 m_{XXX} + a_4 m_{XY} = 0. \quad (21)$$

According to Definition 3, Eq. (21) can be rewritten as

$${}_0^R D_T^\beta m + a_1 m_X + a_2 m^2 m_X + a_3 m_{XXX} + a_4 m_{XY} = 0, \quad (22)$$

$0 \ll \beta \ll 1$ ,  $T \in [0, T_0]$ . Equation (22) is new model, which is obtained by the time-fractional calculus. So, we call it time-fractional mZK equation.

#### 4 The analytical solution of time-fractional mZK equation

In Section 3, we get the time-fractional mZK equation. In this section, based on sech-tanh method, we seek the explicit solution of time-fractional mZK equation. By applying the fractional complex transformations

$$m(X, Y, T) = \varphi(\eta), \quad \eta = a \left( X + Y - \frac{lT^\beta}{\Gamma(\beta + 1)} \right),$$

where  $a, l$  are unknown constants, which are ensured in the later calculation, Eq. (22) can be written in the following form:

$$-l\varphi'(\eta) + a_1\varphi'(\eta) + a_2\varphi^2(\eta)\varphi'(\eta) + a_3\varphi'''(\eta) + a_4\varphi'''(\eta) = 0.$$

Assume that the solution of Eq. (22) can be expressed by a polynomial in  $\lambda$  and will take the following form:

$$\varphi(\eta) = c_0 + \sum_{i=0}^N c_i \lambda^i, \quad \lambda = \tanh(\eta).$$

We know that  $n = 1$  by balancing the highest-order derivative term and nonlinear term of Eq. (22). Then we have

$$\varphi(\eta) = c_0 + c_1 \lambda, \quad \lambda = \tanh(\eta).$$

Thus, we can get

$$\begin{aligned} \frac{d\varphi}{d\eta} &= (1 - \lambda^2) \frac{d\varphi}{d\lambda}, \\ \frac{d^2\varphi}{d\eta^2} &= (1 - \lambda^2) \left( -2\lambda \frac{d\varphi}{d\lambda} + (1 - \lambda^2) \frac{d^2\varphi}{d\lambda^2} \right), \\ \frac{d^3\varphi}{d\eta^3} &= (1 - \lambda^2)^3 \frac{d^3\varphi}{d\lambda^3} - 6\lambda(1 - \lambda^2) \frac{d^2\varphi}{d\lambda^2} + 2(1 - \lambda^2)(3\lambda^2 - 1) \frac{d\varphi}{d\lambda}, \\ &\dots \end{aligned}$$

According to the above equations, we obtain

$$\begin{aligned} c_1(a_1 - k)(1 - \lambda^2) + a_2c_1(c_0 + c_1\lambda)^2(1 - \lambda^2)^2 \\ + 2c_1(a_3 + a_4)(1 - \lambda^2)(3\lambda^2 - 1) = 0. \end{aligned}$$

Gathering all of coefficients of same index about  $\lambda$  in the above equation and making them to 0, respectively, we can have

$$\begin{aligned} \lambda^0: \quad &c_1[(a_1 - k) + a_2c_0^2 - 2(a_3 + a_4)] = 0, \\ \lambda^1: \quad &2a_2c_0c_1^2 = 0, \\ \lambda^2: \quad &c_1[(a_1 - k) + a_2(c_1^2 - c_0^2) + 8(a_3 + a_4)] = 0, \\ \lambda^3: \quad &-2a_2c_0c_1^2 = 0, \\ \lambda^4: \quad &c_1[-a_2c_1^2 - 6(a_3 + a_4)] = 0. \end{aligned}$$

Solving the above equation, we can gain

$$a = 1, \quad c_0 = 0, \quad c_1 = -\frac{\sqrt{6(a_3 + a_4)}}{a_2}, \quad l = a_1 - 2(a_3 + a_4).$$

So,  $m(X, Y, T)$  can be written as

$$\begin{aligned} m(X, Y, T) &= \varphi(\eta) = -\frac{\sqrt{6(a_3 + a_4)}}{a_2} \tanh(\eta) \\ &= -\frac{\sqrt{6(a_3 + a_4)}}{a_2} \tanh\left(X + Y - \frac{a_1 - 2(a_3 + a_4)T^\beta}{\Gamma(1 + \beta)}\right). \end{aligned} \quad (23)$$

Expression (23) is an analytical solution of time-fractional mZK equation.

## 5 The numerical solution of time-fractional mZK equation

In the previous section, we get an analytical solution of time-fractional mZK equation by using sech-tanh method. However, analytical solution sometimes cannot be used to study the property of equation. So, according to the analytical solution, we seek the numerical solution of time-fractional mZK equation by using radial basis function method.

According to Eq. (23), we can get the initial condition of Eq. (22) as follows:

$$m(X, Y, 0) = -\frac{\sqrt{6(a_3 + a_4)}}{a_2} \tanh X = h(X, Y).$$

The boundary conditions of Eq. (22) are as follows:

$$\begin{aligned} m(X_1, Y_1, T) &= -\frac{\sqrt{6(a_3 + a_4)}}{a_2} \tanh\left(X_1 + Y_1 - \frac{a_1 - 2(a_3 + a_4)T^\beta}{\Gamma(1 + \beta)}\right) \\ &= d(X_1, Y_1, T) \end{aligned} \quad (24)$$

and

$$\begin{aligned} m(X_N, Y_N, T) &= -\frac{\sqrt{6(a_3 + a_4)}}{a_2} \tanh\left(X_N + Y_N - \frac{a_1 - 2(a_3 + a_4)T^\beta}{\Gamma(1 + \beta)}\right) \\ &= d(X_N, Y_N, T). \end{aligned} \quad (25)$$

Assuming that

$$m^n = m(X, Y, T^n), \quad T^n = n\Delta T, \quad n = 0, 1, 2, \dots,$$

and using

$$\frac{\partial^\beta m(X, Y, T)}{\partial T^\beta} = \begin{cases} \frac{1}{\Gamma(1-\beta)} \int_0^T \frac{\partial m(X, Y, \eta)}{\partial \eta} \frac{d\eta}{(T-\eta)^\beta}, & 0 < \beta < 1, \\ \frac{\partial m(X, Y, T)}{\partial T}, & \beta = 1, \end{cases}$$

we can get a finite difference approximation

$$\begin{aligned}
\frac{\partial^\beta m(X, Y, T_{n+1})}{\partial T^\beta} &= \frac{1}{\Gamma(1-\beta)} \int_0^{T_{n+1}} \frac{\partial m(X, Y, \eta)}{\partial \eta} \frac{d\eta}{(T_{n+1}-\eta)^\beta} \\
&= \frac{1}{\Gamma(1-\beta)} \sum_{k=0}^n \int_{k\Delta T}^{(k+1)\Delta T} \frac{\partial m(X, Y, \eta)}{\partial \eta} \frac{d\eta}{(T_{k+1}-\eta)^\beta} \\
&\approx \frac{1}{\Gamma(1-\beta)} \sum_{k=0}^n \int_{k\Delta T}^{(k+1)\Delta T} \frac{\partial m(X, Y, \eta_k)}{\partial \eta} \frac{d\eta}{(T_{k+1}-\eta)^\beta}, \quad (26)
\end{aligned}$$

where the first-order derivative is defined by

$$\frac{\partial m(X, Y, T_k)}{\partial \eta} = \frac{m(X, Y, \eta_{k+1}) - m(X, Y, \eta_k)}{\Delta T} + O(\Delta T).$$

So, according to Eq. (26), we obtain

$$\begin{aligned}
&\frac{\partial^\beta m(X, Y, T_{n+1})}{\partial T^\beta} \\
&= \frac{1}{\Gamma(1-\beta)} \sum_{k=0}^n \frac{m(X, Y, \eta_{k+1}) - m(X, Y, \eta_k)}{\Delta T} \int_{k\Delta T}^{(k+1)\Delta T} \frac{d\eta}{(T_{k+1}-\eta)^\beta}.
\end{aligned}$$

That means that

$$\begin{aligned}
&\frac{\partial^\beta m(X, Y, T)}{\partial T^\beta} \\
&= \begin{cases} \frac{(\Delta T)^{-\beta}}{\Gamma(2-\beta)} (m^1 - m^0), & n = 0, \\ \frac{(\Delta T)^{-\beta}}{\Gamma(2-\beta)} (m^{n+1} - m^n) \\ \quad + \frac{(\Delta T)^{-\beta}}{\Gamma(2-\beta)} \sum_{k=1}^n (m^{n+1-k} - m^{n-k}) [(k+1)^{1-\beta} - k^{1-\beta}], & n \gg 1. \end{cases} \quad (27)
\end{aligned}$$

Assuming

$$p_0 = \frac{(\Delta T)^{-\beta}}{\Gamma(2-\beta)}, \quad p_k = (k+1)^{1-\beta} - k^{1-\beta},$$

Eq. (27) can be rewritten as

$$\frac{\partial^\beta m}{\partial T^\beta} = \begin{cases} \eta_0 (m^1 - m^0), & n = 0, \\ \frac{(\Delta T)^{-\beta}}{\Gamma(2-\beta)} (m^{n+1} - m^n) + \eta_0 \sum_{k=1}^n \eta_k (m^{n+1-k} - m^{n-k}), & n \gg 1. \end{cases} \quad (28)$$

Substituting Eq. (28) into Eq. (22), we get two cases in the following form:

$$\begin{aligned} \xi_0(m^1 - m^0) + a_1 \nabla m^0 + a_2(m^0)^2 \nabla m^0 + a_3 \nabla^3 m^1 + a_4 \nabla^3 m^0 &= 0 \quad \text{as } n = 0, \\ \xi_0(m^{n+1} - m^n) + \eta_0 \sum_{k=1}^n \eta_k(m^{n+1-k} - m^{n-k}) \\ + a_1 \nabla m^n + a_2(m^n)^2 \nabla m^n + a_3 \nabla^3 m^{n+1} + a_4 \nabla^3 m^n &= 0 \quad \text{as } n \gg 1. \end{aligned}$$

So, Eq. (22) is converted into the following equation:

$$\begin{aligned} \xi_0 m^{n+1} + a_3 \nabla^3 m^{n+1} &= \xi_0 m^n - \eta_0 \sum_{k=1}^n \eta_k(m^{n+1-k} - m^{n-k}) \\ &\quad - a_1 \nabla m^n - a_2(m^n)^2 \nabla m^n - a_4 \nabla^3 m^n. \end{aligned} \quad (29)$$

According to radial basis function approximation, the function  $m(X, Y, T)$  can be written as a linear combination of  $N$  radial functions as follows:

$$m(X, Y, T) = \sum_{j=1}^N \tau_j^{n+1} \rho(r_{ij}) + \tau_{N+1}^{n+1} (X_i + Y_i) + \tau_{N+2}^{n+1}, \quad (30)$$

where  $\tau_j, j = 1, 2, \dots, N$ , are unknown coefficients, which can be calculated,  $N$  is the numbers of data points,  $\rho(X, X_j) = \rho(r_j), r_j = \|X - X_j\|, j = 1, 2, \dots, N$ , are the Euclidean norm.

Based on  $N$  equations, which are caused by Eq. (30) at  $N$  points, and the regularization conditions, we can have additional two equations

$$\sum_{j=1}^N \tau_j^{n+1} = \sum_{j=1}^N \tau_{N+1}^{n+1} (X_i + Y_i) = 0. \quad (31)$$

When Eqs. (30) and (31) are combined together, we can get the matrix equation

$$(m)^{n+1} = P(\tau)^{n+1},$$

where

$$\begin{aligned} (m)^{n+1} &= (m_1^{n+1}, m_2^{n+1}, m_3^{n+1}, \dots, m_N^{n+1}, 0, 0, 0)^T, \\ (\tau)^{n+1} &= (\tau_1^{n+1}, \tau_2^{n+1}, \tau_3^{n+1}, \dots, \tau_{N+3}^{n+1})^T, \end{aligned}$$

and  $P = (a_{ij})$  is a  $(N + 3) \times (N + 3)$  matrix, which is defined by

$$P = \begin{pmatrix} \rho_{11} & \dots & \rho_{1j} & \dots & \rho_{1N} & X_1 & Y_1 & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \rho_{i1} & \dots & \rho_{ij} & \dots & \rho_{iN} & X_i & Y_i & 1 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \rho_{N1} & \dots & \rho_{Nj} & \dots & \rho_{NN} & X_N & Y_N & 1 \\ X_1 & \dots & X_j & \dots & X_N & 0 & 0 & 0 \\ Y_1 & \dots & Y_j & \dots & Y_N & 0 & 0 & 0 \\ 1 & \dots & 1 & \dots & 1 & 0 & 0 & 0 \end{pmatrix}.$$

According to Eqs. (24), (25) and (31) and substituting Eq. (30) into Eq. (29), we can obtain the discretization equation

$$Q(\tau)^{n+1} = s^{n+1}, \quad (32)$$

where  $Q$  is a  $(N + 3) \times (N + 3)$  matrix, which is defined by

$$Q = \begin{pmatrix} J(\rho_{11}) & \dots & J(\rho_{1j}) & \dots & J(\rho_{1N}) & J(X_1) & J(Y_1) & J(1) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ J(\rho_{i1}) & \dots & J(\rho_{ij}) & \dots & J(\rho_{iN}) & J(X_i) & J(Y_i) & J(1) \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ J(\rho_{N1}) & \dots & J(\rho_{Nj}) & \dots & J(\rho_{NN}) & J(X_N) & J(Y_N) & J(1) \\ X_1 & \dots & X_j & \dots & X_N & 0 & 0 & 0 \\ Y_1 & \dots & Y_j & \dots & Y_N & 0 & 0 & 0 \\ 1 & \dots & 1 & \dots & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Here  $J$  represents an operate, which is defined by

$$J(*) = \begin{cases} (*), & i = 1 \text{ or } i = N, \\ (\eta_0 - a_3 \nabla^3)(*), & 1 < i < N, \end{cases}$$

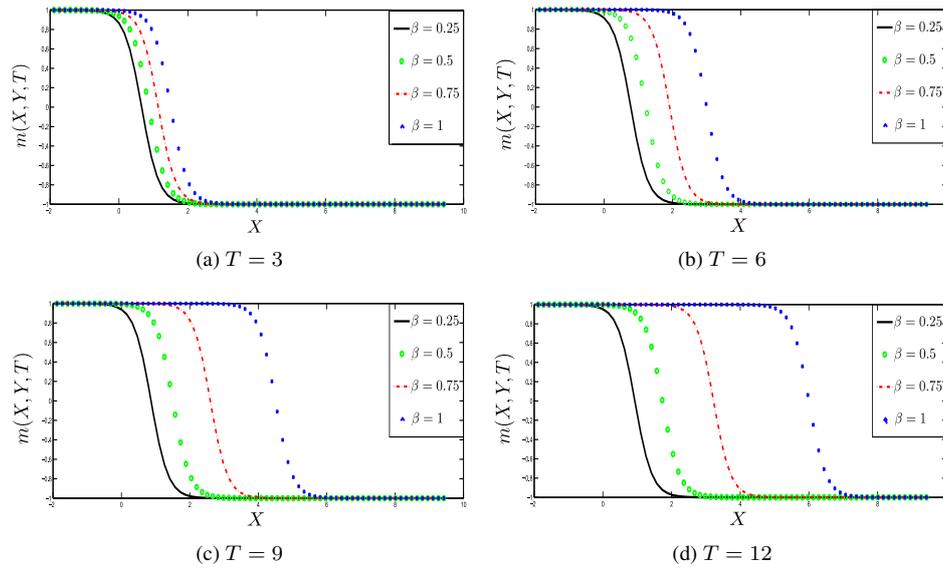
$s^{n+1} = (s_1^{n+1}, s_2^{n+1}, \dots, s_N^{n+1}, 0, 0, 0)^T$ ,  $m_i^{n+1} = m(X_i, Y_i, T_{n+1})$ , and

$$s_i^{n+1} = \begin{cases} \eta_0 m_i^0 + h_i^1, & n = 0, 1 < i < N, \\ \eta_0 [m_i^n - \sum_{k=1}^n (m^{n+1-k} - m^{n-k})] - a_1 \nabla m^n \\ \quad - a_2 (m^n)^2 \nabla m^n - a_4 \nabla^3 m^n, & n \geq 1, 1 < i < N, \\ h(X_i, Y_i, T_{n+1}), & i = 1 \text{ or } i = N. \end{cases}$$

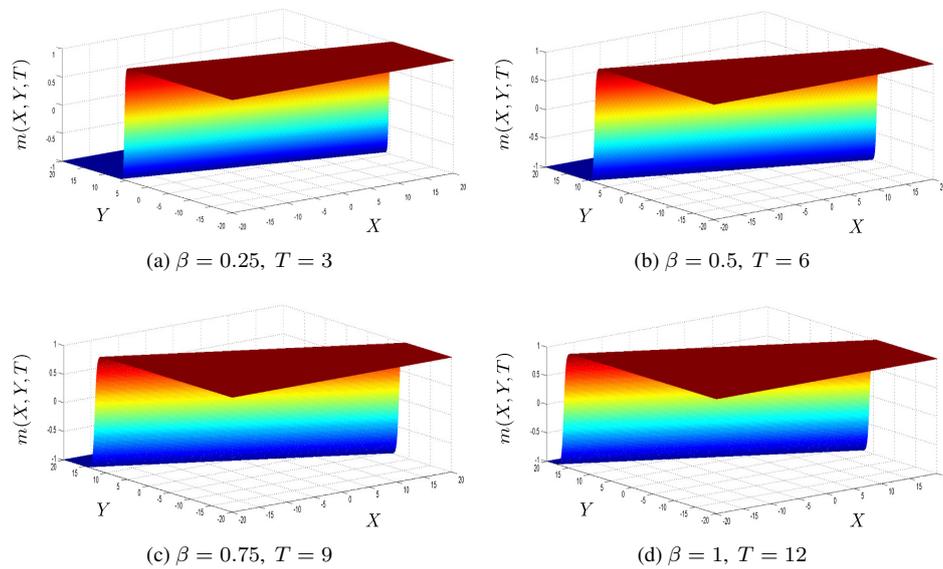
Based on Eq. (32), we can calculate the undetermined coefficient  $(\tau)^{n+1}$ . Substituting the  $(\tau)^{n+1}$  into Eq. (30), we can obtain a numerical solution of time-fractional mZK equation.

Next, we discuss the influence of fractional order ( $\beta$ ) and time ( $T$ ) on the solitary waves solution of mZK equation. According to the numerical solution of time-fractional mZK equation and changing  $\beta$  and  $T$ , we can study the propagation property of gravity solitary waves, and waves forms are represented in different figures. The effect of time-fractional order  $\beta$  on the soliton shapes has been studied in Figs. 1(a)–1(d). These figures show that the fractional order of differentiation has a small effect only on the position of the turning points. And with the march of time, the distances among the turning points become more and more wide. In other words, the time-fractional order do not change the shape of gravity solitary waves and only has a small effect on the position of these waves. Figures 2(a)–2(d) show the evolution of the  $(2 + 1)$ -dimensional gravity solitary waves solution for time-fractional mZK equation when  $\beta = 0.25, 0.5, 0.75$  and  $1$ , respectively.

It is judged that gravity solitary waves maintain its shape during the propagation process. And the main evolving features of gravity solitary waves can be not changed.



**Figure 1.** The (1 + 1)-dimensional plot of numerical solution for gravity solitary waves for different values of  $\beta$ .



**Figure 2.** The (2 + 1)-dimensional plot of numerical solution for gravity solitary waves for different values of  $\beta$  and  $T$ .

## 6 Conclusion

In this paper, we obtain a new  $(2 + 1)$ -dimensional mZK equation by using the multi-scale analysis and perturbation method. Based on the semi-inverse method and Agrawal method, we gain the time-fractional mZK equation. In the end, in order to study the properties of gravity solitary waves, we seek the solution of time-fractional mZK equation. The focus of the article are as follows.

1. We obtain a new  $(2 + 1)$ -dimensional integer-order mZK equation. It can accurately describe the gravity solitary waves. According to semi-inverse and Agrawal methods, we get a time-fractional mZK equation. It is the generalization of integer-order mZK equations.
2. According to the time-fractional mZK equation, we get an analytical solution by using sech-tanh method. According to the analytical solution, a numerical solution can be obtained by using radial basis function method. Then, based on the solution of time-fractional mZK equation, we discuss the effect of fractional order on wave propagation.

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