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Existence and uniqueness of positive solutions for a class of nonlinear fractional differential equations with mixed-type boundary value conditions*

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Abstract. In this article, we study a class of nonlinear fractional differential equations with mixed-type boundary conditions. The fractional derivatives are involved in the nonlinear term and the boundary conditions. By using the properties of the Green function, the fixed point index theory and the Banach contraction mapping principle based on some available operators, we obtain the existence of positive solutions and a unique positive solution of the problem. Finally, two examples are given to demonstrate the validity of our main results.

Keywords: fractional differential equations, mixed-type boundary value problem, positive solution, unique positive solution.

1 Introduction

In this article, we consider the following class of boundary value problem (BVP):

$$\begin{aligned} &D^{\gamma}_{0+}x(t) + f\bigl(t, x(t), D^{\alpha}_{0+}x(t), D^{\beta}_{0+}x(t)\bigr) = 0, \quad 0 < t < 1,\ n-1 < \gamma \leqslant n, \\ &x^{(j)}(0) = 0, \quad D^{\beta}_{0+}x(0) = 0, \quad j = 0, 1, \ldots, n-3, \end{aligned} \tag{1_1}$$

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$$D^{\beta}_{0+}x(1) = a_1 \int\limits_0^1 p_1(s) D^{\beta}_{0+}x(s)\,\mathrm{d}A_1(s) + a_2 \int\limits_0^{\eta} p_2(s) D^{\beta}_{0+}x(s)\,\mathrm{d}A_2(s)$$
$$+ a_3 \sum_{i=1}^{\infty} \mu_i D^{\beta}_{0+}x(\zeta_i), \tag{1_2}$$

where D^{γ}_{0+} is the Riemann–Liouville's fractional derivative, $0 < \alpha < n-2 \leqslant \beta < n-1$, $\gamma - \beta > 1$, $\beta - \alpha \geqslant 1$; $a_j \geqslant 0$ $(j = 1, 2, 3)$; $\mu_i \geqslant 0$, $0 < \eta < \zeta_1 < \zeta_2 < \cdots < \zeta_i < \cdots < 1$ $(i = 1, 2, \dots)$; $1 - a_3 \sum_{i=1}^{\infty} \mu_i \zeta_i^{\delta-1} > 0$; $p_1, p_2 \colon (0, 1) \to \mathbb{R}^+ = [0, +\infty)$ are continuous with $p_1, p_2 \in L^1(0, 1)$; $\int_0^1 p_1(s)u(s)\,\mathrm{d}A_1(s)$ and $\int_0^1 p_2(s)u(s)\,\mathrm{d}A_2(s)$ denote the Riemann–Stieltjes integrals, in which $A_1, A_2 \colon [0, 1] \to \mathbb{R}$ are function of bounded variation. The nonlinearity $f \colon [0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

Fractional differential equations with boundary value conditions have been investigated by many researchers due to its wide range of applications in many fields of sciences, robotics and electrical networks, etc. For details, we refer the reader to [5, 11, 15, 17, 26]. Recently, many results have been obtained for the existence of positive solutions or the uniqueness of solution of fractional differential equations in [1–4, 6–10, 12–14, 16, 18–25, 27–37]. In [12,13,30], the authors studied fractional differential equation with multi-point boundary conditions; [7,8,21] deal with fractional differential equations with infinite point boundary; integral type boundary value conditions of fractional differential equations are investigated in [1, 2, 18, 19, 27, 36]. Moreover, in [1, 2, 4, 7], the conditions for the existence of positive solutions to various fractional differential equations are established; while in [14], the conditions for the existence of two positive solutions has been achieved. For results on the uniqueness of solutions, we refer the reader to [3, 10, 19–21, 27, 33]. Some interesting results obtained by using the reduced order method can be found in [9, 10, 22, 23, 28, 31–33, 35, 36] and the references therein.

In [36], Zhang et al. studied the following fractional differential equation:

$$-D^{\alpha}_t x(t) = f\bigl(t, x(t), D^{\beta}_t x(t)\bigr), \quad 0 < t < 1,$$
$$D^{\beta}_t x(0) = 0, \qquad D^{\beta}_t x(1) = \int\limits_0^1 g(s) D^{\beta}_t x(s)\,\mathrm{d}A(s),$$

where D^{α}_t is Riemann–Liouville's fractional derivative, $0 < \beta \leqslant 1 < \alpha \leqslant 2$, $\alpha - \beta > 1$, A is a function of bounded variation and $\mathrm{d}A$ can be a signed measure, $f \colon (0, 1) \times (0, +\infty) \times (0, +\infty) \to \mathbb{R}^+$ is continuous, and $f(t, x, y)$ may be singular at both $t = 0, 1$ and $x = y = 0$. By analyzing the spectral of the relevant linear operator, they obtained positive solutions of the singular fractional differential equation.

In [30], Zhang studied the following nonlinear fractional differential equation with infinite-point boundary value conditions:

$$D^{\alpha}_{0+}u(t) + q(t)f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \qquad u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_i u(\xi_i),$$

where $\alpha > 2$, $n-1 < \alpha \leqslant n$, $i \in [1, n-2]$ is a fixed integer, $\alpha_j \geqslant 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ ($j = 1, 2, \dots$), $(\alpha-1)(\alpha-2)\cdots(\alpha-i) - \sum_{j=1}^{\infty} \alpha_i u(\xi_i) > 0$, D_{0+}^{α} is the standard Riemann–Liouville derivative, $q\colon (0,1) \to \mathbb{R}^+$ and $f\colon (0,1) \times (0,+\infty) \to \mathbb{R}^+$ are continuous functions, and $q(t)$ may be singular at $t = 0, 1$. By using height functions of the nonlinear term on some bounded sets, the author obtained the positive solutions of the problem.

In [25], Qarout et al. studied the following semi-linear Caputo fractional differential equation:

$$
\begin{aligned}
&{}^cD_{0+}^{q} x(t) = f\bigl(t, x(t)\bigr), \quad 0 < t < 1,\ n-1 < q \leqslant n,\\
&x(0) = x'(0) = x''(0) = \cdots = x^{(n-2)}(0) = 0,\\
&x(1) = a \int_0^{\xi} x(s)\,\mathrm{d}(s) + b \sum_{i=1}^{m-2} \alpha_i x(\eta_i),
\end{aligned}
$$

where ${}^cD_{0+}^{q}$ denotes the Caputo fractional derivative of order q, $f\colon [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, a and b are real constants and α_i are positive real constants. They got the existence of solutions by using some standard tools of fixed point theory.

Motivated by the above mentioned work, the purpose of this article is to investigate the existence of solutions for the BVP (1). The main new features presented in this paper are as follows. Firstly, the boundary value problem has wider and more general boundary conditions; it includes many situations, which were investigated before as special cases. Secondly, the presence of the fractional derivatives in the nonlinear term f and the boundary conditions makes the study extremely difficult. By using some available operators, the BVP (1) is transformed into a class of relatively simple low-order fractional differential equations. Thirdly, our technique and tools are novel. Consequently, conditions for the positive solutions and a unique positive solution of the BVP (1) are obtained.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries and lemmas that are used to prove our main results, and we also develop some properties of the Green function, and reduce the original equation to a class of relatively simple equations by using some available operators. In Section 3, we discuss the existence of positive solutions of the BVP (1) by the tool of the fixed point index theory, and give an example to demonstrate the application of our theoretical results. In Section 4, we create a appropriate operator and discuss a unique positive solution of the BVP (1), and give an example to emphasize our two theories.

2 Preliminaries and lemmas

In this section, for the convenience of reader, we present some necessary definitions and lemmas that will be used in the proof of our main results.

Definition 1. (See [24].) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $y\colon (0,\infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\,\mathrm{d}s,$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2. (See [24].) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $y\colon (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of the number α, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Lemma 1. (*See* [24].) *Let* $\alpha > 0$. *If we assume* $u \in C(0,1) \cap L^1(0,1)$, *then the fractional differential equation* $D_{0+}^{\alpha} u(t) = 0$ *has* $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$, $C_i \in \mathbb{R}$ $(i = 1, 2, \dots, N)$, *as the unique solution, where* N *is the smallest integer greater than or equal to* α.

Lemma 2. (*See* [24].) *Assume that* $u \in C(0,1) \cap L^1(0,1)$ *with a fractional derivative of order* $\alpha > 0$ *that belongs to* $C(0,1) \cap L^1(0,1)$. *Then* $I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N}$ *for some* $C_i \in \mathbb{R}$ $(i = 1, 2, \dots, N)$, *where* N *is the smallest integer greater than or equal to* α.

Now, we consider the following modified boundary value problem (BVP):

$$\begin{aligned}
&D_{0+}^{\delta} u(t) + f\bigl(t, I_{0+}^{\beta} u(t), I_{0+}^{\beta-\alpha} u(t), u(t)\bigr) = 0, \quad 0 < t < 1,\ 1 < \delta \leqslant 2, \\
&u(0) = 0, \\
&u(1) = a_1 \int_0^1 p_1(s) u(s)\,\mathrm{d}A_1(s) + a_2 \int_0^{\eta} p_2(s) u(s)\,\mathrm{d}A_2(s) + a_3 \sum_{i=1}^{\infty} \mu_i u(\zeta_i),
\end{aligned} \tag{2}$$

where $\delta = \gamma - \beta$.

Lemma 3. *If* $u \in C[0,1]$ *is a solution of BVP* (2), *then* $I_{0+}^{\beta} u(t)$ *is a solution of BVP* (1).

Proof. We assume $u \in C[0,1]$ is a solution for BVP (2). Let $x(t) = I_{0+}^{\beta} u(t)$. We have

$$\begin{aligned}
&D_{0+}^{\beta} x(t) = u(t), \qquad D_{0+}^{\alpha} x(t) = D_{0+}^{\alpha} I_{0+}^{\beta} u(t) = I_{0+}^{\beta-\alpha} u(t), \\
&D_{0+}^{\gamma} x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} I_{0+}^{n-\gamma} x(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} I_{0+}^{n-\gamma} I^{\beta} u(t) = \frac{\mathrm{d}^n}{\mathrm{d}t^n} I_{0+}^{n-\gamma+\beta} u(t) = D_{0+}^{\gamma-\beta} u(t),
\end{aligned}$$

which means that

$$\begin{aligned} & D_{0+}^{\gamma} x(t) + f\big(t, x(t), D_{0+}^{\alpha} x(t), D_{0+}^{\beta} x(t)\big) \\ & \quad = D_{0+}^{\gamma-\beta} u(t) + f\big(t, I_{0+}^{\beta} u(t), I_{0+}^{\beta-\alpha} u(t), u(t)\big) \\ & \quad = D_{0+}^{\delta} u(t) + f\big(t, I_{0+}^{\beta} u(t), I_{0+}^{\beta-\alpha} u(t), u(t)\big) = 0, \\ & x^{(i)}(t) = D_{0+}^{i} I_{0+}^{\beta} u(t) = I_{0+}^{\beta-i} u(t), \quad i = 1, 2, \dots, n-3. \end{aligned} \tag{3}$$

By $x(t) = I_{0+}^{\beta} u(t)$ and (3), we obtain

$$\begin{aligned} & D_{0+}^{\beta} x(0) = 0, \qquad x(0) = x'(0) = \cdots = x^{(n-3)}(0) = 0, \\ & D_{0+}^{\beta} x(1) = a_1 \int_0^1 p_1(s) D_{0+}^{\beta} x(t) \, \mathrm{d}A_1(s) + a_2 \int_0^{\eta} p_2(s) D_{0+}^{\beta} x(t) \, \mathrm{d}A_2(s) \\ & \qquad + a_3 \sum_{i=1}^{\infty} \mu_i D_{0+}^{\beta} x(\zeta_i). \end{aligned}$$

Hence, we claim that $I_{0+}^{\beta} u(t)$ is a solution of the BVP (1). The proof is completed. □

Lemma 4. *Let $y \in C(0,1) \cap L^1(0,1)$ be a given function. Then the function $u \in C[0,1]$ given by*

$$u(t) = \int_0^1 G(t,s) y(s) \, \mathrm{d}s, \quad t \in [0,1],$$

is a solution of the following boundary value problem:

$$\begin{aligned} & D_{0+}^{\delta} u(t) + y(t) = 0, \quad 0 < t < 1, \ 1 < \delta \leqslant 2, \\ & u(0) = 0, \\ & u(1) = a_1 \int_0^1 p_1(s) u(s) \, \mathrm{d}A_1(s) + a_2 \int_0^{\eta} p_2(s) u(s) \, \mathrm{d}A_2(s) + a_3 \sum_{i=1}^{\infty} \mu_i u(\zeta_i), \end{aligned} \tag{4}$$

where

$$G(t,s) = G_1(t,s) + \kappa_1 t^{\delta-1} P_1(s) + \kappa_2 t^{\delta-1} P_2(s), \tag{5}$$

$$G_1(t,s) = \frac{1}{\sigma\Gamma(\delta)} \begin{cases} [t(1-s)]^{\delta-1} l(s) - \sigma(t-s)^{\delta-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ [t(1-s)]^{\delta-1} l(s), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$P_1(s) = \int_0^1 G_1(t,s) p_1(t) \, \mathrm{d}A_1(t), \quad P_2(s) = \int_0^{\eta} G_1(t,s) p_2(t) \, \mathrm{d}A_2(t), \quad s \in [0,1],$$

$$\sigma = 1 - a_3 \sum_{i=1}^{\infty} \mu_i \zeta_i^{\delta-1} > 0, \qquad l(s) = 1 - a_3 \frac{b(s)}{(1-s)^{\delta-1}}, \quad s \in [0,1],$$

$$b(s) = \begin{cases} \sum_{i=1}^{\infty} \mu_i (\zeta_i - s)^{\delta-1}, & 0 \leqslant s < \zeta_1, \\ \sum_{i=2}^{\infty} \mu_i (\zeta_i - s)^{\delta-1}, & \zeta_1 \leqslant s < \zeta_2, \\ \dots \\ \sum_{i=i_0}^{\infty} \mu_i (\zeta_i - s)^{\delta-1}, & \zeta_{i_0-1} \leqslant s < \zeta_{i_0}, \\ \dots \\ 0, & \lim_{i\to\infty} \zeta_i \leqslant s \leqslant 1, \end{cases}$$

$$\kappa_1 = \frac{a_1}{\sigma\bar{\rho}_1}\left(1 + \left(\frac{a_1\rho_1}{\sigma\bar{\rho}_1} + 1\right)\frac{a_2\rho_2}{\sigma\bar{\rho}_2}\right), \qquad \kappa_2 = \left(\frac{a_1\rho_1}{\sigma\bar{\rho}_1} + 1\right)\frac{a_2}{\sigma\bar{\rho}_2},$$

$$\rho_1 = \int_0^1 t^{\delta-1} p_1(t)\,\mathrm{d}A_1(t) > 0, \qquad \bar{\rho}_1 = 1 - \frac{a_1\rho_1}{\sigma} > 0,$$

$$\rho_2 = \int_0^\eta t^{\delta-1} p_2(t)\,\mathrm{d}A_2(t) > 0, \qquad \bar{\rho}_2 = 1 - \left(\frac{a_1 a_2 \rho_1}{\sigma^2\bar{\rho}_1} + \frac{a_2}{\sigma}\right)\rho_2 > 0.$$

Obviously, $G(t,s)$ is a continuous function on $[0,1]\times[0,1]$.

Proof. By means of Lemma 2, we can turn (4) to an equivalent integral equation

$$u(t) = c_1 t^{\delta-1} + c_2 t^{\delta-2} - \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} y(s)\,\mathrm{d}s.$$

Considering the fact that $u(0) = 0$, we get that $c_2 = 0$. Then

$$u(t) = c_1 t^{\delta-1} - \int_0^t \frac{(t-s)^{\delta-1}}{\Gamma(\delta)} y(s)\,\mathrm{d}s. \tag{6}$$

On the other hand, by the condition

$$u(1) = a_1 \int_0^1 p_1(s)u(s)\,\mathrm{d}A_1(s) + a_2 \int_0^\eta p_2(s)u(s)\,\mathrm{d}A_2(s) + a_3 \sum_{i=1}^{\infty} \mu_i u(\zeta_i),$$

we have

$$c_1 = \frac{1}{\sigma\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} y(s)\,\mathrm{d}s + \frac{a_1}{\sigma} \int_0^1 p_1(s)u(s)\,\mathrm{d}A_1(s)$$
$$+ \frac{a_2}{\sigma} \int_0^\eta p_2(s)u(s)\,\mathrm{d}A_2(s) - \frac{a_3}{\sigma\Gamma(\delta)} \sum_{i=1}^{\infty} \mu_i \int_0^{\zeta_i} (\zeta_i - s)^{\delta-1} y(s)\,\mathrm{d}s. \tag{7}$$

By (6) and (7), we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}y(s)\,\mathrm{d}s + \frac{1}{\sigma\Gamma(\delta)}\int_0^1 \big(t(1-s)\big)^{\delta-1}y(s)\,\mathrm{d}s \\ &\quad + \frac{a_1t^{\delta-1}}{\sigma}\int_0^1 p_1(s)u(s)\,\mathrm{d}A_1(s) + \frac{a_2t^{\delta-1}}{\sigma}\int_0^\eta p_2(s)u(s)\,\mathrm{d}A_2(s) \\ &\quad - \frac{a_3}{\sigma\Gamma(\delta)}\sum_{i=1}^{\infty}\mu_i\int_0^{\zeta_i}\big(t(\zeta_i-s)\big)^{\delta-1}y(s)\,\mathrm{d}s \\ &= \frac{1}{\sigma\Gamma(\delta)}\int_0^t t^{\delta-1}(1-s)^{\delta-1}\left(1-a_3\sum_{i=1}^{\infty}\frac{b_i(s)}{(1-s)^{\delta-1}}\right)y(s)\,\mathrm{d}s \\ &\quad - \frac{1}{\sigma\Gamma(\delta)}\int_0^t \sigma(t-s)^{\delta-1}y(s)\,\mathrm{d}s \\ &\quad + \frac{1}{\sigma\Gamma(\delta)}\int_t^1 t^{\delta-1}(1-s)^{\delta-1}\left(1-a_3\sum_{i=1}^{\infty}\frac{b_i(s)}{(1-s)^{\delta-1}}\right)y(s)\,\mathrm{d}s \\ &\quad + \frac{a_1t^{\delta-1}}{\sigma}\int_0^1 p_1(s)u(s)\,\mathrm{d}A_1(s) + \frac{a_2t^{\delta-1}}{\sigma}\int_0^\eta p_2(s)u(s)\,\mathrm{d}A_2(s), \end{aligned} \tag{8}$$

where

$$b_i(s) = \begin{cases} \mu_i(\zeta_i-s)^{\delta-1}, & 0\leqslant s<\zeta_i, \\ 0, & \zeta_i\leqslant s\leqslant 1, \end{cases} \qquad i=1,2,\dots .$$

Then, by (8), we have

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,s)y(s)\,\mathrm{d}s + \frac{a_1t^{\delta-1}}{\sigma}\int_0^1 p_1(s)u(s)\,\mathrm{d}A_1(s) \\ &\quad + \frac{a_2t^{\delta-1}}{\sigma}\int_0^\eta p_2(s)u(s)\,\mathrm{d}A_2(s). \end{aligned} \tag{9}$$

Multiplying both sides of (9) by $p_1(t)$ and integrating from 0 to 1, we have

$$\int_0^1 p_1(t)u(t)\,\mathrm{d}A_1(t) = \int_0^1 p_1(t)\left(\int_0^1 G_1(t,s)y(s)\,\mathrm{d}s\right)\mathrm{d}A_1(t)$$

$$+\frac{a_1}{\sigma}\int_0^1 t^{\delta-1}p_1(t)\,\mathrm{d}A_1(t)\int_0^1 p_1(s)u(s)\,\mathrm{d}A_1(s)$$
$$+\frac{a_2}{\sigma}\int_0^1 t^{\delta-1}p_1(t)\,\mathrm{d}A_1(t)\int_0^\eta p_2(s)u(s)\,\mathrm{d}A_2(s). \tag{10}$$

Then, from (10) we obtain

$$\int_0^1 p_1(t)u(t)\,\mathrm{d}A_1(t)=\frac{1}{\bar\rho_1}\int_0^1 P_1(s)y(s)\,\mathrm{d}s+\frac{a_2\rho_1}{\sigma\bar\rho_1}\int_0^\eta p_2(s)u(s)\,\mathrm{d}A_2(s). \tag{11}$$

Substituting (11) into (9), we have

$$u(t)=\int_0^1 G_1(t,s)y(s)\,\mathrm{d}s+\frac{a_1t^{\delta-1}}{\sigma\bar\rho_1}\int_0^1 P_1(s)y(s)\,\mathrm{d}s$$
$$+\left(\frac{a_1a_2\rho_1}{\sigma^2\bar\rho_1}+\frac{a_2}{\sigma}\right)t^{\delta-1}\int_0^\eta p_2(s)u(s)\,\mathrm{d}A_2(s). \tag{12}$$

Multiplying both sides of (12) by $p_2(t)$ and integrating from 0 to η, we have

$$\int_0^\eta p_2(t)u(t)\,\mathrm{d}A_2(t)=\frac{1}{\bar\rho_2}\int_0^1 P_2(s)y(s)\,\mathrm{d}s+\frac{a_1\rho_2}{\sigma\bar\rho_1\bar\rho_2}\int_0^1 P_1(s)y(s)\,\mathrm{d}s. \tag{13}$$

From (12) and (13) we have

$$u(t)=\int_0^1 G_1(t,s)y(s)\,\mathrm{d}s+\kappa_1\int_0^1 t^{\delta-1}P_1(s)y(s)\,\mathrm{d}s+\kappa_2\int_0^1 t^{\delta-1}P_2(s)y(s)\,\mathrm{d}s$$
$$=\int_0^1 G(t,s)y(s)\,\mathrm{d}s.$$

The proof is complete. □

Lemma 5. *Suppose that* $a_3\sum_{i=1}^\infty \mu_i\zeta_i^{\delta-1}<1$*, then the function* $l(s)=1-a_3b(s)/(1-s)^{\delta-1}$ *defined in Lemma* 4 *satisfies* $l(s)>0$*,* $s\in[0,1]$*.*

Proof. According to the property in convergence of sequence, there exists $0\leqslant\zeta_0\leqslant1$ such that $\lim_{i\to\infty}\zeta_i=\zeta_0$. For $s\in[0,1]$, we may discuss in two aspects:

(i) If $0 \leqslant s < \zeta_0$, then there exists $N \in \mathbb{N}_+$ such that $\zeta_{N-1} \leqslant s < \zeta_N$. Hence,

$$l(s) = 1 - a_3 \frac{b(s)}{(1-s)^{\delta-1}} = 1 - a_3 \sum_{i=N}^{\infty} \mu_i \frac{(\zeta_i - s)^{\delta-1}}{(1-s)^{\delta-1}}.$$

The assumption $a_3 \sum_{i=1}^{\infty} \mu_i \zeta_i^{\delta-1} < 1$ gives a guarantee that the above function is well defined. Moreover,

$$\begin{aligned} l'(s) &= a_3 \sum_{i=N}^{\infty} \mu_i \big[(\delta-1)(\zeta_i - s)^{\delta-2}(1-s)^{1-\delta} + (1-\delta)(\zeta_i - s)^{\delta-1}(1-s)^{-\delta}\big] \\ &= a_3 \sum_{i=N}^{\infty} \mu_i (\zeta_i - s)^{\delta-2}(1-s)^{-\delta}(\delta-1)(1-\zeta_i) \geqslant 0. \end{aligned}$$

(ii) If $s \geqslant \zeta_0$, then $s > \zeta_i$ for all $i \in \mathbb{N}_+$. In view of the definition of $b(s)$ in Lemma 4, we know that $b(s) = 0$, $s \in [\zeta_0, 1]$. Thus, $l(s) = 1$, $s \in [\zeta_0, 1]$.

Hence, from (1) and (2) we know that $l(s)$ is nondecreasing on $[0,1]$, and $l(s) \geqslant l(0) > 0$, $s \in [0,1]$. The proof is complete. □

Lemma 6. *Let $a_3 \sum_{i=1}^{\infty} \mu_i \zeta_i^{\delta-1} < 1$, and $A_i(t)$ is increasing on $t \in [0,1]$ $(i = 1, 2)$. Then the Green function $G(t,s)$ defined by* (5) *satisfies*:

(i) $G(t,s) \geqslant 0$, $(t,s) \in [0,1] \times [0,1]$;
(ii) $t^{\delta-1} G(1,s) \leqslant G(t,s) \leqslant t^{\delta-1} Q(s)$, $t, s \in [0,1]$, *where*

$$Q(s) = \frac{(1-s)^{\delta-1}}{\sigma\Gamma(\delta)} l(s) + \kappa_1 P_1(s) + \kappa_2 P_2(s).$$

Proof. (i) For $0 \leqslant s \leqslant t \leqslant 1$,

$$\begin{aligned} G(t,s) &= \frac{1}{\sigma\Gamma(\delta)} \big[\big(t(1-s)\big)^{\delta-1} l(s) - \sigma(t-s)^{\delta-1}\big] + \kappa_1 t^{\delta-1} P_1(s) + \kappa_2 t^{\delta-1} P_2(s) \\ &\geqslant \frac{1}{\sigma\Gamma(\delta)} \big[\big(t(1-s)\big)^{\delta-1} l(s) - \sigma(t(1-s))^{\delta-1}\big] + \kappa_1 t^{\delta-1} P_1(s) + \kappa_2 t^{\delta-1} P_2(s) \\ &= \frac{(t(1-s))^{\delta-1}}{\sigma\Gamma(\delta)} [l(s) - l(0)] + \kappa_1 t^{\delta-1} P_1(s) + \kappa_2 t^{\delta-1} P_2(s) \geqslant 0. \end{aligned}$$

For $0 \leqslant t \leqslant s \leqslant 1$,

$$G(t,s) = \frac{1}{\sigma\Gamma(\delta)} \big(t(1-s)\big)^{\delta-1} l(s) + \kappa_1 t^{\delta-1} P_1(s) + \kappa_2 t^{\delta-1} P_2(s) \geqslant 0.$$

(ii) For $0 \leqslant s \leqslant t \leqslant 1$,

$$\begin{aligned} G(t,s) &= \frac{1}{\sigma\Gamma(\delta)} \big[\big(t(1-s)\big)^{\delta-1} l(s) - \sigma(t-s)^{\delta-1}\big] + \kappa_1 t^{\delta-1} P_1(s) + \kappa_2 t^{\delta-1} P_2(s) \\ &\geqslant \frac{1}{\sigma\Gamma(\delta)} \big[\big(t(1-s)\big)^{\delta-1} l(s) - \sigma\big(t(1-s)\big)^{\delta-1}\big] + \kappa_1 t^{\delta-1} P_1(s) + \kappa_2 t^{\delta-1} P_2(s) \\ &= t^{\delta-1} G(1,s), \end{aligned}$$

$$
\begin{aligned}
G(t,s) &= \frac{1}{\sigma\Gamma(\delta)}\big[\big(t(1-s)\big)^{\delta-1}l(s) - \sigma(t-s)^{\delta-1}\big] + \kappa_1 t^{\delta-1}P_1(s) + \kappa_2 t^{\delta-1}P_2(s) \\
&\leqslant \frac{1}{\sigma\Gamma(\delta)}\big(t(1-s)\big)^{\delta-1}\big[l(s)-l(0)\big] + \frac{l(0)(\delta-1)}{\sigma\Gamma(\delta)}\int\limits_{t-s}^{t(1-s)} x^{\delta-2}\,\mathrm{d}x \\
&\quad + \kappa_1 t^{\delta-1}P_1(s) + \kappa_2 t^{\delta-1}P_2(s) \\
&\leqslant \frac{1}{\sigma\Gamma(\delta)}\big(t(1-s)\big)^{\delta-1}\big[l(s)-l(0)\big] + \frac{l(0)(\delta-1)}{\sigma\Gamma(\delta)}\big(t(1-s)\big)^{\delta-2}s(1-t) \\
&\quad + \kappa_1 t^{\delta-1}P_1(s) + \kappa_2 t^{\delta-1}P_2(s) \\
&\leqslant \frac{1}{\sigma\Gamma(\delta)}\big(t(1-s)\big)^{\delta-1}\big[l(s)-(2-\delta)l(0)\big] + \kappa_1 t^{\delta-1}P_1(s) + \kappa_2 t^{\delta-1}P_2(s) \\
&= t^{\delta-1}Q(s).
\end{aligned}
$$

For $0 \leqslant t \leqslant s \leqslant 1$,

$$
\begin{aligned}
G(t,s) &= \frac{1}{\sigma\Gamma(\delta)}\big(t(1-s)\big)^{\delta-1}l(s) + \kappa_1 t^{\delta-1}P_1(s) + \kappa_2 t^{\delta-1}P_2(s) = t^{\delta-1}G(1,s), \\
G(t,s) &= \frac{1}{\sigma\Gamma(\delta)}\big(t(1-s)\big)^{\delta-1}l(s) + \kappa_1 t^{\delta-1}P_1(s) + \kappa_2 t^{\delta-1}P_2(s) = t^{\delta-1}Q(s).
\end{aligned}
$$

The proof is complete. □

Lemma 7. (*See* [6].) *Let E be a real Banach space, P be a cone of E. Let $\Omega \subset E$ be a bounded open set, $T\colon \overline{\Omega} \cap P \to P$ be a completely continuous. If there exists $u_0 \in P\setminus\{\theta\}$ such that $u - Tu \neq \mu u_0$ for all $\mu \geqslant 0$, $u \in \partial\Omega \cap P$, then $i(T, \Omega\cap P, P) = 0$.*

Lemma 8. (*See* [6].) *Let E be a real Banach space, P be a cone of E. Let $\Omega \subset E$ be a bounded open set with $\theta \in \Omega$, and $T\colon \overline{\Omega} \cap P \to P$ be a completely continuous. If $\mu u \neq Tu$ for all $\mu \geqslant 1$, $u \in \partial\Omega \cap P$, then $i(T, \Omega \cap P, P) = 1$.*

Lemma 9. (*See* [16].) *Let E be a real Banach space, P be a cone of E. Suppose that $L\colon E \to E$ is a completely continuous linear operator, and $L(P) \subset P$. If there exist $\psi \in P - P$, $\psi \notin -P$ and a constant $c > 0$ such that $cL\psi \geqslant \psi$, then the spectral radius $r(L) \neq 0$, and L has a positive eigenfunction φ^* corresponding to its first eigenvalue $\lambda_1 = (r(L))^{-1}$ such that $\lambda_1 L\varphi^* = \varphi^*$.*

Definition 3. (See [6, 16].) Let E be a real Banach space, P be a cone of E. Let $T\colon E \to E$ be a linear operator, and $T\colon P \to P$. If there exists $u_0 \in P \setminus \{\theta\}$ such that for any $x \in P \setminus \{\theta\}$, there exist a natural number n and real numbers α, $\beta > 0$, satisfying $\alpha u_0 \leqslant T^n x \leqslant \beta u_0$, then T is called a u_0-bounded linear operator on E.

Lemma 10. (*See* [6, 16].) *Let E be a real Banach space, P be a cone of E. Let T be a completely continuous u_0-bounded operator, λ_1 be the first eigenvalue of T. Then T must have a positive eigenfunction φ^*, which belongs to $P \setminus \{\theta\}$ such that $\lambda_1 L\varphi^* = \varphi^*$; and λ_1 is the unique positive eigenvalue of T corresponding to the positive eigenfunction.*

Let $E = C[0,1]$, $\|u\| = \sup_{0\leqslant t\leqslant 1}|u(t)|$. Then $(E, \|\cdot\|)$ is a Banach space. Let $P = \{u \in E\colon\ u(t) \geqslant 0,\ t \in [0,1]\}$. It is easy to see that P is a cone in E.

In what follows, two operators $T, L_1\colon E \to E$ are defined respectively by

$$(Tu)(t) = \int_0^1 G(t,s) f\bigl(s, I_{0+}^{\beta} u(s), I_{0+}^{\beta-\alpha} u(s), u(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1],$$

$$(L_1 u)(t) = \int_0^1 G(t,s)\bigl(I_{0+}^{\beta} u(s) + I_{0+}^{\beta-\alpha} u(s) + u(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1]. \tag{14}$$

Lemma 11. *As is defined by* (14)*,* $(L_1 u)(t) = \int_0^1 \overline{G}_1(t,s)u(s)\,\mathrm{d}s$*,* $t \in [0,1]$*, where*

$$\overline{G}_1(t,s) = \frac{1}{\Gamma(\beta)} \int_s^1 G(t,\tau)(\tau - s)^{\beta-1}\,\mathrm{d}\tau + \frac{1}{\Gamma(\beta-\alpha)} \int_s^1 G(t,\tau)(\tau - s)^{\beta-\alpha-1}\,\mathrm{d}\tau$$
$$+ G(t,s), \quad (t,s) \in [0,1] \times [0,1].$$

Proof. For any $u \in E$, $t \in [0,1]$, by (14) we obtain

$$\begin{aligned}
(L_1 u)(t) &= \int_0^1 G(t,s)\bigl(I_{0+}^{\beta} u(s) + I_{0+}^{\beta-\alpha} u(s) + u(s)\bigr)\,\mathrm{d}s \\
&= \frac{1}{\Gamma(\beta)} \int_0^1 \left(\int_s^1 G(t,\tau)(\tau - s)^{\beta-1}\,\mathrm{d}\tau \right) u(s)\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\beta-\alpha)} \int_0^1 \left(\int_s^1 G(t,\tau)(\tau - s)^{\beta-\alpha-1}\,\mathrm{d}\tau \right) u(s)\,\mathrm{d}s + \int_0^1 G(t,s)u(s)\,\mathrm{d}s \\
&= \int_0^1 \overline{G}_1(t,s)u(s)\,\mathrm{d}s.
\end{aligned}$$

□

It is easy to verify that $T\colon P \to P$ and $L_1\colon P \to P$ are completely continuous operators.

Lemma 12. *The spectral radius* $r(L_1) \neq 0$*, and* L_1 *has a positive eigenfunction* φ^* *corresponding to the first eigenvalue* $\lambda_1 = (r(L_1))^{-1}$ *such that* $\lambda_1 L_1 \varphi^* = \varphi^*$.

Proof. It is easy to check that $L_1\colon P \to P$ is a completely continuous operator. In fact, by Lemma 6, there exists $[a,b] \in (0,1)$ such that $G(t,s) > 0$ for $t, s \in [a,b]$. On the other hand, P is generating, i.e. $C[0,1] = P - P$. Take $\psi \in C[0,1]$ such that $\psi(t) > 0$

for $t \in (a,b)$, and $\psi(t) = 0$ for $t \notin (a,b)$. Then for all $t \in [a,b]$,

$$(L_1\psi)(t) = \int_0^1 G(t,s)\big(I_{0+}^{\beta}\psi(s) + I_{0+}^{\beta-\alpha}\psi(s) + \psi(s)\big)\,\mathrm{d}s \geqslant \int_a^b G(t,s)\psi(s)\,\mathrm{d}s > 0.$$

According to the density of $\mathbb{R}$, there exists a constant $c > 0$ such that $c(L_1\psi)(t) \geqslant \psi(t)$, $t \in [0,1]$. In view of Lemma 9, the spectral radius $r(L_1) \neq 0$, and L_1 has a positive eigenfunction φ^* corresponding to its first eigenvalue $\lambda_1 = (r(L_1))^{-1}$ such that $\lambda_1 L_1 \varphi^* = \varphi^*$. The proof is complete. □

3 Existence of a positive solution

In this section, let λ_1 be the first eigenvalue of operator L_1. We need the following conditions:

(H1)
$$\liminf_{\substack{u+v+w\to 0^+\\ u,v,w\geqslant 0}} \min_{t\in[0,1]} \frac{f(t,u,v,w)}{u+v+w} > \lambda_1, \tag{H1$_1$}$$

$$\limsup_{\substack{u+v+w\to +\infty\\ u,v,w\geqslant 0}} \max_{t\in[0,1]} \frac{f(t,u,v,w)}{u+v+w} < \lambda_1, \tag{H1$_2$}$$

(H2)
$$\limsup_{\substack{u+v+w\to 0^+\\ u,v,w\geqslant 0}} \max_{t\in[0,1]} \frac{f(t,u,v,w)}{u+v+w} < \lambda_1, \tag{H2$_1$}$$

$$\liminf_{\substack{u+v+w\to +\infty\\ u,v,w\geqslant 0}} \min_{t\in[0,1]} \frac{f(t,u,v,w)}{u+v+w} > \lambda_1. \tag{H2$_2$}$$

Theorem 1. *Assume that* (H1) *or* (H2) *holds, then the BVP* (1) *has at least one positive solution.*

Proof. It follows from (H1$_1$) that there exists $r_1 > 0$ such that, for all $t \in [0,1]$,

$$f(t,u,v,w) \geqslant \lambda_1(u+v+w), \quad 0 \leqslant u+v+w \leqslant r_1,\ u,v,w \geqslant 0. \tag{15}$$

For any $u \in P$, $t \in [0,1]$, we have

$$0 \leqslant I_{0+}^{\beta}u(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}u(s)\,\mathrm{d}s \leqslant \frac{\|u\|}{\Gamma(\beta+1)} \leqslant \|u\|, \tag{16}$$

$$0 \leqslant I_{0+}^{\beta-\alpha}u(t) = \frac{1}{\Gamma(\beta-\alpha)}\int_0^t (t-s)^{\beta-\alpha-1}u(s)\,\mathrm{d}s \leqslant \frac{\|u\|}{\Gamma(\beta-\alpha+1)} \leqslant \|u\|. \tag{17}$$

Let $B_r = \{u \in E\colon \|u\| < r \leqslant r_1/3\}$. Then, by (16) and (17), we have

$$0 \leqslant I_{0_+}^{\beta} u(t) + I_{0_+}^{\beta-\alpha} u(t) + u(t) \leqslant 3\|u\| \leqslant r_1, \quad u \in \overline{\Omega}_r \cap P,\ t \in [0,1],$$

which, together with (15), for all $u \in \overline{\Omega}_r \cap P$, $t \in [0,1]$, yields that

$$f\bigl(t, I_{0_+}^{\beta} u(t), I_{0_+}^{\beta-\alpha} u(t), u(t)\bigr) \geqslant \lambda_1 \bigl(I_{0_+}^{\beta} u(t) + I_{0_+}^{\beta-\alpha} u(t) + u(t)\bigr). \tag{18}$$

From (18) and the definition of T we know that, for every $u \in \overline{\Omega}_r \cap P$, $t \in [0,1]$,

$$\begin{aligned}
(Tu)(t) &= \int_0^1 G(t,s) f\bigl(s, I_{0_+}^{\beta} u(s), I_{0_+}^{\beta-\alpha} u(s), u(s)\bigr)\,\mathrm{d}s \\
&\geqslant \lambda_1 \int_0^1 G(t,s) \bigl|I_{0_+}^{\beta} u(s) + I_{0_+}^{\beta-\alpha} u(s) + u(s)\bigr|\,\mathrm{d}s \\
&= \lambda_1 \int_0^1 G(t,s) \bigl(I_{0_+}^{\beta} u(s) + I_{0_+}^{\beta-\alpha} u(s) + u(s)\bigr)\,\mathrm{d}s \\
&= \lambda_1 (L_1 u)(t).
\end{aligned} \tag{19}$$

In the following, we prove that

$$u - Tu \neq \mu\varphi^* \quad \forall u \in \partial B_r \cap P,\ \mu \geqslant 0. \tag{20}$$

If not, then there exist $u_1 \in \partial B_r \cap P$ and $\mu_1 \geqslant 0$ such that $u_1 - Tu_1 = \mu_1 \varphi^*$. Then $\mu_1 > 0$, and $u_1 = Tu_1 + \mu_1\varphi^* \geqslant \mu_1\varphi^*$. Let $\overline{\mu} = \sup\{\mu | u_1 \geqslant \mu\varphi^*\}$. Obviously, $\overline{\mu} \geqslant \mu_1 > 0$ and $u_1 \geqslant \overline{\mu}\varphi^*$, then $\lambda_1 L_1 u_1 \geqslant \overline{\mu}\lambda_1 L_1 \varphi^* = \overline{\mu}\varphi^*$. So, by (19) we have

$$u_1 = Tu_1 + \mu_1\varphi^* \geqslant \lambda_1 L_1 u_1 + \mu_1\varphi^* \geqslant \overline{\mu}\varphi^* + \mu_1\varphi^* = (\overline{\mu} + \mu_1)\varphi^*,$$

which contradicts the definition of $\overline{\mu}$. So, (20) holds. It follows from Lemma 7 that

$$i(T, B_r \cap P, P) = 0. \tag{21}$$

By $(\mathrm{H1}_2)$, there exist $R_1 > r$ and $0 < \kappa < 1$ such that

$$f(t,u,v,w) \leqslant \kappa\lambda_1 (u + v + w), \quad u + v + w \geqslant R_1,\ u,v,w \geqslant 0,\ t \in [0,1]. \tag{22}$$

Now we define a linear operator $\widetilde{L}_1\colon E \to E$ as $\widetilde{L}_1 u = \kappa\lambda_1 L_1 u$, $t \in [0,1]$. It is obvious that $\widetilde{L}_1\colon P \to P$ is a bounded linear operator, and the spectral radius of $\widetilde{L}_1$ is $r(\widetilde{L}_1) = \kappa < 1$.

Let

$$Z = \{u \in P\colon \mu u = Tu,\ \mu \geqslant 1\}.$$

For any $u \in E$, we set

$$D(u) = \{t \in [0,1]:\ u(t) \geqslant R_1\}. \tag{23}$$

From (23) we know that for any $u \in E \cap P$, $I_{0_+}^{\beta} u(t) + I_{0_+}^{\beta-\alpha} u(t) + u(t) \geqslant R_1$, $t \in D(u)$, which, together with (22), implies that

$$f\big(t, I_{0_+}^{\beta} u(t), I_{0_+}^{\beta-\alpha} u(t), u(t)\big) \leqslant \kappa\lambda_1 \big(I_{0_+}^{\beta} u(t) + I_{0_+}^{\beta-\alpha} u(t) + u(t)\big), \quad t \in D(u). \tag{24}$$

From (24) and the definition of T, for any $u \in Z$, $\mu \geqslant 1$, $t \in [0,1]$, we have

$$\begin{aligned}
u(t) &\leqslant \mu u(t) = (Tu)(t) \\
&= \int_0^1 G(t,s) f\big(s, I_{0_+}^{\beta} u(s), I_{0_+}^{\beta-\alpha} u(s), u(s)\big)\,\mathrm{d}s \\
&= \int_{D(u)} G(t,s) f\big(s, I_{0_+}^{\beta} u(s), I_{0_+}^{\beta-\alpha} u(s), u(s)\big)\,\mathrm{d}s \\
&\quad + \int_{[0,1]\setminus D(u)} G(t,s) f\big(s, I_{0_+}^{\beta} u(s), I_{0_+}^{\beta-\alpha} u(s), u(s)\big)\,\mathrm{d}s \\
&\leqslant \kappa\lambda_1 \int_{D(u)} G(t,s)\big(I_{0_+}^{\beta} u(s) + I_{0_+}^{\beta-\alpha} u(s) + u(s)\big)\,\mathrm{d}s + G f_{R_1} \\
&\leqslant \kappa\lambda_1 \int_0^1 G(t,s)\big(I_{0_+}^{\beta} u(s) + I_{0_+}^{\beta-\alpha} u(s) + u(s)\big)\,\mathrm{d}s + G f_{R_1} \\
&= (\widetilde{L}_1 u)(t) + G f_{R_1},
\end{aligned} \tag{25}$$

where $G = \max_{t,s\in[0,1]} |G(t,s)|$, $f_{R_1} = \max\{f(t,u,v,w):\ t \in [0,1],\ 0 \leqslant u,v,w \leqslant R_1\}$. By the Gelfand's formula, we know that $(I - \widetilde{L}_1)^{-1}$ exists and $(I - \widetilde{L}_1)^{-1} = \sum_{i=1}^{\infty} L_1^i$, which also implies $(I - \widetilde{L}_1)^{-1}(P) \subset P$. This, together with (25), yields that

$$u(t) \leqslant (I - \widetilde{L}_1)^{-1} G f_{R_1},$$

which means Z is bounded. Now we choose $R > \max\{R_1, \sup\{\|u\|:\ u \in Z\}\}$. We can get that

$$\mu u \neq Tu \quad \forall u \in \partial B_R \cap P,\ \mu \in [0,1].$$

By Lemma 8, we know

$$i(T, B_R \cap P, P) = 1. \tag{26}$$

It follows from (21) and (26) that $i(T, (B_R \setminus \overline{B}_r) \cap P, P) = i(T, B_R \cap P, P) - i(T, B_r \cap P, P) = 1$. So, the operator T has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$. This implies that BVP (1) has at least one positive solution.

If (H2) holds, similar to the proof of above, there exist $R > r > 0$ such that $i(T, B_r \cap P, P) = 1$, $i(T, B_R \cap P, P) = 0$. Therefore $i(T, B_R \setminus \overline{B}_r \cap P, P) = i(T, B_R \cap P, P) - i(T, B_r \cap P, P) = -1$. It implies that T has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$. This implies that the BVP (1) has at least one positive solution. The proof of Theorem 1 is completed. $\square$

Example 1. We consider the following fractional equations:

$$\begin{aligned}
&D_{0+}^{9/2} x(t) + \left(x + x' + D_{0+}^{5/2} x\right)^{-1/3} + \left|\sin\left(D_{0+}^{\beta} x\right)\right| = 0, \quad 0 < t < 1, \\
&x^{(j)}(0) = 0, \quad j = 0, 1, 2, 3, \\
&x'''(1) = \frac{\sqrt{2}}{5} \int_0^1 \frac{4}{4s^2+1} x'''(s)\,\mathrm{d}A_1(s) + 4 \int_0^{1/40} \frac{4}{4s^2+3} x'''(s)\,\mathrm{d}A_2(s) \\
&\qquad + 3 \sum_{i=1}^{\infty} \frac{2}{i^2} x'''\left(\left(1 - \frac{1}{i+1}\right)^5\right),
\end{aligned} \tag{27}$$

where $f(t, u, v, w) = (u+v+w)^{-1/3} + |\sin w|$, $\gamma = 9/2$, $\beta = 5/2$, $\alpha = 1$, $a_1 = \sqrt{2}/5$, $a_2 = 4$, $a_3 = 3$, $\eta = 1/40$, $\mu_i = 2/i^2$, $\zeta_i = (1 - 1/(i+1))^5$, $p_1(t) = 4/(4t^2+1)$, $p_2(t) = 4/(4t^2+3)$, and

$$A_1(t) = \begin{cases} 2, & t \in [0, \frac{1}{2}), \\ 3, & t \in [\frac{1}{2}, 1], \end{cases} \qquad A_2(t) = \begin{cases} 3, & t \in [0, \frac{1}{2}), \\ 4, & t \in [\frac{1}{2}, 1]. \end{cases}$$

It is obvious that

$$\begin{aligned}
&\liminf_{u+v+w \to 0^+} \min_{t \in [0,1]} \frac{f(t,u,v,w)}{u+v+w} \\
&\quad = \liminf_{u+v+w \to 0^+} \min_{t \in [0,1]} \frac{(u+v+w)^{-1/3} + |\sin w|}{u+v+w} = +\infty > \lambda_1, \\
&\limsup_{u+v+w \to \infty} \max_{t \in [0,1]} \frac{f(t,u,v,w)}{u+v+w} \\
&\quad = \limsup_{u+v+w \to \infty} \max_{t \in [0,1]} \frac{(u+v+w)^{-1/3} + |\sin w|}{u+v+w} = 0 < \lambda_1.
\end{aligned}$$

Thus, the assumptions of Theorem 1 are satisfied, therefore the BVP (27) has at least one positive solution.

4 Existence of the unique positive solution

In this section, we need the following condition:

(H3) There exist nonnegative functions $l_i \in L^1[0,1]$ ($i = 1,2,3$) such that, for any $u_i, v_i, w_i \in \mathbb{R}^+$ ($i = 1,2$), $t \in [0,1]$,

$$\begin{aligned}&\bigl|f(t,u_1,v_1,w_1) - f(t,u_2,v_2,w_2)\bigr|\\&\quad \leqslant l_1(t)|u_1-u_2| + l_2(t)|v_1-v_2| + l_3(t)|w_1-w_2|.\end{aligned}$$

Now, for $t \in [0,1]$, we define an operator $L_2\colon P \to P$ as follows:

$$(L_2u)(t) = \int_0^1 G(t,s)\bigl(l_1(s)I_{0_+}^{\beta}u(s) + l_2(s)I_{0_+}^{\beta-\alpha}u(s) + l_3(s)u(s)\bigr)\,\mathrm{d}s. \tag{28}$$

For convenience, we set

$$\begin{aligned}w_1 &= \max_{t\in[0,1]} \int_0^1 \left(\int_s^1 G(t,\tau)l_1(\tau)(\tau-s)^{\beta-1}\,\mathrm{d}\tau\right)\mathrm{d}s,\\ w_2 &= \max_{t\in[0,1]} \int_0^1 \left(\int_s^1 G(t,\tau)l_2(\tau)(\tau-s)^{\beta-\alpha-1}\,\mathrm{d}\tau\right)\mathrm{d}s,\\ w_3 &= \max_{t\in[0,1]} \int_0^1 G(t,s)l_3(s)\,\mathrm{d}s.\end{aligned}$$

Lemma 13. *The operator L_2 defined by* (28) *is a linear operator with $\|L_2\| = w_1/\Gamma(\beta) + w_2/\Gamma(\beta-\alpha) + w_3$.*

Proof. Let

$$\begin{aligned}\overline{G}_2(t,s) = {}&\frac{1}{\Gamma(\beta)}\int_s^1 G(t,\tau)l_1(\tau)(\tau-s)^{\beta-1}\,\mathrm{d}\tau\\ &+ \frac{1}{\Gamma(\beta-\alpha)}\int_s^1 G(t,\tau)l_2(\tau)(\tau-s)^{\beta-\alpha-1}\,\mathrm{d}\tau\\ &+ G(t,s)l_3(s), \quad (t,s)\in[0,1]\times[0,1].\end{aligned} \tag{29}$$

Then for any $u \in E$, $t \in [0,1]$, we have

$$\begin{aligned}(L_2u)(t) &= \int_0^1 G(t,s)\bigl(l_1(s)I_{0_+}^{\beta}u(s) + l_2(s)I_{0_+}^{\beta-\alpha}u(s) + l_3(s)u(s)\bigr)\,\mathrm{d}s\\ &= \frac{1}{\Gamma(\beta)}\int_0^1 \left(\int_s^1 G(t,\tau)l_1(\tau)(\tau-s)^{\beta-1}\,\mathrm{d}\tau\right)u(s)\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{\Gamma(\beta-\alpha)} \int_0^1 \left(\int_s^1 G(t,\tau) l_2(\tau)(\tau - s)^{\beta-\alpha-1}\,\mathrm{d}\tau \right) u(s)\,\mathrm{d}s \\
&+ \int_0^1 G(t,s) l_3(s) u(s)\,\mathrm{d}s \\
&= \int_0^1 \overline{G}_2(t,s) u(s)\,\mathrm{d}s, \quad t \in [0,1].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|L_2\| &= \max_{t\in[0,1]} \int_0^1 \bigl|\overline{G}_2(t,s)\bigr|\,\mathrm{d}s \\
&= \max_{t\in[0,1]} \left\{ \frac{1}{\Gamma(\beta)} \int_0^1 \left(\int_s^1 G(t,\tau) l_1(\tau)(\tau - s)^{\beta-1}\,\mathrm{d}\tau \right) \mathrm{d}s. \right. \\
&\quad \left. + \frac{1}{\Gamma(\beta-\alpha)} \int_0^1 \left(\int_s^1 G(t,\tau) l_2(\tau)(\tau - s)^{\beta-\alpha-1}\,\mathrm{d}\tau \right) \mathrm{d}s + \int_0^1 G(t,s) l_3(s)\,\mathrm{d}s \right\} \\
&= \frac{1}{\Gamma(\beta)} w_1 + \frac{1}{\Gamma(\beta-\alpha)} w_2 + w_3.
\end{aligned}
$$

□

Theorem 2. *Assume that* (H3) *holds and* $\|L_2\| = w_1/\Gamma(\beta) + w_2/\Gamma(\beta-\alpha) + w_3 < 1$. *Then the BVP* (1) *has a unique positive solution in* $C([0,1], \mathbb{R}^+)$.

Proof. For any $u, v \in P$, by (H3)we have

$$
\begin{aligned}
\|Tu - Tv\| &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s) f\bigl(s, I_{0+}^{\beta} u(s), I_{0+}^{\beta-\alpha} u(s), u(s)\bigr) \right. \\
&\qquad \left. - f\bigl(s, I_{0+}^{\beta} v(s), I_{0+}^{\beta-\alpha} v(s), v(s)\bigr)\,\mathrm{d}s \right| \\
&\leqslant \max_{t\in[0,1]} \int_0^1 \bigl|G(t,s)\bigr| \bigl| f\bigl(s, I_{0+}^{\beta} u(s), I_{0+}^{\beta-\alpha} u(s), u(s)\bigr) \\
&\qquad - f\bigl(s, I_{0+}^{\beta} v(s), I_{0+}^{\beta-\alpha} v(s), v(s)\bigr) \bigr|\,\mathrm{d}s
\end{aligned}
$$

$$\leqslant \max_{t\in[0,1]}\left\{\frac{1}{\Gamma(\beta)}\int\limits_0^1\left(\int\limits_s^1 G(t,\tau)l_1(\tau)(\tau-s)^{\beta-1}\,\mathrm{d}\tau\right)\bigl|u(s)-v(s)\bigr|\,\mathrm{d}s\right.$$
$$+\frac{1}{\Gamma(\beta-\alpha)}\int\limits_0^1\left(\int\limits_s^1 G(t,\tau)l_2(\tau)(\tau-s)^{\beta-\alpha-1}\,\mathrm{d}\tau\right)\bigl|u(s)-v(s)\bigr|\,\mathrm{d}s$$
$$\left.+\int\limits_0^1 G(t,s)l_3(s)\bigl|u(s)-v(s)\bigr|\,\mathrm{d}s\right\}$$
$$\leqslant\left(\frac{1}{\Gamma(\beta)}w_1+\frac{1}{\Gamma(\beta-\alpha)}w_2+w_3\right)\|u-v\|=\|L_2\|\|u-v\|,$$

which means that the operator T is a contraction mapping on P. So, by the Banach contraction mapping principle, T has a fixed point in P, and also the BVP (1) has a unique positive solution. The proof of Theorem 2 is completed. □

Theorem 3. *Assume that* (H3) *holds and*

$$\lim_{n\to\infty}\|L_2^n\|^{1/n}$$
$$=\lim_{n\to\infty}\left(\max_{t\in[0,1]}\int\limits_0^1\cdots\int\limits_0^1\left(\int\limits_0^1\overline{G}_2(t,s_1)\overline{G}_2(s_1,s_2)\cdots\overline{G}_2(s_{n-1},s_n)\,\mathrm{d}s_1\right)\mathrm{d}s_2\cdots\mathrm{d}s_n\right)^{1/n}$$
$$<1.$$

Then the BVP (1) *has a unique positive solution in* $C([0,1],\mathbb{R}^+)$.

Proof. By the Gelfand's formula, we know $r(L_2)=\lim_{n\to\infty}\|L_2^n\|^{1/n}<1$. Let $\varepsilon_0=(1-r(L_2))/3$. There exists a sufficiently large natural number N such that for $n\geqslant N$, $\|L_2^n\|\leqslant(r(L_2)+\varepsilon_0)^n$. For any $u\in E$, we define

$$\|u\|^*=\sum_{i=1}^N\bigl(r(L_2)+\varepsilon_0\bigr)^{N-i}\bigl\|L_2^{i-1}u\bigr\|,\tag{30}$$

where $L_2^0=I$ is the identity operator. Clearly, $\|\cdot\|^*$ is also a norm of E.

Then for any $u,v\in P$, by (30) we get

$$\|Tu-Tv\|^*=\sum_{i=1}^N\bigl(r(L_2)+\varepsilon_0\bigr)^{N-i}\bigl\|L_2^{i-1}(Tu-Tv)\bigr\|$$
$$=\sum_{i=1}^N\bigl(r(L_2)+\varepsilon_0\bigr)^{N-i}\max_{t\in[0,1]}\bigl|L_2^{i-1}(Tu-Tv)(t)\bigr|$$
$$\leqslant\sum_{i=1}^N\bigl(r(L_2)+\varepsilon_0\bigr)^{N-i}\max_{t\in[0,1]}\bigl|\bigl(L_2^{i}|u-v|\bigr)(t)\bigr|$$

$$
\begin{aligned}
&= \sum_{i=1}^{N} \bigl(r(L_2)+\varepsilon_0\bigr)^{N-i} \bigl\| \bigl(L_2^i |u-v|\bigr) \bigr\| \\
&= (r(L_2)+\varepsilon_0) \sum_{i=1}^{N-1} \bigl(r(L_2)+\varepsilon_0\bigr)^{N-i-1} \bigl\| \bigl(L_2^i |u-v|\bigr) \bigr\| \\
&\quad + \bigl\| \bigl(L_2^N |u-v|\bigr) \bigr\| \\
&\leqslant \bigl(r(L_2)+\varepsilon_0\bigr) \sum_{i=1}^{N-1} \bigl(r(L_2)+\varepsilon_0\bigr)^{N-i-1} \bigl\| \bigl(L_2^i |u-v|\bigr) \bigr\| \\
&\quad + \bigl(r(L_2)+\varepsilon_0\bigr)^{N} \|u-v\| \\
&= \bigl(r(L_2)+\varepsilon_0\bigr) \sum_{i=0}^{N-1} \bigl(r(L_2)+\varepsilon_0\bigr)^{N-i-1} \bigl\| \bigl(L_2^i |u-v|\bigr) \bigr\| \\
&= \bigl(r(L_2)+\varepsilon_0\bigr) \sum_{i=1}^{N} (r(L_2)+\varepsilon_0)^{N-i} \bigl\| \bigl(L_2^{i-1} |u-v|\bigr) \bigr\| \\
&= \bigl(r(L_2)+\varepsilon_0\bigr) \|u-v\|^* = \frac{1+2r(L_2)}{3} \|u-v\|^*.
\end{aligned}
$$

This means that the operator T is a contraction mapping on P. By the Banach contraction mapping principle, T has a fixed point in P. That is, the BVP (1) has a unique positive solution on E. The proof of Theorem 3 is completed. □

Remark 1. We end the paper with the following examples. In Example 3, the BVP (1) has a unique positive solution under the condition that $r(L_2) < 1$, but $\|L_2\| > 1$. As it is well known, $\|L_2\| < 1$ implies that $r(L_2) < 1$. That is, Theorem 3 is an extension of Theorem 2. But, it is very difficult to calculate the value of $r(L_2)$ in most cases, and the value of $\|L_2\|$ is relatively easy to calculate. Example 2 shows that if we verify the condition of Theorem 2, We will avoid calculating the value of $r(L_2)$, which is an extremely complex work.

Example 2. We consider BVP (1) with $\gamma = 5$, $\beta = 3$, $\alpha = 3/2$, $a_1 = a_2 = a_3 = 0$,

$$
f(t,u,v,w) = \begin{cases} \sin^2 tu + \sin^4 tv + \frac{1}{2}\arctan w, & t \in \mathbb{R} \setminus (0,1], \\ \frac{1}{2}\arctan w, & t \in (0,1]. \end{cases}
$$

Then the problem can be transformed to the following two-point boundary value problem:

$$
\begin{aligned}
&u''(t) + f\bigl(t, I_{0+}^{3} u(t), I_{0+}^{3/2} u(t), u(t)\bigr) = 0, \quad 0 < t < 1, \\
&u(0) = u(1) = 0.
\end{aligned}
$$

The corresponding Green's function is

$$
G(t,s) = \begin{cases} s(1-t), & 0 \leqslant s \leqslant t \leqslant 1, \\ t(1-s), & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}
$$

For any $u_i, v_i, w_i \in \mathbb{R}^+ (i = 1, 2)$, $t \in [0, 1]$, we have

$$\left|f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)\right| \leqslant \frac{1}{2}|w_1 - w_2|.$$

This means

$$\|L_2\| = \frac{1}{\Gamma(\beta)} w_1 + \frac{1}{\Gamma(\beta - \alpha)} w_2 + w_3 \leqslant 0 + 0 + \frac{1}{2} < 1,$$

and, consequently, the assumptions of Theorem 2 are satisfied. Thus, the BVP (1) has a unique positive solution.

Example 3. We consider BVP (1) with $\gamma = 5$, $\beta = 3$, $\alpha = 3/2$, $a_1 = a_2 = a_3 = 0$,

$$f(t, u, v, w) = \begin{cases} \sin^2 tu + \sin^4 tv + 9w, & t \in \mathbb{R} \setminus (0, 1], \\ 9w, & t \in (0, 1]. \end{cases}$$

Then the problem can be transformed to the following two-point boundary value problem

$$u''(t) + f\big(t, I_{0+}^3 u(t), I_{0+}^{3/2} u(t), u(t)\big) = 0, \quad 0 < t < 1,$$
$$u(0) = u(1) = 0.$$

The corresponding Green's function is

$$G(t, s) = \begin{cases} s(1 - t), & 0 \leqslant s \leqslant t \leqslant 1, \\ t(1 - s), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

Obviously, for any $u_i, v_i, w_i \in \mathbb{R}^+$ $(i = 1, 2)$, $t \in [0, 1]$, we have

$$\left|f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2)\right| \leqslant 9|w_1 - w_2|,$$

which implies $l_1(t) = l_2(t) = 0$, $l_3(t) = 9$. Then, by definition (29), we have $\overline{G}_2(t, s) = 9G(t, s)$. It is easy to check that $\|L_2\| = 9/8 > 1$. However, by computation, we have

$$\int_0^1 \overline{G}_2(t, s) \sin(\pi s)\, \mathrm{d}s = \frac{9}{\pi^2} \sin(\pi t), \quad t \in [0, 1],$$

which means that $9/\pi^2$ is a positive eigenvalue of L_2. On the other hand, in view of Definition 3, we get that L_2 is a u_0-bounded operator with $u_0(t) = t(1 - t)$. It follows from Lemma 10 that L_2 has no positive eigenvalue except for the first eigenvalue $\lambda_1 = (r(L))^{-1}$. Thus, we conclude that $r(L_2) = 9/\pi^2 < 1$. Then, by Theorem 3, we know that the BVP (1) has a unique positive solution.

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