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A note about the deterministic property of characteristic functions

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Abstract. We study an extension property for characteristic functions $f : \mathbb{R}^n \to \mathbb{C}$ of probability measures. More precisely, let f be the characteristic function of a probability density φ on $\mathbb{R}^n$, and let $U_\sigma = \{x \in \mathbb{R}^n\colon \min_k |x_k| > \sigma\}$, $\sigma > 0$, be a neighborhood of infinity. We say that f has the σ-deterministic property if for any other characteristic function g such that $f = g$ on U_σ, it follows that $f \equiv g$. A sufficient condition on f to has the σ-deterministic property is given. We also discuss the question about how precise our sufficient condition is? These results show that the σ-deterministic property of f depends on an arithmetic structure of the support of φ.

Keywords: characteristic function, density function, entire function, probability measure, Bernstein space.

1 Introduction

Let $M(\mathbb{R}^n)$ be the family of finite complex-valued regular Borel measures on $\mathbb{R}^n$. Given a measure $\mu \in M(\mathbb{R}^n)$, we define its Fourier transform by

$$\widehat{\mu}(x) = \int\limits_{\mathbb{R}^n} \mathrm{e}^{-i(x,t)}\,\mathrm{d}\mu(t),$$

$x \in \mathbb{R}^n$. Here and subsequently, (x,t) denotes the scalar product $\sum_{k=1}^n x_k t_k$ of vectors $x, t \in \mathbb{R}^n$. If the norm in $M(\mathbb{R}^n)$ is given by the total variation of $\mu \in M(\mathbb{R}^n)$, then this allow us to identify the usual Lebesgue Banach space $L^1(\mathbb{R}^n)$ with the closed ideal in $M(\mathbb{R}^n)$ of all measures, which are absolutely continuous with respect to the Lebesgue measure $\mathrm{d}t = \mathrm{d}t_1 \cdots \mathrm{d}t_n$ on $\mathbb{R}^n$.

Assume that $\mu \in M(\mathbb{R}^n)$ is a positive measure. If, in addition, $\|\mu\| = 1$, then in the language of probability theory, this μ and the function $f(x) := \widehat{\mu}(-x)$, $x \in \mathbb{R}^n$, are called a probability measure and its characteristic function, respectively. In particular, if $\mu = \varphi\,\mathrm{d}t$ with $\varphi \in L^1(\mathbb{R}^n)$ such that $\|\varphi\|_{L^1(\mathbb{R}^n)} = 1$ and $\varphi \geqslant 0$ on $\mathbb{R}^n$, then φ is called the probability density function of μ, or the probability density for short. Let us note that if φ is a measurable function on $\mathbb{R}^n$, then we write here and in the sequel $\varphi \geqslant 0$ on $\mathbb{R}^n$ if $\varphi \geqslant 0$ $\mathrm{d}t$-almost everywhere on $\mathbb{R}^n$.

For a characteristic function f and a subset U of $\mathbb{R}^n$, we study the problem: is it true that there exists a characteristic function g on $\mathbb{R}^n$ such that $g = f$ on U but $g \not\equiv f$? Our interest to this question is initiated by a similar problem posed by N.G. Ushakov in [9, p. 276]: Is it true that for any neighborhoods of infinity $U \subset \mathbb{R}$ with $0 \notin U$, there exists the characteristic function g such that $g \not\equiv \mathrm{e}^{-x^2/2}$ but $g(x) = \mathrm{e}^{-x^2/2}$ for all $x \in U$? A positive answer to this question was given by Gneiting [1, p. 360]:

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{C}$ be the characteristic function of a distribution with a continuous and strictly positive density. Then there exists, for each $\sigma > 0$, a characteristic function g such that $f(x) = g(x)$ if $x = 0$ or $|x| \geqslant \sigma$ and $f(x) \neq g(x)$ otherwise.*

Moreover, in [1, p. 361], the author also conjectured that the same statement holds for any characteristic function with an absolutely continuous component. This conjecture was disproved in [3]. Indeed, for $a > 0$,

$$\varphi(t) = \begin{cases} \frac{2(a-2|t|)}{a^2}, & |t| \leqslant \frac{a}{2}, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

is the density of the usual triangular probability distribution. Let $\sigma > 0$ and assume that g is a characteristic function such that $g(x) = \widehat{\varphi}(-x)$ for $|x| > \sigma$. If

$$a\sigma \leqslant \pi, \tag{2}$$

then $g(x) = \widehat{\varphi}(-x)$ for all $x \in \mathbb{R}$ (see [3, Ex. 1]).

In this paper, a problem of uniqueness for extensions of characteristic functions of several variables is studied. More precisely, given a characteristic function $f : \mathbb{R}^n \to \mathbb{C}$, we consider characteristic extensions of f in a manner indicated by the above mentioned Ushakov's problem, from a neighborhoods U of infinity to the whole $\mathbb{R}^n$. In particular, we obtain that estimate (2) can be weakened. Our Theorems 2 and 3 show that the exact estimate in (2) is $a\sigma \leqslant 2\pi$. Any characteristic function $f : \mathbb{R}^n \to \mathbb{C}$ satisfies $f(-x) = \overline{f(x)}$ for each $x \in \mathbb{R}^n$. Hence, it is enough to study the extensions only from symmetric neighborhoods U. For $\sigma > 0$, set $Q_\sigma^n = \{x \in \mathbb{R}^n\colon\ |x_k| \leqslant \sigma,\ k = 1, \dots, n\}$. Then

$$U_\sigma = \mathbb{R}^n \setminus Q_\sigma^n$$

denotes such a neighborhood. Also, we say that f has the σ-deterministic property if there exists no other characteristic function g such that $f(x) = g(x)$ for all $x \in U_\sigma$.

Let $\tau \in \mathbb{R}$, and let A and B be subsets of $\mathbb{R}^n$. Then $A + B$ and τA denote the sets $\{a + b\colon\ a \in A,\ b \in B\}$ and $\{\tau a\colon\ a \in A\}$, respectively. If, in addition, A is measurable, then we denote the Lebesgue measure of A by $|A|$. Given a measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$, we denote by S_φ the essential support of φ. By definition, a point $x \in \mathbb{R}^n$ belongs to S_φ if for any $\delta > 0$, the set

$$\bigl(x + Q_\delta^n\bigr) \cap \bigl\{t \in \mathbb{R}^n\colon\ \bigl|\varphi(t)\bigr| > 0\bigr\}$$

has positive Lebesgue measure. Note that if φ is continuous on $\mathbb{R}^n$, then S_φ coincides with the usual support of φ. As usual, $\mathbb{Z}^n$ is the n-dimensional integer lattice.

The following theorem is the main result of this paper.

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be the characteristic function of a probability density φ. Assume that there exist $a \in \mathbb{R}^n$, $\varrho > 0$ and $\tau > 0$ such that*

$$\left|S_\varphi \cap \left(a + Q_\varrho^n + \tau\mathbb{Z}^n\right)\right| = 0. \tag{3}$$

If

$$\sigma\tau \leqslant 2\pi, \tag{4}$$

then f has the σ-deterministic property.

Note that if a pair of positive numbers ϱ and τ satisfy (3), then it is necessary that

$$\varrho < \frac{\tau}{2}. \tag{5}$$

Indeed, in the converse case, we have that $Q_\delta^n + \tau\mathbb{Z}^n = \mathbb{R}^n$. On the other hand, it is clear that, for any probability density φ, we have $|S_\varphi| > 0$. Hence, $|S_\varphi \cap (a + Q_\varrho^n + \tau\mathbb{Z}^n)| > 0$ contrary to condition (3).

The statement of Theorem 2 is sharp in the sense that the right-hand side of (4) cannot be replaced by $2\pi + \varepsilon$ for any positive ε. This follows from the next theorem.

Theorem 3. *For any positive σ and τ such that*

$$\sigma\tau > 2\pi, \tag{6}$$

there exist $\varrho > 0$ and a probability density φ such that (3) *is satisfied but the characteristic function of φ has no the σ-deterministic property.*

We conclude this section by presenting our previous paper [4], where a similar extension problem was studied in the case of continuous density functions of one variable. The main result of [4] states that if φ is a continuous probability density on $\mathbb{R}$ such that there exist lattices $\Lambda_j = \tau_j + \alpha_j\mathbb{Z}$, $\tau_j \in \mathbb{R}$, $\alpha_j > 0$, $\alpha_j\sigma \leqslant 2\pi$, $j = 1, 2$, $\Lambda_1 \cap \Lambda_2 = \emptyset$, and φ vanishes on $\Lambda_1 \cup \Lambda_2$, then, for any characteristic function $g : \mathbb{R} \to \mathbb{C}$ such that it coincides on U_σ with the characteristic function f of φ, we have that $g \equiv f$. It is easy to see that for continuous density φ, this statement is more general than our Theorem 2. On the other hand, the formulation of this statement (as also its proof) uses substantially the property that φ is continuous.

2 Preliminaries and proofs

As usual, we write $S(\mathbb{R}^n)$ for the Schwartz space of test functions on $\mathbb{R}^n$ and $S'(\mathbb{R}^n)$ for the dual space of tempered distributions. We define the inverse Fourier transform

$$\check{\chi}(t) = \frac{1}{(2\pi)^n} \int\limits_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}(t,x)} \chi(x)\,\mathrm{d}x$$

so that the inversion formula $\widehat{(\check{\chi})} = \chi$ holds for suitable $\chi \in L^1(\mathbb{R}^n)$. Given a closed subset $\Omega \subset \mathbb{R}^n$, a function $\omega : \mathbb{R}^n \to \mathbb{C}$ is called bandlimited to Ω if $\widehat{\omega}$ vanishes outside Ω. Note that here we understand $\widehat{\omega}$ in a distributional sense.

Let $B(\mathbb{R}^n) = \{\widehat{\mu}\colon \mu \in M(\mathbb{R}^n)\}$ denote the Fourier–Stieltjes algebra with the usual pointwise multiplication. The norm in $B(\mathbb{R}^n)$ is inherited from $M(\mathbb{R}^n)$, in such a way,

$$\|\widehat{\mu}\|_{B(\mathbb{R}^n)} := \|\mu\|_{M(\mathbb{R}^n)}.$$

Note that the Fourier algebra $A(\mathbb{R}^n) = \{\widehat{\varphi}\colon \varphi \in L^1(\mathbb{R}^n)\}$ is an ideal in $B(\mathbb{R}^n)$.

The closed subspace B^p_Ω in $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, of all $F \in L^p(\mathbb{R})$ such that F is bandlimited to Ω, is called the Bernstein space. The Banach space B^p_Ω is equipped with the norm

$$\|F\|_p = \left(\int\limits_{\mathbb{R}^n} \bigl|F(x)\bigr|^p \,\mathrm{d}x\right)^{1/p}, \quad 1 \leqslant p < \infty,$$

and $\|F\|_\infty = \sup_{x\in\mathbb{R}^n} |F(x)|$. By the Paley–Wiener–Schwartz theorem (see [2, p. 181]), if Ω is a compact subset of $\mathbb{R}^n$, then any $F \in B^p_\Omega$ is infinitely differentiable on $\mathbb{R}^n$ and has an extension onto the complex space $\mathbb{C}^n$ to an entire function. For $a > 0$ and each $F \in B^p_{Q^n_\sigma}$, $1 \leqslant p < \infty$, there exists a positive number M such that the Plancherel–Polya–Nikol'skii-type inequality

$$\sum_{k\in\mathbb{Z}^n} \bigl|F(x + ak)\bigr|^p \leqslant M\|F\|^p_{L^p(\mathbb{R}^n)} \tag{7}$$

is satisfied for all $x \in \mathbb{R}^n$ (see [8, p. 19]). If $F \in B^1_{Q^n_\sigma}$, then the Poisson summation formula

$$\sum_{\omega\in\mathbb{Z}^n} F(x + \nu\omega) = \frac{1}{\nu^n} \sum_{\theta\in\mathbb{Z}^n} \widehat{F}\left(\frac{2\pi}{\nu}\theta\right) \mathrm{e}^{2\pi \mathrm{i}(x,\theta)/\nu} \tag{8}$$

holds for all $x \in \mathbb{R}^n$ and each $\nu > 0$ (see, e.g., [5, p. 166]).

Proof of Theorem 2. We start with the simple observation that we can consider, without loss of generality, condition (3) with $a = 0$, i.e., the case if S_φ, ϱ and τ satisfy

$$\bigl|S_\varphi \cap \bigl(Q^n_\varrho + \tau\mathbb{Z}^n\bigr)\bigr| = 0. \tag{9}$$

Indeed, define $\varphi_a(x) := \varphi(x + a)$, $x \in \mathbb{R}^n$. Then φ_a is a probability density and satisfies (9) if and only if φ satisfies (3). Moreover,

$$\widehat{\varphi_a}(-x) = \mathrm{e}^{-\mathrm{i}(a,x)}\widehat{\varphi}(-x)$$

for $x \in \mathbb{R}^n$. Hence, $\widehat{\varphi_a}$ has the σ-deterministic property if $\widehat{\varphi}$ also has this property.

Assume that g is any characteristic function such that $g = f$ on U_σ. It remains to prove that $f \equiv g$. Our proof starts with the observation that this g is also the characteristic function of a probability density. Indeed, let $g(x) = \widehat{\mu}(-x)$ for certain probability measure μ. Take any $u \in S(\mathbb{R}^n)$ such that $u(x) = 1$ for all $x \in Q^n_\sigma$. Then

$$\widehat{\mu} - \widehat{\varphi} \equiv u(\widehat{\mu} - \widehat{\varphi}). \tag{10}$$

Since $S(\mathbb{R}^n) \subset A(\mathbb{R}^n)$ and $A(\mathbb{R}^n)$ is an ideal in $B(\mathbb{R}^n)$, we conclude from (10) that $\widehat{\mu} \in A(\mathbb{R}^n)$. Hence, there is a probability density ψ such that $g(x) = \widehat{\psi}(-x)$.

Define

$$\xi = \varphi - \psi. \tag{11}$$

Using the fact that $\operatorname{supp}(f - g) \subset Q^n_\sigma$, we see that $\xi \in B^1_{Q^n_\sigma}$. Moreover, from (11) it follows that

$$\xi \leqslant \varphi \tag{12}$$

almost everywhere on $\mathbb{R}^n$ and

$$\int\limits_{\mathbb{R}^n} \xi(x)\,\mathrm{d}x = 0. \tag{13}$$

Now we claim that (9) implies

$$\int\limits_{Q^n_\varrho} \xi(x)\,\mathrm{d}x = 0. \tag{14}$$

To that end, we write E_k for the set $Q^n_\varrho + \tau k$, $k \in \mathbb{Z}^n$. According to (9) and (12), we have that

$$\int\limits_{E_k} \xi(x)\,\mathrm{d}x \leqslant 0 \tag{15}$$

for all $k \in \mathbb{Z}^n$. Since $\xi \in B^1_{Q^n_\sigma}$, the Poisson summation formula (8) holds for $F = \xi$, $\nu = \tau$ and all $x \in \mathbb{R}^n$:

$$\sum_{k\in\mathbb{Z}^n} \xi(x + \tau k) = \frac{1}{\tau^n} \sum_{\theta\in\mathbb{Z}^n} \widehat{\xi}\left(\frac{2\pi}{\tau}\theta\right) \mathrm{e}^{2\pi\mathrm{i}(x,\theta)/\tau}. \tag{16}$$

$\widehat{\xi}$ is continuous on $\mathbb{R}^n$, and $\operatorname{supp}\widehat{\xi} \subset Q^n_\sigma$. Hence, condition (4) implies that

$$\widehat{\xi}\left(\frac{2\pi}{\tau}\theta\right) = 0$$

for all $\theta \in \mathbb{Z}^n \setminus \{0\}$. Moreover, from (13) it follows that also $\widehat{\xi}(0) = \int_{\mathbb{R}^n} \xi(x)\,\mathrm{d}x = 0$. Altogether, (16) reduces to

$$\sum_{k\in\mathbb{Z}^n} \xi(x + \tau k) = 0, \tag{17}$$

$x \in \mathbb{R}^n$. From (7) it follows that this series converges absolutely on $\mathbb{R}^n$. Also, if we consider (16) only for $x \in Q^n_\tau$, i.e., on Q^n_τ, then the left-hand side of (17) converges in the norm of $L^1(Q^n_\tau)$ (see [7, p. 251]). According to (5), we see that the left-hand side of (17) converges also in the norm of $L^1(Q^n_\varrho)$. Then

$$0 = \int\limits_{Q^n_\varrho} \sum_{k\in\mathbb{Z}^n} \xi(x + \tau k)\,\mathrm{d}x = \sum_{k\in\mathbb{Z}^n} \int\limits_{Q^n_\varrho} \xi(x + \tau k)\,\mathrm{d}x = \sum_{k\in\mathbb{Z}^n} \int\limits_{E_k} \xi(x)\,\mathrm{d}x.$$

Combining this with (15) gives $\int_{E_k} \xi(x)\,\mathrm{d}x = 0$ for all $k \in \mathbb{Z}^n$. Hence, this proves our claim (14) since $Q_\varrho^n = E_0$.

Next, we claim that

$$\xi(x) = 0 \tag{18}$$

for all $x \in Q_\varrho^n$. Indeed, set

$$I_{(+)} = \{x \in Q_\varrho^n\colon \xi(x) > 0\}, \quad I_{(-)} = \{x \in Q_\varrho^n\colon \xi(x) < 0\}$$

and

$$I_0 = \{x \in Q_\varrho^n\colon \xi(x) = 0\}.$$

Take into account (9) and (12), we obtain

$$\int_{I_{(+)}} \xi(x)\,\mathrm{d}x \leqslant \int_{I_{(+)}} \varphi(x)\,\mathrm{d}x \leqslant \int_{Q_\varrho^n} \varphi(x)\,\mathrm{d}x = 0.$$

Since ξ is continuous on $\mathbb{R}^n$, it follows that $I_{(+)}$ is an open subset of $\mathbb{R}^n$. Hence, $I_{(+)} = \emptyset$. Using the obvious equality $\int_{I_0} \xi(x)\,\mathrm{d}x = 0$ and (14), we get $\int_{I_{(-)}} \xi(x)\,\mathrm{d}x = 0$. Hence, $I_{(-)} = \emptyset$. This completes the proof of claim (18).

On the other hand, (18) shows that the entire function ξ vanishes on the nonempty and open subset set Q_ϱ^n in $\mathbb{R}^n$. In particular, this implies that the function ξ vanishes at $z = 0$ together with all its partial derivatives. Thus, by the uniqueness theorem for analytic functions (see, e.g., [6, p. 21]), we have that ξ is the zero function. Hence, $\varphi \equiv \psi$. Thus $f \equiv g$. Theorem 2 is proved. □

Proof of Theorem 3. According to (6), we may take a number θ such that

$$\frac{2\pi}{\sigma} < \theta < \tau. \tag{19}$$

Next, let ϱ be any positive number, which satisfies

$$\varrho < \frac{\tau - \theta}{2}. \tag{20}$$

Take an arbitrary continuous on $\mathbb{R}^n$ probability density φ with

$$S_\varphi = B^n_{\theta\sqrt{n}/2}, \tag{21}$$

where B_r^n denotes the ball $\{x \in \mathbb{R}^n\colon \sum_{k=1}^n x_k^2 \leqslant r^2\}$. Combining (6) with (20) and (21), it is a simple calculation to see that for these θ, ϱ, φ and

$$a = \left(\frac{\tau}{2}, \frac{\tau}{2}, \dots, \frac{\tau}{2}\right) \in \mathbb{R}^n,$$

condition (3) is satisfied.

The next step of our proof consists in the construction of a function $\xi \in B^1_{Q^n_\sigma}$, $\xi \not\equiv 0$, satisfying (12) and (13). Put

$$\omega_1(t) = \left(\frac{\sigma}{2\pi}\cos\frac{\pi t}{\sigma}\right)\cdot\chi_{[-\sigma/2,\,\sigma/2]}(t) \tag{22}$$

and

$$\omega_2(t) = \left(\frac{\mathrm{i}}{2}\sin\frac{\pi t}{\sigma}\right)\cdot\chi_{[-\sigma/2,\,\sigma/2]}(t), \tag{23}$$

where χ_A is the indicator function of the subset $A \subset \mathbb{R}$. It is straightforward to verify that

$$\widehat{\omega_1}(x) = \frac{\cos\frac{\sigma x}{2}}{(\frac{\pi}{\sigma})^2 - x^2} \tag{24}$$

and

$$\widehat{\omega_2}(x) = x\,\widehat{\omega_1}(x). \tag{25}$$

According to our definitions of the Fourier transform and its inverse transform, the following Plancherel formula holds

$$2\pi\|\gamma\|^2_{L^2(\mathbb{R})} = \|\widehat{\gamma}\|^2_{L^2(\mathbb{R})}$$

for each $\widehat{\gamma} \in L^2(\mathbb{R})$. Hence, using (22) and (23), we get

$$\|\widehat{\omega_1}\|^2_{L^2(\mathbb{R})} = 2\pi\|\omega_1\|^2_{L^2(\mathbb{R})} = \frac{\sigma^2}{2\pi}\int\limits_{-\sigma/2}^{\sigma/2}\cos^2\frac{\pi t}{\sigma}\,\mathrm{d}t = \frac{\sigma^3}{4\pi} \tag{26}$$

and

$$\|\widehat{\omega_2}\|^2_{L^2(\mathbb{R})} = \frac{\pi}{2}\int\limits_{-\sigma/2}^{\sigma/2}\sin^2\frac{\pi t}{\sigma}\,\mathrm{d}t = \frac{\pi\sigma}{4} \tag{27}$$

since $\omega_k \in L^2(\mathbb{R})$, $k = 1, 2$.

For $x \in \mathbb{R}^n$, let us define

$$\begin{aligned}\xi_0(x) &= \frac{\pi^2 n}{\sigma^2}\prod_{k=1}^n \widehat{\omega_1}^2(x_k) - \sum_{k=1}^n\left[\widehat{\omega_2}^2(x_k)\cdot\prod_{j=1,\,j\neq k}^n \widehat{\omega_1}^2(x_j)\right] \\ &= \prod_{k=1}^n\left(\frac{\cos\frac{\sigma x_k}{2}}{(\frac{\pi}{\sigma})^2 - x_k^2}\right)^2\left[\frac{\pi^2 n}{\sigma^2} - \sum_{k=1}^n x_k^2\right].\end{aligned} \tag{28}$$

Obviously, $\xi_0 \in L^1(\mathbb{R}^n)$. On the other hand, from (22), (23), (24) and (25) it follows that $\widehat{\xi_0}$ is supported on $[-\sigma,\sigma]^n = Q^n_\sigma$. Hence, $\xi_0 \in B^1_{Q^n_\sigma}$.

We claim that there exists $\varepsilon > 0$ such that the function

$$\xi := \varepsilon \cdot \xi_0 \tag{29}$$

satisfies (12) and (13). Indeed, using (26), (27) and (28), we get

$$\begin{aligned}
&\int\limits_{\mathbb{R}^n} \xi_0(x)\,\mathrm{d}x \\
&\quad= \frac{\pi^2 n}{\sigma^2} \prod_{k=1}^{n} \int\limits_{\mathbb{R}} \widehat{\omega_1}^2(x_k)\,\mathrm{d}x_k - \sum_{k=1}^{n} \Biggl[\int\limits_{\mathbb{R}} \widehat{\omega_2}^2(x_k)\,\mathrm{d}x_k \cdot \prod_{j=1,\, j\neq k}^{n} \int\limits_{\mathbb{R}} \widehat{\omega_1}^2(x_j)\,\mathrm{d}x_j \Biggr] \\
&\quad= \frac{\pi^2 n}{\sigma^2} \left(\frac{\sigma^3}{4\pi}\right)^n - n\frac{\pi\sigma}{4}\left(\frac{\sigma^3}{4\pi}\right)^{n-1} = \left(\frac{\sigma^3}{4\pi}\right)^{n-1} \left[\frac{\pi^2 n}{\sigma^2} \cdot \frac{\sigma^3}{4\pi} - n\frac{\pi\sigma}{4}\right] = 0.
\end{aligned}$$

Therefore, $\int_{\mathbb{R}^n} \xi(x)\,\mathrm{d}x = 0$.

Next, from (19) it follows that

$$B^n_{\pi\sqrt{n}/\sigma} \subsetneqq B^n_{\theta\sqrt{n}/2}.$$

This implies that we can take a continuous probability density φ such that it satisfies (21) and also the following additional condition:

$$\min\bigl\{\varphi(x)\colon\, x \in B^n_{\pi\sqrt{n}/\sigma}\bigr\} > 0.$$

Then, since ξ (see formula (29)) is also continuous on $\mathbb{R}^n$, we have that there exists $\varepsilon > 0$ in (29) such that (12) is satisfied for this ξ and all $x \in B^n_{\pi\sqrt{n}/\sigma}$. On the other hand, from (28) and (29) it follows that ξ is nonpositive on $\mathbb{R}^n \setminus B^n_{\pi\sqrt{n}/\sigma}$. Therefore, ξ satisfies (12) for all $x \in \mathbb{R}^n$.

Finally, if we set $\psi := \varphi - \xi$, then (12) and (13) show that ψ is a probability density. Moreover, we have that $\widehat{\psi} = \widehat{\varphi}$ on U_σ but $\psi \not\equiv \varphi$. Theorem 3 is proved. □

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