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Solvability of control problem for fractional nonlinear differential inclusions with nonlocal conditions

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Abstract. In this paper, we study the approximate controllability of nonlocal fractional differential inclusions involving the Caputo fractional derivative of order $q \in (1, 2)$ in a Hilbert space. Utilizing measure of noncompactness and multivalued fixed point strategy, a new set of sufficient conditions is obtained to ensure the approximate controllability of nonlocal fractional differential inclusions when the multivalued maps are convex. Precisely, the results are developed under the assumption that the corresponding linear system is approximately controllable.

Keywords: approximate controllability, measure of noncompactness, multivalued map, fractional differential inclusions, resolvent operator.

1 Introduction

In recent years, fractional calculus has been applied in many real processes, and notable contributions have been made to both theory and applications of fractional differential equations. Fractional differential equations have been used in various fields such as fluid flow, viscoelasticity, electrical networks, dynamical processes in porous structures, optics and signal processing, hydraulics of dams, diffusion problems and so on [13–18]. On the other hand, fractional evolution inclusions are an important form of differential inclusions within a nonlinear mathematical analysis [20]. Compared to fractional evolution equations, research on the theory of fractional differential inclusions is however only in its initial stage of development. This is essential since differential models involving the fractional derivative give a brilliant tool for depiction of memory and genetic properties, and have recently been demonstrated as significant tools in the modeling of many physical phenomena. Applied problems requiring definitions of fractional derivatives are those, which can be physically interpreted for initial conditionscontaining $u(0)$, $u'(0)$, etc.

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Similar requirements are valid for boundary conditions. Caputo's fractional derivative fulfills these requests. Recently, the existence results for different kind of fractional differential inclusions have been reported in [6–8, 16, 21, 22].

The notion of controllability is closely related to the theory of optimal control and minimal realization [14, 15]. In particular, two important concepts such as exact and approximate controllability are developed in the case of infinite-dimensional systems. It should be noted that it is not easy to understand the conditions of exact controllability for infinite-dimensional systems, and hence the approximate controllability becomes an essential topic for dynamical systems. Therefore, in fact, it is necessary to study the approximate controllability for nonlinear dynamical control systems [9, 23]. Several authors [2, 17] studied the approximate controllability results of various class of nonlinear systems utilizing compact semigroup, compact sectorial operator under the assumption that corresponding linear system is approximately controllable. In this work, we attempt to consider the approximate controllability of fractional nonlocal differential inclusion using measure of noncompactness instead of assuming compactness of sectorial operator. Byszewski [5] first considered a differential equation with nonlocal initial conditions and proved that the corresponding models more precisely describe some physical phenomena better than the standard initial condition since more data was used in its design. For example, several physical phenomena in engineering, physics and life sciences can be described with the help of differential equations subject to nonlocal boundary conditions [10]. On the other hand, measure of noncompactness is an important tool in the wide areas of functional analysis, topology, operator theory, for example, metric fixed point theory, theory of operators in Banach spaces, optimizations, differential equations and so on [1, 11].

Motivated by the previous works, in this paper, we investigate the approximate controllability of the following integro-differential inclusions involving nonlocal conditions in a separable Banach space $(E, \|\cdot\|)$ in the following form:

$$^{C}D_t^q\left[u(t) - \int_0^t (t-s)G\left(s,\, u\big(h_1(t)\big), \int_0^s a_1\big(s,\tau,u\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\right] \in Au(t) + Bx(t) + F\big(t,u(t)\big) + H\big(t,u(t)\big), \quad t \in [0,T], \tag{1}$$

$$u(0) = u_0 + g(u), \qquad u'(0) = u_1 + h(u) \in E, \tag{2}$$

where $1 < q < 2$, $^{C}D_t^q$ denotes the generalized fractional derivative in Caputo sense, $A : D(A) \subset E \to E$ is a closed and linear operator with the dense domain $D(A)$ defined in a Hilbert space E, the state $u(t)$ takes its values in E, $x(t)$ represents the control function given in $L^2([0,T], X)$, a Banach space of admissible control functions with X as a Hilbert space, $B : X \to E$ is a bounded linear operator, $F, H : [0,T] \times E \to \mathcal{P}(E)$ are multivalued functions, $g, h : E \to E$, $a_1 : D_1 \times E \to E$ and $G : [0,T] \times E \times E \to E$ are continuous functions satisfying certain conditions to be mentioned later, where $D_1 = \{(t,s) \in [0,T] \times [0,T]\colon\ s \leqslant t\}$ and $h_1, h_2 : [0,T] \to [0,T]$ are continuous functions such that $h_1(t), h_2(t) \leqslant t$ for $t \in [0,T]$.

2 Preliminaries

In this section, we provide some definitions and lemmas, which are needed to establish our main results. Throughout this paper, it is assumed that E and X are Hilbert spaces. Let $C([0,T];E)$ denote the Banach space of all the continuous functions from $[0,T]$ into E equipped with the norm $\|z(t)\|_C = \sup_{t\in[0,T]} \|z(t)\|_E$. Let $L^p((0,T);E)$ denote the Banach space of all Bochner-measurable functions from $(0,T)$ to E with the norm $\|z\|_{L^p} = (\int_{(0,T)} \|z(s)\|_E^p \,\mathrm{d}s)^{1/p}$. Denote $\mathcal{P}(E) = \{Z \subset E\colon Z \neq \emptyset\}$, $\mathcal{PC}_{\mathrm{cv}}(E) = \{Z \in \mathcal{P}(E)\colon Z \text{ is convex}\}$, $\mathcal{P}_{\mathrm{cp}}(E) = \{Z \in \mathcal{P}(E)\colon Y \text{ is compact}\}$, $\mathcal{P}_{\mathrm{cv,cp}}(E) = \mathcal{P}_{\mathrm{cv}}(E) \cap \mathcal{P}_{\mathrm{cp}}$.

A multivalued map $H : E \to \mathcal{P}(E)$ has convex values if $H(u)$ is convex for all $u \in E$. H is bounded on bounded set if $H(C) = \cup_{u\in C} H(u)$ is bounded in E for any bounded set $C \subset E$ (i.e. $\sup_{u\in C}\{\sup\{|z|\colon z \in H(u)\}\} < +\infty$).

The multivalued map H is upper semicontinuous (u.s.c.) on E if for each $u_0 \in E$, the set $H(u_0)$ is a nonempty, closed subset of E, and if for each open set N of E containing $H(u_0)$, there exists an open neighborhood M of u_0 such that $H(M) \subset N$. Also, H is lower semicontinuous (l.s.c.) if $H : E \to \mathcal{P}(E)$ is a multivalued operator with nonempty closed values, and if the set $\{u \in E\colon H(u) \cap C \neq \emptyset\}$ is open for any open set $C \subset E$, H is completely continuous if $H(C)$ is relatively compact for every bounded subset $C \subset E$. If the multivalued function H is completely continuous with nonempty compact values, then H is u.s.c. if and only if H has a closed graph. The multivalued function H has a fixed point if there exists $u \in E$ such that $u \in H(u)$. A multivalued function $H : [0,T] \to \mathcal{P}_{\mathrm{cl}}(E)$ is called measurable if for each $u \in E$, the function $\mathcal{Y} : [0,T] \to \mathbb{R}^+$ defined by $\mathcal{Y}(t) = d(u, H(t)) = \inf\{\|u - z\|\colon z \in H(t)\}$ is measurable.

Definition 1. (See [4].) The Hausdorff measure of noncompactness χ of bounded subset W of E is given by $\chi(W) = \inf\{\epsilon > 0\colon W \text{ admits a finite cover by balls of radius } \leqslant \epsilon\}$.

Definition 2. (See [3].) A sequence $\{F_n\}_{n\geqslant 1}$ is called semicompact if

(i) it is integrable bounded;
(ii) the set $\{F_n\}_{n\geqslant 1}$ is relatively compact in E for almost all $t \in [0,T]$.

Lemma 1. (*See* [3].) *Let $\{W_n\}_{n\geqslant 1}$ be a sequence of subsets of E. Assume that there is a compact and convex subset $W \subset E$ such that for any neighborhood M of W, there is an N with $W_m \subset M$ for any $m \geqslant N$. Then $\cap_{N>0} \overline{\mathrm{conv}}(\cup_{n\geqslant N} W_n) \subset W$.*

Let (E,d) denote a metric space induced from the normed space $(E, \|\cdot\|)$. Define $H_d : \mathcal{P}(E) \times \mathcal{P}(E) \to \mathbb{R}^+ \cup \{\infty\}$ by $H_d(W_1, W_2) = \max\{\sup_{w_1\in W_1} d(w_1, W_2), \sup_{w_2\in W_2} d(W_1, w_2)\}$, where $d(w_1, W_2) = \inf_{w_2\in W_2} d(w_1, w_2)$ and $d(W_1, w_2) = \inf_{w_1\in W_1} d(w_1, w_2)$. Then $(\mathcal{P}_{\mathrm{bd,cl}}(E), H_d)$ is a metric space and $(\mathcal{P}_{\mathrm{cl}}(E), H_d)$ is a generalized metric space.

Definition 3. A multivalued map $G : E \to \mathcal{P}_{\mathrm{cl}}(E)$ is said to be

(i) γ-Lipschitz if and only if there exists $\mu > 0$ such that $H_d(G(u_1) - G(u_2)) \leqslant \mu d(u_1, u_2)$ for every $u_1, u_2 \in E$;
(ii) a contraction if and only if it is μ-Lipschitz with $\mu < 1$.

Lemma 2. (*See* [12].) *Let W be a closed convex subset of a Banach space E and $G : W \to \mathcal{P}_{\mathrm{cv,cp}}(W)$ be a closed multivalued function, which is χ-condensing. Then G has a fixed point, where χ means a nonsingular measure of noncompactness defined on subsets of W.*

Now, we provide some basic definitions and properties of fractional calculus.

Definition 4. The Riemann–Liouville fractional integral operator J of order $q > 0$ is defined as

$$J_t^q F(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} F(s)\,\mathrm{d}s,$$

where $F \in L^1((0,T),E)$.

Definition 5. The Riemann–Liouville fractional derivative as $D_t^q F(t) = D_t^m J_t^{m-q} F(t)$, $m-1 < q < m$, $m \in \mathbb{N}$, where $D_t^m = \mathrm{d}^m/\mathrm{d}t^m$, $F \in L^1((0,T);E)$, $J_t^{m-q} \in W^{m,1}((0,T);E)$. Here the notation $W^{m,1}((0,T);E)$ stands for the Sobolev space defined as $W^{m,1}((0,T);E) = \{y \in E\colon \exists z \in L^1((0,T);E)\colon\ y(t) = \sum_{k=0}^{m-1} d_k t^k/k! + t^{m-1}/(m-1)! \cdot z(t),\ t \in (0,T)\}$. Note that $z(t) = y^m(t)$, $d_k = y^k(0)$.

Definition 6. The Caputo fractional derivative is given as ${}^C D_t^\alpha F(t) = (1/\Gamma(m-\alpha)) \times \int_0^t (t-s)^{m-\alpha-1} F^m(t)\,\mathrm{d}t$, $m-1 < \alpha < m$, where $F \in C^{m-1}((0,T),E) \cap L^1((0,T),E)$.

Definition 7. The definition of one parameter Mittag–Leffler function is given by $E_\alpha(z) = \sum_{k=0}^\infty z^k/\Gamma(\alpha k+1)$, and two parameter function of Mittag–Leffler type is defined by $E_{\alpha,\beta}(z) = \sum_{k=0}^\infty z^k/\Gamma(\alpha k+\beta) = (2\pi \mathrm{i})^{-1} \int_C \mu^{\alpha-\beta} \mathrm{e}^\mu/(\mu^a - z)\,\mathrm{d}\mu$, $0 < \alpha, \beta$, $z \in \mathbb{C}$, where C is a contour, which starts and ends at $-\infty$ and encircles the disc $|\mu| \leqslant |z|^{1/2}$ counter clockwise. The Laplace transform of Mittag–Leffler function is defined by $L(t^{\beta-1} E_{\alpha,\beta}(-\rho^\alpha t^\alpha)) = \lambda^{\alpha-\beta}/(\lambda^\alpha + \rho^\alpha)$, $\mathrm{Re}\lambda > \rho^{1/\alpha}$, $\rho > 0$.

Definition 8. (See [19].) Let $A : D(A) \subset E \to E$ be a closed linear operator. A is said to be sectorial operator of type (M, θ, μ) if there exist $0 < \theta < \pi/2$, $M > 0$ and $\mu \in \mathbb{R}$ such that the q-resolvent of A exists outside the sector $\mu + S_\theta = \{\mu + \hat{\lambda}\colon\ \hat{\lambda} \in \mathbb{C}$, $|\arg(-\hat{\lambda})| < \theta\}$, and $\|(\hat{\lambda} I - A)^{-1}\| \leqslant M/|\hat{\lambda} - \mu|$, $\hat{\lambda} \notin \mu + S_\theta$.

Definition 9. (See [19].) Let A be a densely defined operator in E that satisfies the conditions:

(i) For some $0 < \theta < \pi/2$, $\mu + S_\theta = \{\mu + \hat{\lambda}\colon\ \hat{\lambda} \in \mathbb{C},\ |\arg(-\hat{\lambda})| < \theta\}$;
(ii) There is a constant $M > 0$ such that $\|(\hat{\lambda} I - A)^{-1}\| \leqslant M/(|\hat{\lambda} - \mu|)$, $\hat{\lambda} \notin \mu + S_\theta$.

Then A is the infinitesimal generator of a semigroup $\mathcal{T}(t)$ satisfying $\|\mathcal{T}(t)\| \leqslant C$. Moreover, $\mathcal{T}(t) = (2\pi \mathrm{i})^{-1} \int_{\widetilde{\Gamma}} \mathrm{e}^{\hat{\lambda} t} R(\hat{\lambda}, A)\,\mathrm{d}\hat{\lambda}$, where $\widetilde{\Gamma}$ is a suitable path for $\hat{\lambda} \notin \mu + S_\theta$ and $\hat{\lambda} \in \widetilde{\Gamma}$.

Definition 10. (See [19].) A closed linear operator $A : D(A) \subset E \to E$ said to be a sectorial operator of type (M, θ, q, μ) if there exist $0 < \theta < \pi/2$, $M > 0$ and $\mu \in \mathbb{R}$ such that the q-resolvent of A exists outside the sector $\mu+S_\theta = \{\mu+\hat{\lambda}^q\colon \hat{\lambda} \in \mathbb{C},\ |\arg(-\hat{\lambda}^q)| < \theta\}$, and $\|(\hat{\lambda}^q I - A)^{-1}\| \leqslant M/|\hat{\lambda}^q - \mu|$, $\hat{\lambda}^q \notin \mu + S_\theta$.

Remark 1. If A is a sectorial operator of type (M,θ,q,μ), then it is easy to see that A is the infinitesimal generator of a q-resolvent family $\{S_q(t)\}_{t\geqslant 0}$ in a Banach space, and $S_q(t) = (2\pi\mathrm{i})^{-1}\int_{\widetilde{\Gamma}} \mathrm{e}^{\hat{\lambda}t}\hat{\lambda}^{q-1}R(\hat{\lambda}^q, A)\,\mathrm{d}\hat{\lambda}$, $K_q(t) = (2\pi\mathrm{i})^{-1}\int_{\widetilde{\Gamma}} \mathrm{e}^{\hat{\lambda}t}\hat{\lambda}^{q-2}R(\hat{\lambda}^q, A)\,\mathrm{d}\hat{\lambda}$, $R_q(t) = (2\pi\mathrm{i})^{-1}\int_{\widetilde{\Gamma}} \mathrm{e}^{\hat{\lambda}t}R(\hat{\lambda}^q, A)\,\mathrm{d}\hat{\lambda}$, and $\widetilde{\Gamma}$ is a suitable path with $\hat{\lambda}^q \notin \mu + S_\theta$, where $\hat{\lambda} \in \mathbb{C}$.

Throughout this paper, we assume that $S_q(t)$, $K_q(t)$ and $R_q(t)$ are equicontinuous and there exists a positive constant $\widetilde{M}$ such that $\|S_q(t)\|, \|K_q(t)\|, \|R_q(t)\| \leqslant \widetilde{M}$ for all $t \geqslant 0$.

Lemma 3. *Let f be a function that satisfies the uniform Hölder condition with the exponent $\beta \in (0,1]$, and let A be a sectorial operator of type (M,θ,q,μ). Then the unique solution of the fractional system $D^q u(t) = Au(t) + f(t)$, $t \in [0,T]$, $1 < q < 2$, $u(0) = u_0 \in E$, $u'(0) = u_1 \in E$ is given by*

$$u(t) = S_q(t)u_0 + K_q(t)u_1 + \int_0^t R_q(t-s)f(s)\,\mathrm{d}s, \quad t \in [0,T].$$

For any $u \in E$, define the sets as follows:

$$S_{H,u} = \{v \in L^1([0,T],E)\colon v(t) \in H(t,u(t)) \text{ for a.e. } t \in [0,T]\},$$
$$S_{F,u} = \{f \in L^1([0,T],E)\colon f(t) \in F(t,u(t)) \text{ for a.e. } t \in [0,T]\}.$$

Consider the infinite-dimensional linear control system in the following form:

$$D^q u(t) = Au(t) + Bx(t), \quad t \in [0,T],\ 1 < q < 2,$$
$$u(0) = u_0 \in E, \qquad u'(0) = u_1 \in E,$$

where $x(t) \in L^2([0,T],X)$, $A : E \to E$, $B : X \to E$ and $T > 0$. It is appropriate at this point to define the operator $\Gamma_\tau^T : E \to E$ by

$$\Gamma_\tau^T = \int_\tau^T R_q(T-s)BB^*R_q^*(T-s)\,\mathrm{d}s,$$
$$R(\lambda, \Gamma_\tau^T) = (\lambda I + \Gamma_\tau^T)^{-1}, \quad \lambda > 0,$$

where B^* represents the adjoint of B, $\|B\| = M_B$ and $R_q^*(t)$ is the self adjoint of $R_q(t)$. It is clear that Γ_0^T is a linear bounded operator for $\tau = 0$.

Lemma 4. (*See* [17].) *The linear system* (1)–(2) *is approximately controllable if and only if* $\lambda R(\lambda, \Gamma_0^T) = \lambda(\lambda I + \Gamma_0^T)^{-1} \to 0$ *as* $\lambda \to 0^+$ *in the strong operator topology.*

Now, the definition of the mild solution of (1)–(2) is presented.

Definition 11. A continuous function $u : [0,T] \to E$ is called a mild solution for the fractional control system (1)–(2) if for each $t \geqslant 0$ and $x \in L^2([0,T],E)$, $u(t)$ satisfies the following integral equation:

$$\begin{aligned} u(t) &= S_q(t)\big[u_0 + g(u)\big] + K_q(t)\big[u_1 + h(u)\big] \\ &\quad + \int_0^t K_q(t-s) G\left(s,\, u\big(h_1(s)\big),\, \int_0^s a_1\big(s,\tau,u\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\right) \mathrm{d}s \\ &\quad + \int_0^t R_q(t-s) Bx(s)\,\mathrm{d}s + \int_0^t R_q(t-s) f(s)\,\mathrm{d}s \\ &\quad + \int_0^t R_q(t-s)\rho(s)\,\mathrm{d}s, \quad t \in [0,T], \end{aligned} \tag{3}$$

where $f \in S_{F,u}$, and $\rho \in S_{H,u}$.

3 Main results

To prove the results, we need to impose the following conditions on the data of the system (1)–(2).

(A1) (i) The function $G : [0,T] \times E \times E \to E$ is continuous, compact and there exists a constant $L_G > 0$ such that $\|G(t_1,u_1,v_1) - G(t_2,u_2,v_2)\| \leqslant L_G[|t_1 - t_2| + \|u_1 - u_2\| + \|v_1 - v_2\|]$ for all $t_1, t_2 \in [0,T]$ and (u_1,v_1), $(u_2,v_2) \in E \times E$, $C_1 = \sup_{t\in[0,T]} \|G(t,0,0)\|$.

(ii) The map $a_1 : D \times E \to E$ is a continuous mapping and there exists a positive constant L_{a_1} such that

$$\left\| \int_0^t \big[a_1(t,s,u_1) - a_1(t,s,u_2)\big]\,\mathrm{d}s \right\| \leqslant L_{a_1}\|u_1 - u_2\|$$

for $(t,s) \in D$, $u_1, u_2 \in E$, and $C_2 = T\sup_{(t,s)\in D_1} \|a_1(t,s,0)\|$.

(A2) (i) The map $t \to H(t,u)$ is measurable for each $u \in E$ and a.e. $t \in [0,T]$, and $u \to H(t,u)$ is upper semicontinuous for almost all $t \in [0,T]$.

(ii) There exists a function $m_H(t) \in L^1([0,T],\mathbb{R}^+)$ and a continuous increasing function $W_H : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|H(t,u)\|_E := \sup\{\|\rho\|\colon \rho \in H(t,u)\} \leqslant m_H(t) W_H(\|u\|_E)$ for a.e. $t \in [0,T]$ and for each $u \in E$.

(iii) There exists a function $\alpha_1 \in L^1([0,T],\mathbb{R}^+)$ such that for every bounded subset $W \subset E$, $\chi(H(t,W)) < \alpha_1(t)\chi(W)$.

(A3) (i) The map $t \to F(t,u)$ is measurable for each $u \in E$ and a.e. $t \in [0,T]$, and $u \to F(t,u)$ is upper semicontinuous for almost all $t \in [0,T]$.

(ii) There exists a function $m_F(t) \in L^1([0,T],\mathbb{R}^+)$ and a continuous increasing function $W_F : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\|F(t,u)\|_E := \sup\{\|f\|: f \in F(t,u)\} \leqslant m_F(t)W_F(\|u\|_E)$ for a.e. $t \in [0,T]$ and for each $u \in E$.

(iii) There exists a function $\alpha_2 \in L^1([0,T],\mathbb{R}^+)$ such that for every bounded subset $W \subset E$, $\chi(F(t,W)) < \alpha_2(t)\chi(W)$.

(A4) The functions $g,h : E \to E$ are continuous, compact, and there exist constants $L_g^1, L_g^2, L_h^1, L_h^2 > 0$ such that $\|g(u)\|_E \leqslant L_g^1\|u\|_E + L_g^2$, for all $u \in E$, and $\|h(u)\|_E \leqslant L_h^1\|u\|_E + L_h^2$ for all $u \in E$.

(A5) For $\lambda > 0$, $\lim_{r_0\to\infty}\sup(b/\lambda + c/\lambda + dW(r_0)/(\lambda r_0)) < 1$ and $2T\widetilde{M} \times (1 + \sqrt{T}M_B^2\widetilde{M}^2/\lambda)(\|\alpha_1\|_{L^1} + \|\alpha_2\|_{L^1}) < 1$, where $W(r_0) = \max\{W_F(r_0), W_H(r_0)\}$.

In the following derivation, it will be proved that the fractional control system (1)–(2) is approximately controllable if for all $\lambda > 0$, there is a continuous function $u(\cdot) \in E$ defined in (3) and a control function $x(t)$ such that

$$x(t) = x^\lambda(t,u) = B^*R_q^*(T-t)R\big(\lambda, \varGamma_0^T\big)p\big(u(\cdot)\big),$$

where

$$\begin{aligned} p\big(u(\cdot)\big) &= u_T - S_q(T)\big[u_0 + g(u)\big] - K_q(T)\big[u_1 + h(u)\big] \\ &\quad - \int_0^T K_q(T-s)G\left(s,\, u\big(h_1(s)\big), \int_0^s a_1\big(s,\tau,u\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad - \int_0^T R_q(T-s)f(s)\,\mathrm{d}s - \int_0^T R_q(T-s)\rho(s)\,\mathrm{d}s. \end{aligned}$$

For convenience, let us introduce some other notations.

$$\begin{aligned} &\mathfrak{L} = \max\big\{1, M_B\widetilde{M}, \widetilde{M}M_B^2, \widetilde{M}^2, M_B\widetilde{M}\sqrt{T}\big\}, \\ &b_1 = 3\mathfrak{L}\big(\|u_T\| + \widetilde{M}\big(\|u_0\| + L_g^2 + \|u_1\| + L_h^2\big) + T\widetilde{M}(L_GC_2 + C_1)\big), \\ &b_2 = 3\widetilde{M}\big(\|u_0\| + L_g^2 + \|u_1\| + L_h^2 + T(L_GC_2 + C_1)\big), \\ &c_1 = 3\mathfrak{L}\widetilde{M}\big(L_g^1 + L_h^1 + TL_G(1 + L_{a_1})\big), \\ &c_2 = 3\widetilde{M}\big(L_g^1 + L_h^1 + TL_G(1 + L_{a_1})\big), \\ &d_1 = 3\mathfrak{L}\widetilde{M}M_BT\big(\|m_F\|_{L^1} + \|m_H\|_{L^1}\big), \\ &d_2 = 3\widetilde{M}T\big(\|m_F\|_{L^1} + \|m_H\|_{L^1}\big), \\ &b = \max\{b_1, b_2\}, \quad c = \max\{c_1, c_2\}, \qquad d = \max\{d_1, d_2\}. \end{aligned}$$

Theorem 1. *If conditions* (A1)–(A5) *hold, then the set of solution for fractional control system* (1)–(2) *is nonempty.*

Proof. For $\lambda > 0$, the multivalued operator $\Pi : E \to 2^E$ is defined by

$$\Pi(u) = \Biggl\{ y \in E\colon\ y(t) = S_q(t)\bigl[u_0 + g(u)\bigr] + K_q(t)\bigl[u_1 + h(u)\bigr] + \int_0^t K_q(t-s) G\Biggl(s,\ u\bigl(h_1(s)\bigr),\ \int_0^s a_1\bigl(s,\tau,u\bigl(h_2(\tau)\bigr)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s + \int_0^t R_q(t-s) Bx(s)\,\mathrm{d}s + \int_0^t R_q(t-s) f(s)\,\mathrm{d}s + \int_0^t R_q(t-s)\rho(s)\,\mathrm{d}s,\ f \in S_{F,u},\ \rho \in S_{H,u} \Biggr\}.$$

The proof of this theorem will be divided into several steps.

Step 1. To prove that the values of $\Pi(u)$ are closed and convex.

Let $u \in \mathcal{B}_r = \{u \in E\colon\ \|u\| \leqslant r\}$ and $\{y_n,\ n \geqslant 1\}$ be a sequence in $\Pi(u)$ such that $y_n \to y \in E$ as $n \to \infty$. Then there exist sequences $\{f_n,\ n \geqslant 1\}$ in $S_{F,u}$ and $\{\rho_n,\ n \geqslant 1\}$ in $S_{H,u}$ such that

$$\begin{aligned} y_n(t) &= S_q(t)\bigl[u_0 + g(u)\bigr] + K_q(t)\bigl[u_1 + h(u)\bigr] \\ &\quad + \int_0^t K_q(t-s) G\Biggl(s,\ u\bigl(h_1(s)\bigr),\ \int_0^s a_1\bigl(s,\tau,u\bigl(h_2(\tau)\bigr)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\ &\quad + \int_0^t R_q(t-\xi) B\Biggl\{ B^* R_q^*(T-\xi)\bigl[\bigl(\lambda I + \Gamma_0^T\bigr)^{-1} u_T - S_q(T)\bigl[u_0 + g(u)\bigr] \\ &\qquad - K_q(T)\bigl[u_1 + h(u)\bigr]\bigr] - B^* R_q^*(T-\xi) \\ &\qquad \times \int_0^T \bigl(\lambda I + \Gamma_s^T\bigr)^{-1} K_q(T-s) G\Biggl(s,\ u\bigl(h_1(s)\bigr),\ \int_0^s a_1\bigl(s,\tau,u\bigl(h_2(\tau)\bigr)\bigr)\,\mathrm{d}\tau\Biggr)\,\mathrm{d}s \\ &\qquad - B^* R_q^*(T-\xi) \int_0^T \bigl(\lambda I + \Gamma_s^T\bigr)^{-1} R_q(T-s) f_n(s)\,\mathrm{d}s \\ &\qquad - B^* R_q^*(T-\xi) \int_0^T \bigl(\lambda I + \Gamma_s^T\bigr)^{-1} R_q(T-s) \rho_n(s)\,\mathrm{d}s \Biggr\}\,\mathrm{d}\xi \\ &\quad + \int_0^t R_q(t-s) f_n(s)\,\mathrm{d}s + \int_0^t R_q(t-s)\rho_n(s)\,\mathrm{d}s. \end{aligned} \tag{4}$$

It follows from (A2) and (A3) that $\|f_n(t)\| \leqslant m_F(t)W_F(\|u(t)\|_E)$, $\|\rho_n(t)\| \leqslant m_H(t)W_H(\|u(t)\|_E)$ for every $n \geqslant 1$ and a.e. $t \in [0,T]$. This implies that the sets $\{f_n,\, n \geqslant 1\}$ and $\{\rho_n,\, n \geqslant 1\}$ are integral bounded. Since $\{f_n(t),\, n \geqslant 1\} \subset F(t,u(t))$ and $\{\rho_n(t),\, n \geqslant 1\} \subset H(t,u(t))$ for a.e. $t \in [0,T]$, the sets $\{f_n(t),\, n \geqslant 1\}$ and $\{\rho_n(t),\, n \geqslant 1\}$ are relatively compact in E for a.e. $t \in [0,T]$. Therefore, the sets $\{f_n,\, n \geqslant 1\}$ and $\{\rho_n,\, n \geqslant 1\}$ are semicompact. Also, without loss of generality, we can assume that f_n and ρ_n converge weakly to the functions $f \in L^1([0,T],E)$ and $\rho \in L^1([0,T],E)$, respectively. From Mazur's lemma, for every number $j \in \mathbb{N}$, there are natural number $m_0(j) > j$ and a sequence of positive real numbers $\lambda_{j,m}$, $m = j,\dots,m_0(j)$, such that $\sum_{m=j}^{m_0(j)} \lambda_{j,m} = 1$. Moreover, the sequence of convex combinations $z_j = \sum_{m=j}^{m_0(j)} \lambda_{j,m} f_m$ and $w_j = \sum_{m=j}^{m_0(j)} \lambda_{j,m}\rho_m$, $j \geqslant 1$, converge strongly to $f \in L^1([0,T],E)$ and $\rho \in L^1([0,T],E)$ as $j \to \infty$, respectively. Therefore, we assume that $z_j(t) \to f(t)$ and $w_j(t) \to \rho(t)$ for a.e. $t \in [0,T]$. Since F and H take convex and closed values, we have

$$f(t) \in \bigcap_{j\geqslant 1} \overline{\{z_m(t),\, m \geqslant j\}} \subseteq \bigcap_{j\geqslant 1} \overline{\operatorname{conv}}\{f_m,\, m \geqslant j\} \subseteq F\big(t,u(t)\big) \quad \text{for a.e. } t \in [0,T],$$

and

$$\rho(t) \in \bigcap_{j\geqslant 1} \overline{\{w_m(t),\, m \geqslant j\}} \subseteq \bigcap_{j\geqslant 1} \overline{\operatorname{conv}}\{\rho_m,\, m \geqslant j\} \subseteq H\big(t,u(t)\big) \quad \text{for a.e.. } t \in [0,T].$$

For every $t, s \in [0,T]$ with $s \in (0,t]$ and every $n \geqslant 1$, we have $\|R_q(t-s)z_n(s)\| \leqslant \widetilde{M} \times m_F(s)W_F(\|u(s)\|_E) \in L^1([0,T],\mathbb{R}^+)$, $\|R_q(t-s)w_n(s)\| \leqslant \widetilde{M} m_H(s)W_H(\|u(s)\|_E) \in L^1([0,T],\mathbb{R}^+)$. Taking $\overline{y}_j = \sum_{m=j}^{m_0(j)} \lambda_{j,m} y_m$, from equation (4) it follows that

$$\begin{aligned}
\overline{y}_n(t) &= S_q(t)\big[u_0 + g(u)\big] + K_q(t)\big[u_1 + h(u)\big] \\
&\quad + \int_0^t K_q(t-s)G\left(s,\, u\big(h_1(s)\big), \int_0^s a_1\big(s,\tau,u\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\
&\quad + \int_0^t R_q(t-\xi)B\bigg\{B^* R_q^*(T-\xi)\big[\big(\lambda I + \Gamma_0^T\big)^{-1} u_T - S_q(T)\big[u_0 + g(u)\big] \\
&\qquad - K_q(T)\big[u_1 + h(u)\big]\big] - B^* R_q^*(T-\xi) \\
&\qquad \times \int_0^T \big(\lambda I + \Gamma_s^T\big)^{-1} K_q(T-s)G\left(s,\, u\big(h_1(s)\big), \int_0^s a_1\big(s,\tau,u\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\right)\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\quad - B^* R_q^*(T-\xi) \int_0^T \big(\lambda I + \varGamma_s^T\big)^{-1} R_q(T-s) z_n(s)\,\mathrm{d}s \\
&\quad - B^* R_q^*(T-\xi) \int_0^T \big(\lambda I + \varGamma_s^T\big)^{-1} R_q(T-s) w_n(s)\,\mathrm{d}s \bigg\}\,\mathrm{d}\xi \\
&+ \int_0^t R_q(t-s) z_n(s)\,\mathrm{d}s + \int_0^t R_q(t-s) w_n(s)\,\mathrm{d}s, \quad t \in [0,T].
\end{aligned} \tag{5}
$$

Since $\overline{y}_n \to y(t)$, $z_n(t) \to f(t)$ and $w_n(t) \to \rho(t)$ as $n \to \infty$, from equation (5) and Lebesgue's dominated convergence theorem we get

$$
\begin{aligned}
y(t) &= S_q(t)\big[u_0 + g(u)\big] + K_q(t)\big[u_1 + h(u)\big] \\
&+ \int_0^t K_q(t-s) G\bigg(s,\, u\big(h_1(s)\big), \int_0^s a_1\big(s,\tau,u\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\bigg)\,\mathrm{d}s \\
&+ \int_0^t R_q(t-\xi) B \bigg\{ B^* R_q^*(T-\xi) \big[\big(\lambda I + \varGamma_0^T\big)^{-1} u_T \\
&\quad - S_q(T)\big[u_0 + g(u)\big] - K_q(T)\big[u_1 + h(u)\big]\big] - B^* R_q^*(T-\xi) \\
&\quad \times \int_0^T \big(\lambda I + \varGamma_s^T\big)^{-1} K_q(T-s) G\bigg(s,\, u\big(h_1(s)\big), \int_0^s a_1\big(s,\tau,u\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\bigg)\,\mathrm{d}s \\
&\quad - B^* R_q^*(T-\xi) \int_0^T \big(\lambda I + \varGamma_s^T\big)^{-1} R_q(T-s) f(s)\,\mathrm{d}s \\
&\quad - B^* R_q^*(T-\xi) \int_0^T \big(\lambda I + \varGamma_s^T\big)^{-1} R_q(T-s) \rho(s)\,\mathrm{d}s \bigg\}\,\mathrm{d}\xi \\
&+ \int_0^t R_q(t-s) f(s)\,\mathrm{d}s + \int_0^t R_q(t-s) \rho(s)\,\mathrm{d}s,
\end{aligned}
$$

where $f \in S_{F,u}$ and $\rho \in S_{H,u}$. This demonstrates that $\varPi(u)$ is closed. Since $S_{F,u}$ and $S_{H,u}$ are convex, therefore the convexity of $\varPi(u)$ is obvious.

Step 2. To construct a nonincreasing sequence $\{S_n,\, n \geqslant 1\}$ of nonempty, bounded, closed and convex subsets of E.

By assumption (A5), we have that for any $\lambda > 0$, there exists a positive constant $r = r_0(\lambda)$ such that $b/\lambda + cr/\lambda + dW(r)/\lambda \leqslant r$. Take $S_0 = \mathcal{B}_r = \{u \in E\colon \|u\| \leqslant r\}$. It is clear that S_0 is a bounded, closed and convex subset of E. We claim that $\Pi(S_0) \subseteq S_0$. To prove this, let $u \in S_0$ and $y \in \Pi(u)$. Now it follows from (A1)–(A4) and Hölder inequality that

$$\begin{aligned}\|y(t)\| \leqslant \widetilde{M}\bigl\{&\bigl[\|u_0\| + L_g^1\|u\| + L_g^2\bigr] + \bigl(\|u_1\| + L_h^1\|u\| + L_g^2\bigr)\\&+ T\bigl[L_G\bigl(\|u(t)\| + L_{a_1}\|u(t)\| + C_2\bigr) + C_1\bigr] + TM_B\|x(t)\|\\&+ T\|m_F\|_{L^1}W_F(r) + T\|m_H\|_{L^1}W_H(r)\bigr\}.\end{aligned}$$

For $t \in [0,T]$, we have

$$\begin{aligned}\|x(t)\| \leqslant{}& \|B^*\| \cdot \|R_q^*(T-t)R(\lambda, \Gamma_0^T)\|\bigl[\|u_T\| + \widetilde{M}\bigl[\|u_0\| + L_g^1\|u\| + L_g^2\bigr]\\&+ \widetilde{M}\bigl[\|u_1\| + L_h^1\|u\| + L_h^2\bigr] + \widetilde{M}T\bigl[L_G\bigl(\|u(t)\| + L_{a_1}\|u(t)\| + C_2\bigr) + C_1\bigr]\\&+ \widetilde{M}T\|m_F\|_{L^1}W_F(r) + \widetilde{M}T\|m_H\|_{L^1}W_H(r)\bigr]\\\leqslant{}& \frac{M_B\widetilde{M}}{\lambda}\bigl[\|u_T\| + \widetilde{M}\bigl(\|u_0\| + L_g^2 + \|u_1\| + L_h^2\bigr) + T\widetilde{M}(L_GC_2 + C_1)\bigr]\\&+ \frac{M_B\widetilde{M}^2}{\lambda}\bigl[L_g^1 + L_h^1 + TL_G(1 + L_{a_1})\bigr]r\\&+ \frac{M_B\widetilde{M}}{\lambda}\bigl[\widetilde{M} \cdot T\|m_F\|_{L^1} + T\widetilde{M}\|m_H\|_{L^1}\bigr]W(r)\\\leqslant{}& \frac{b}{3\lambda\mathfrak{L}} + \frac{c}{3\lambda\mathfrak{L}}r + \frac{d}{3\lambda\mathfrak{L}}W(r) \leqslant \frac{r}{3\mathfrak{L}}.\end{aligned}$$

Therefore, we get

$$\begin{aligned}\|y(t)\| \leqslant{}& \widetilde{M}\bigl\{\bigl[\|u_0\| + L_g^1\|u\| + L_g^2\bigr] + \bigl(\|u_1\| + L_h^1\|u\| + L_h^2\bigr)\\&+ T\bigl[L_G\bigl(\|u(t)\| + C_2 + L_{a_1}\|u(t)\|\bigr) + C_1\bigr]\\&+ \sqrt{T}M_B\|x\|_{L^2} + T\|m_F\|_{L^1}W_F(r) + T\|m_H\|_{L^1}W_H(r)\bigr\}\\\leqslant{}& \frac{b}{3} + \frac{c}{3}r + \frac{d}{3}W(r) + \frac{r}{3}\\\leqslant{}& \frac{1}{3}\bigl[b + cr + dW(r)\bigr] + \frac{r}{3}\\\leqslant{}& \frac{2r}{3} \leqslant r.\end{aligned}$$

Therefore, we get $\Pi(S_0) \subseteq S_0$. Next, define $S_n = \overline{\operatorname{conv}}\,\Pi(S_{n-1})$, $n \geqslant 1$. For every $n \geqslant 1$, it is clear that the set S_n is nonempty, closed and convex in E. By induction, we have that the sequence $\{S_n, n \geqslant 1\}$ is decreasing. Since S_0 is convex and closed, therefore we get $S_1 \subseteq S_0$. Hence $S_2 \subseteq \overline{\operatorname{conv}}\,S_1 \subseteq \overline{\operatorname{conv}}\,S_0 = S_1$. Assume $S_n \subset S_{n-1}$. Then it is obvious $S_{n+1} \subseteq \overline{\operatorname{conv}}\,S_n \subseteq \overline{\operatorname{conv}}\,S_{n-1} = S_n$. Furthermore, S_0 being bounded, S_n is also bounded for every $n \geqslant 1$.

Step 3. Let $V = \Pi(S_0)$. We claim that V is equicontinuous. Now, we show that S_n is equicontinuous for every $n \geqslant 1$.

Let $u \in V$, then there exist $u \in S_0$ with $y \in \Pi(u)$. Therefore, there exist $f \in S_{F,u}$ and $\rho \in S_{H,u}$ such that

$$\begin{aligned} y(t) = {} & S_q(t)\big[u_0 + g(u)\big] + K_q(t)\big[u_1 + h(u)\big] \\ & + \int_0^t K_q(t-s) G\Bigg(s,\, u\big(h_1(s)\big),\, \int_0^s a_1\big(s, \tau, u\big(h_2(\tau)\big)\big)\, \mathrm{d}\tau\Bigg)\, \mathrm{d}s \\ & + \int_0^t R_q(t-s) Bx(s)\, \mathrm{d}s + \int_0^t R_q(t-s) f(s)\, \mathrm{d}s \\ & + \int_0^t R_q(t-s) \rho(s)\, \mathrm{d}s, \quad t \in [0,T]. \end{aligned}$$

Let $t,\ t+\epsilon \in [0,T]$ with $\epsilon > 0$. Then we get

$$\begin{aligned} & \big\|y(t+\epsilon) - y(t)\big\| \\ & \quad \leqslant \big\|S_q(t+\epsilon) - S_q(t)\big\| \big[\|u_0\| + L_g^1\|u\| + L_g^2\big] \big\| \\ & \qquad + \big\|K_q(t+\epsilon) - K_q(t)\big[\|u_1\| + L_h^1\|u\| + L_h^2\big] \\ & \qquad + \epsilon\widetilde{M}\big[L_G\big(\big\|u(t)\big\| + L_{a_1}\big\|u(t)\big\| + C_2\big) + C_1\big] \\ & \qquad + \int_0^t \big\|K_q(t+\epsilon-s) - K_q(t-s)\big\| \Bigg\|G\Bigg(s,\, u\big(h_1(s)\big),\, \int_0^s a_1\big(s, \tau, u\big(h_2(\tau)\big)\big)\, \mathrm{d}\tau\Bigg)\Bigg\|\, \mathrm{d}s \\ & \qquad + \epsilon\widetilde{M} W_F(r) \int_t^{t+\epsilon} m_F(s)\, \mathrm{d}s + W_F(r) \int_0^t \big\|R_q(t+\epsilon) - R_q(t-s)\big\| m_F(s)\, \mathrm{d}s \\ & \qquad + \epsilon\widetilde{M} W_H(r) \int_t^{t+\epsilon} m_H(s)\, \mathrm{d}s + W_H(r) \int_0^t \big\|R_q(t+\epsilon) - R_q(t-s)\big\| m_F(s)\, \mathrm{d}s. \end{aligned} \tag{6}$$

Utilizing the continuity of $S_q(t), K_q(t)$ and $R_q(t)$ in t in the uniform operator topology, it can be obtained that righthand side of (6) tends to zero independently of $u \in E$, which gives that U is equicontinuous. Then we can prove that S_1 is equicontinuous for each $t \in [0,T]$. Evidenced by the same as above, S_n is equicontinuous for each $n \geqslant 1$.

Step 4. By using the previous results, we make a subset $S = \cap_{n=1}^{\infty} S_n$, which is nonempty in E. Now, it is sufficient to prove that $\lim_{n\to\infty} \chi_C(S_n) = 0$.

Let $\epsilon > 0$ and $n \geqslant 1$ be a fixed natural number. Now, there exists a sequence $\{y_m,\, m \geqslant 1\}$ in $\Pi(S_{n-1})$ such that $\chi_C(S_n) = \chi_C \Pi(S_{n-1}) \leqslant 2\chi_C\{y_k,\, k \geqslant 1\} + \epsilon$.

For $t \in [0,T]$, from the above inequality, we get

$$\chi_C(S_n) \leqslant 2\chi(W) + \epsilon, \tag{7}$$

where $W = \{y_m,\, m \geqslant 1\}$. For every $n \geqslant 1$, S_n is equicontinuous. Then we get $\chi(W) = \sup_{t\in[0,T]} \chi(W(t))$. Thus, utilizing the nonsingularity of χ, inequality (7) becomes

$$\chi_C(S_n) \leqslant 2 \sup_{t\in[0,T]} \chi\big(W(t)\big) + \epsilon = 2 \sup_{t\in[0,T]} \chi\big(\{y_m(t),\, m \geqslant 1\}\big) + \epsilon. \tag{8}$$

Since $y_m \in \Pi(S_{n-1})$, $m \geqslant 1$, there exists $u_m \in S_{n-1}$ such that $y_m \in \Pi(u_m)$, $m \geqslant 1$. Then there exist $f_m \in S_{F,u_m}$ and $\rho_m \in S_{H,u_m}$ such that

$$\begin{aligned}
&\chi\big\{y_m(t),\, m \geqslant 1\big\} \\
&\quad\leqslant \chi\big\{S_q(t)\big[u_0 + g(u_m)\big]\big\} + \chi\big\{K_q(t)\big[u_1 + h(u_m)\big]\big\} \\
&\qquad + \chi\Bigg\{\int_0^t K_q(t-s)G\bigg(s,\, u_m\big(h_1(s)\big),\, \int_0^s a_1\big(s,\tau,u_m\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\bigg)\,\mathrm{d}s\Bigg\} \\
&\qquad + \chi\Bigg\{\int_0^t R_q(t-s)Bx_m(s)\,\mathrm{d}s\Bigg\} + \chi\Bigg\{\int_0^t R_q(t-s)f_m(s)\,\mathrm{d}s\Bigg\} \\
&\qquad + \chi\Bigg\{\int_0^t R_q(t-s)\rho_m(s)\,\mathrm{d}s\Bigg\},
\end{aligned}$$

where

$$\begin{aligned}
x_m(s) &= B^*R_q^*(T-\xi)\big[\big(\lambda I + \Gamma_0^T\big)^{-1}u_T - S_q(T)\big[u_0 + g(u_m)\big] \\
&\quad - K_q(T)\big[u_1 + h(u_m)\big]\big] - B^*R_q^*(T-\xi) \\
&\quad \times \int_0^T \big(\lambda I + \Gamma_s^T\big)^{-1} K_q(T-s)G\bigg(s,\, u_m\big(h_1(s)\big),\, \int_0^s a_1\big(s,\tau,u_m\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\bigg)\,\mathrm{d}s \\
&\quad - B^*R_q^*(T-\xi)\int_0^T \big(\lambda I + \Gamma_s^T\big)^{-1} R_q(T-s)f_m(s)\,\mathrm{d}s \\
&\quad - B^*R_q^*(T-\xi)\int_0^T \big(\lambda I + \Gamma_s^T\big)^{-1} R_q(T-s)\rho_m(s)\,\mathrm{d}s.
\end{aligned}$$

Since g, h and G are compact, thus we only need to estimate $\chi\{\int_0^t R_q(t-s)f_m(s)\,\mathrm{d}s\}$, $\chi\{\int_0^t R_q(t-s)\rho_m(s)\,\mathrm{d}s\}$ and $\chi\{\int_0^t R_q(t-s)Bx(s)\,\mathrm{d}s\}$. From (A2) and (A3) we have

$$\begin{aligned}
\chi\big\{f_m(t),\, m \geqslant 1\big\} &\leqslant \chi\big\{F\big(t,u_m(t)\big),\, m \geqslant 1\big\} \leqslant \alpha_2(t)\chi\big\{u_m(t),\, m \geqslant 1\big\} \\
&\leqslant \alpha_2(t)\chi_C(S_{n-1}) = \gamma_1(t),
\end{aligned}$$

$$\chi\{\rho_m(t),\, m \geqslant 1\} \leqslant \chi\{H(t, u_m(t)),\, m \geqslant 1\} \leqslant \alpha_1(t)\chi\{u_m(t),\, m \geqslant 1\}$$
$$\leqslant \alpha_1(t)\chi_C(S_{n-1}) = \gamma_2(t),$$

where $\gamma_1, \gamma_2 \in L^1([0,T], \mathbb{R}^+)$. Moreover, for any $m \geqslant 1$, from assumptions (A2)(ii) and (A3)(ii) we have $\|f_m(t)\| \leqslant m_F(t)W_F(r)$ and $\|\rho_m(t)\| \leqslant m_H(t)W_H(r)$ for almost $t \in [0,T]$. Consequently, $f_m, \rho_m \in L^1([0,T], E)$. Therefore, we get $\alpha_2(t)\chi_C(S_{n-1})$, $\alpha_1(t)\chi_C(S_{n-1}) \in L^1([0,T], E)$. Then there is a compact set $V_\epsilon \subseteq E$, a measurable set $J_\epsilon \subset [0,T]$ with measure less than ϵ and sequences of functions $\{\widetilde{f}_m^\epsilon\}, \{\widetilde{\rho}_m^\epsilon\} \in L^1([0,T], E)$ such that $\{\widetilde{f}_m^\epsilon(s),\, m \geqslant 1\}$, $\{\rho_m(s),\, m \geqslant 1\} \subset V_\epsilon$ for all $s \in [0,T]$ and $\|f_m(s) - \widetilde{f}_m^\epsilon(s)\| < 2\gamma_1(s) + \epsilon$ for every $m \geqslant 1$ and every $s \in J'_\epsilon = [0,T] - J_\epsilon$, $\|\rho_m(s) - \widetilde{\rho}_m^\epsilon(s)\| < 2\gamma_2(s) + \epsilon$ for every $m \geqslant 1$ and every $s \in J'_\epsilon = [0,T] - J_\epsilon$. Therefore, we can get

$$\left\| \int_0^t R_q(t-s)\left[f_m(s) - \widetilde{f}_m^\epsilon(s)\right] \mathrm{d}s \right\|$$
$$\leqslant \widetilde{M}T\left\|2\gamma_1(s) + \epsilon\right\|_{L^1} \leqslant 2\widetilde{M}T\left(\|\alpha_2\|_{L^1}\chi_C(S_{n-1}) + \epsilon T\right) \tag{9}$$

and

$$\left\| \int_0^t R_q(t-s)\left[\rho_m(s) - \widetilde{\rho}_m^\epsilon(s)\right] \mathrm{d}s \right\|$$
$$\leqslant \widetilde{M}T\left\|2\gamma_2(s) + \epsilon\right\|_{L^1} \leqslant 2\widetilde{M}T\left(\|\alpha_1\|_{L^1}\chi_C(S_{n-1}) + \epsilon T\right), \tag{10}$$

also, $\|\int_{J_\epsilon} R_q(t-s)f_m(s)\,\mathrm{d}s\| \leqslant \widetilde{M}TW_F(r)\int_{J_\epsilon} m_F(s)\,\mathrm{d}s$, $\|\int_{J_\epsilon} R_q(t-s)\rho_m(s)\,\mathrm{d}s\| \leqslant \widetilde{M}TW_H(r)\int_{J_\epsilon} m_H(s)\,\mathrm{d}s$. From (9)–(10) we obtain

$$\chi\left\{\int_0^t R_q(t-s)f_m(s)\,\mathrm{d}s\right\}$$
$$\leqslant \chi\left\{\int_{J'_\epsilon} R_q(t-s)f_m(s)\,\mathrm{d}s,\, m \geqslant 1\right\} + \chi\left\{\int_{J_\epsilon} R_q(t-s)f_m(s)\,\mathrm{d}s,\, m \geqslant 1\right\}$$
$$\leqslant 2\widetilde{M}T\left(\|\alpha_2\|_{L^1}\chi_C(S_{n-1}) + \epsilon T\right) + T\widetilde{M}W_F(r)\int_{J_\epsilon} m_F(s)\,\mathrm{d}s,$$

and similarly, we can get $\chi\{\int_0^t R_q(t-s)\rho_m(s)\,\mathrm{d}s\} \leqslant 2\widetilde{M}T(\|\alpha_1\|_{L^1} \cdot \chi_C(S_{n-1}) + \epsilon T) + T\widetilde{M}W_H(r)\int_{J_\epsilon} m_H(s)\,\mathrm{d}s$. Since ϵ is arbitrary and that the measure of J_ϵ is less than ϵ, we conclude that for all $t \in [0,T]$, $\chi\{\int_0^t R_q(t-s)f_m(s)\,\mathrm{d}s\} \leqslant 2\widetilde{M}T\|\alpha_2\|_{L^1} \times \chi_C(S_{n-1})$, $\chi\{\int_0^t R_q(t-s)\rho_m(s)\,\mathrm{d}s\} \leqslant 2\widetilde{M}T\|\alpha_1\|_{L^1} \cdot \chi_C(S_{n-1})$. Next, we estimate $\chi_C\{\int_0^t R_q(t-s)Bx_k(s)\,\mathrm{d}s\}$. Let $t \in [0,T]$ and let $\chi\{x_m,\, m \geqslant 1\} = \beta$. Then for all $\beta' > \beta$, there exists a finite family $\{v_1, v_2, \ldots, v_j\} \subset L^2([0,T], X)$ such that for any

$x_m \in L^2([0,T],X)$, there is $i \in \{1,2,\dots,j\}$ with $\|x_m - v_i\|_{L^2([0,T],X)} \leqslant \beta'$. Thus, we conclude that

$$\left\| \int_0^t R_q(t-s)Bx_m(s)\,\mathrm{d}s - \int_0^t R_q(t-s)Bv_i(s)\,\mathrm{d}s \right\| \\ \leqslant \widetilde{M}M_B \int_0^t \big\|x_m(s) - v_i(s)\big\|\,\mathrm{d}s \leqslant \widetilde{M}M_B\sqrt{T}\|x_m - v_i\|_{L^2([0,T],X)} \\ \leqslant \widetilde{M}M_B\sqrt{T}\beta',$$

and hence $\chi_C(\{\int_0^t R_q(t-s)Bx_m(s)\,\mathrm{d}s,\ m \geqslant 1\}) \leqslant \widetilde{M}M_B\sqrt{T} \cdot \chi(\{x_m,\ m \geqslant 1\})$. Since $\chi_C\{x_m,\ m \geqslant 1\} \leqslant M_B\widetilde{M}/\lambda \cdot (2T\widetilde{M}(\|\alpha\|_{L^1} + \|\alpha_2\|_{L^1})) \cdot \chi_C(S_{n-1})$, which gives that $\chi\{\int_0^t R_q(t-s)Bx_k(s)\,\mathrm{d}s\} \leqslant (2M_B\widetilde{M}^3/\lambda)T^{3/2}(\|\alpha_1\|_{L^1}+\|\alpha_2\|_{L^1})\cdot\chi(S_{n-1})$. Therefore, for all $t \in [0,T]$, we have $\chi\{y_m(t),\ m \geqslant 1\} \leqslant 2T\widetilde{M}(1 + \sqrt{T}M_B^2\widetilde{M}^2/\lambda) \times (\|\alpha_1\|_{L^1} + \|\alpha_2\|_{L^1}) \cdot \chi(S_{n-1})$. Using the fact that ϵ is arbitrary and from inequality (8), we obtain

$$\chi_C(S_n) \leqslant 2T\widetilde{M}\left(1 + \frac{\sqrt{T}M_B^2\widetilde{M}^2}{\lambda}\right)\big(\|\alpha_1\|_{L^1} + \|\alpha_2\|_{L^1}\big)\chi_C(S_{n-1}).$$

By means of a finite number of steps, we can write

$$0 \leqslant \chi_C(S_n) \\ \leqslant \left(2T\widetilde{M}\left(1 + \frac{\sqrt{T}M_B^2\widetilde{M}^2}{\lambda}\right)\big(\|\alpha_1\|_{L^1} + \|\alpha_2\|_{L^1}\big)\right)^{n-1}\chi_C(S_1), \quad n \geqslant 2.$$

From (A2) and (A3) we obtain $\lim_{n\to\infty}\chi_C(S_n) = 0$.

Step 5. The multivalued function $\Pi|_S : S \to 2^S$ has a closed graph.

Let $u^n \to \overline{u}$ in $S \subset E$, $y^n \in \Pi(u^n)$ and $y^n \to \overline{y}$. We show that $\overline{y} \in \Pi(\overline{u})$. Indeed, $y^n \in \Pi(u^n)$, it gives that there exist $f_n \in S_{F,u^n}$ and $\rho_n \in S_{H,u^n}$ such that

$$\begin{aligned} y^n(t) = {} & S_q(t)\big[u_0 + g\big(u^n\big)\big] + K_q(t)\big[u_1 + h\big(u^n\big)\big] \\ & + \int_0^t K_q(t-s)G\left(s,\ u^n\big(h_1(s)\big), \int_0^s a_1\big(s,\tau,u^n\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\right)\mathrm{d}s \\ & + \int_0^t R_q(t-\xi)B\bigg\{B^*R_q^*(T-\xi)\big[\big(\lambda I + \Gamma_0^T\big)^{-1}u_T \\ & \quad - S_q(T)\big[u_0 + g\big(u^n\big)\big] - K_q(T)\big[u_1 + h\big(u^n\big)\big]\big] - B^*R_q^*(T-\xi) \\ & \quad \times \int_0^T \big(\lambda I + \Gamma_s^T\big)^{-1}K_q(T-s)G\left(s, u^n\big(h_1(s)\big), \int_0^s a_1\big(s,\tau,u^n\big(h_2(\tau)\big)\big)\mathrm{d}\tau\right)\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\quad - B^* R_q^*(T-\xi) \int_0^T \big(\lambda I + \Gamma_s^T\big)^{-1} R_q(T-s) f^n(s)\, \mathrm{d}s \\
&\quad - B^* R_q^*(T-\xi) \int_0^T \big(\lambda I + \Gamma_s^T\big)^{-1} R_q(T-s) \rho^n(s)\, \mathrm{d}s \Bigg\} \mathrm{d}\xi \\
&+ \int_0^t R_q(t-s) f^n(s)\, \mathrm{d}s + \int_0^t R_q(t-s) \rho^n(s)\, \mathrm{d}s. \qquad (11)
\end{aligned}
$$

Now, we must show that there exist $\overline{f} \in S_{F,u}$ and $\overline{\rho} \in S_{H,u}$ such that

$$
\begin{aligned}
\overline{y}(t) &= S_q(t)\big[u_0 + g(\overline{u})\big] + K_q(t)\big[u_1 + h(\overline{u})\big] \\
&\quad + \int_0^t K_q(t-s) G\Bigg(s,\, \overline{u}\big(h_1(s)\big), \int_0^s a_1\big(s, \tau, \overline{u}\big(h_2(\tau)\big)\big)\, \mathrm{d}\tau\Bigg) \mathrm{d}s \\
&\quad + \int_0^t R_q(t-\xi) B \Bigg\{ B^* R_q^*(T-\xi) \big[\big(\lambda I + \Gamma_0^T\big)^{-1} u_T \\
&\qquad - S_q(T)\big[u_0 + g(\overline{u})\big] - K_q(T)\big[u_1 + h(\overline{u})\big]\big] - B^* R_q^*(T-\xi) \\
&\qquad \times \int_0^T \big(\lambda I + \Gamma_s^T\big)^{-1} K_q(T-s) G\Bigg(s,\, \overline{u}\big(h_1(s)\big), \int_0^s a_1\big(s, \tau, \overline{u}\big(h_2(\tau)\big)\big)\, \mathrm{d}\tau\Bigg) \mathrm{d}s \\
&\qquad - B^* R_q^*(T-\xi) \int_0^T \big(\lambda I + \Gamma_s^T\big)^{-1} R_q(T-s) \overline{f}(s)\, \mathrm{d}s \\
&\qquad - B^* R_q^*(T-\xi) \int_0^T \big(\lambda I + \Gamma_s^T\big)^{-1} R_q(T-s) \overline{\rho}(s)\, \mathrm{d}s \Bigg\} \mathrm{d}\xi \\
&\quad + \int_0^t R_q(t-s) \overline{f}(s)\, \mathrm{d}s + \int_0^t R_q(t-s) \overline{\rho}(s)\, \mathrm{d}s. \qquad (12)
\end{aligned}
$$

For every $n \geqslant 2$ and for a.e. $t \in [0,T]$, we have $\|f^n(t)\| \leqslant m_F(t) W_F(r)$, $\|\rho^n(t)\| \leqslant m_H(t) W_H(r)$, which gives that the sets $\{f^n,\, n \geqslant 1\}$ and $\{\rho^n,\, n \geqslant 1\}$ are integral bounded. Additionally, conditions (A2), (A3) and convergence of $\{u^n\}_{n\geqslant 1}$ imply that $\chi\{f^n(t),\, n \geqslant 1\} \leqslant \chi\{F(t, u^n(t)),\, n \geqslant 1\} \leqslant \alpha_2(t)\chi\{u^n(t),\, n \geqslant 1\} = 0$, $\chi\{\rho^n(t),\, n \geqslant 1\} \leqslant \chi\{H(t, u^n(t)),\, n \geqslant 1\} \leqslant \alpha_1(t)\chi\{u^n(t),\, n \geqslant 1\} = 0$. Therefore, the set $\{f^n,\, n \geqslant 1\}$ and $\{\rho^n,\, n \geqslant 1\}$ are relatively compact for a.e. $t \in [0,T]$. Therefore, the sequence $\{f^n\}_{n\geqslant 1}$ and $\{\rho^n\}_{n\geqslant 1}$ are semicompact, and then we have that $\{f^n\}_{n\geqslant 1}$ and $\{\rho^n\}_{n\geqslant 1}$ are weakly compact in $L^1([0,T], E)$. Without loss of generality, it can be assumed that f^n and ρ^n converge weakly to function f and ρ, respectively. By Mazur's lemma, for every natural number j, there exists a natural number $m_0(j) > j$ and a sequence of nonnegative real numbers $\lambda_{j,m}$, $m = j, \dots, m_0(j)$, such that $\sum_{m=j}^{m_0(j)} \lambda_{j,m} = 1$. Thus, the sequences of convex combinations $z_j = \sum_{m=j}^{m_0(j)} \lambda_{j,m} f^m$, $j \geqslant 1$, and $w_j = \sum_{m=j}^{m_0(j)} \lambda_{j,m} \rho^m$, $j \geqslant 1$, converge strongly to $f \in L^1([0,T], E)$ and $\rho \in L^1([0,T], E)$ as $j \to \infty$, respectively. So, we may assume that $z_j \to f(t) := \overline{f}(t)$ and $w_j(t) \to \rho(t) := \overline{\rho}(t)$ for a.e. $t \in [0,T]$.

Let t be such that $F(t,\cdot)$ and $H(t,\cdot)$ are upper semicontinuous. Then, for any neighborhood $\mathcal{N}_1$ of $F(t,\cdot)$ and neighborhood $\mathcal{N}_2$ of $H(t,\cdot)$, there exists a natural number n_0 such that for any $n \geqslant n_0$, $F(t, u^n(t)) \subseteq \mathcal{N}_1$, and $H(t, u^n(t)) \subseteq \mathcal{N}_2$. Since F and H take convex and compact values, Lemma 1 gives that $\bigcap_{j\geqslant 1} \overline{\operatorname{conv}}(\bigcup_{n\geqslant j} F(t, u^n(t))) \subset F(t, \overline{u}(t))$, $\bigcap_{j\geqslant 1} \overline{\operatorname{conv}}(\bigcup_{n\geqslant j} H(t, u^n(t))) \subset H(t, \overline{u}(t))$. From Mazur's theorem, there exist sequences $\{z_n\}_{n\geqslant 1}$ of convex combinations of f^n and $\{w_n\}_{n\geqslant 1}$ of convex combinations of ρ^n such that $\overline{f}(t) \in \bigcap_{j\geqslant 1} \overline{\{z_n(t),\, n \geqslant j\}} \subseteq \bigcap_{j\geqslant 1} \overline{\operatorname{conv}}\{f^n(t),\, n \geqslant j\}$ for a.e. $t \in [0,T]$ and z_n converges strongly to $f \in L^1([0,T], E)$. Also, $\overline{\rho}(t) \in \bigcap_{j\geqslant 1} \overline{\{w_n(t),\, n \geqslant j\}} \subseteq \bigcap_{j\geqslant 1} \overline{\operatorname{conv}}\{\rho^n(t),\, n \geqslant j\}$ and w_n converges strongly to $\rho \in L^1([0,T], E)$. Then, for a.e. $t \in [0,T]$, $\overline{f}(t) \in \bigcap_{j\geqslant 1} \overline{\operatorname{conv}}\{f^n(t),\, n \geqslant j\} \subseteq \bigcap_{j\geqslant 1} \overline{\operatorname{conv}}\{F(t, u^n(t)),\, n \geqslant j\} \subseteq F(t, \overline{u}(t))$, and $\overline{\rho}(t) \in \bigcap_{j\geqslant 1} \overline{\operatorname{conv}}\{\rho^n(t),\, n \geqslant j\} \subseteq \bigcap_{j\geqslant 1} \overline{\operatorname{conv}}\{H(t, u^n(t)),\, n \geqslant j\} \subseteq H(t, \overline{u}(t))$. In other words, we have that there exist $\overline{f} \in S_{F,\overline{u}}$ and $\overline{\rho} \in S_{H,\overline{u}}$. Moreover, by the continuity of G, h, g, S_q, K_q and T_q, from equation (11) it is concluded that equation (12) holds true. Hence, $\varPi|_S$ has a closed graph. As a consequence of above steps, it is concluded that the multivalued $\varPi|_S : S \to 2^S$ is closed and χ_C-condensing with nonempty convex compact values. Thus, from Lemma 2 there exists a $u \in S$, which is a fixed point of $\varPi$, such that $u \in \varPi(u)$. Hence, $u(\cdot)$ is a mild solution for system (1)–(2). □

Next, we make some additional assumptions to prove approximate controllability of the fractional control system.

(A2) G is uniformly bounded.

(A3) The associated fractional linear differential inclusion of (1)–(2) is approximately controllable.

(A4) $R(\lambda, \varGamma_0^T) = (\lambda I + \varGamma_0^T)^{-1}$. For each $t \in [0,T]$, the operator $\lambda R(\lambda I, \varGamma_0^T) \to 0$ as $\lambda \to 0$ in the strong operator topology.

Theorem 2. *Assume that assumptions* (A1)–(A7) *hold. Then the control problem* (1)–(2) *is approximately controllable.*

Proof. Let $u_\lambda(\cdot)$ be a fixed point of $\varPi$. Any fixed point of $\varPi$ is a mild solution of (1)–(2) under the control

$$x_\lambda(t) = B^* R_q^*(T-t) R\bigl(\lambda, \varGamma_0^T\bigr) p\bigl(u_\lambda(\cdot)\bigr),$$

such that

$$\begin{aligned} u_\lambda(t) &= S_q(t)\bigl[u_0 + g(u_\lambda)\bigr] + K_q(t)\bigl[u_1 + h(u_\lambda)\bigr] \\ &\quad + \int_0^t K_q(t-s) G\left(s,\, u_\lambda\bigl(h_1(s)\bigr),\, \int_0^s a_1\bigl(s,\tau,u_\lambda\bigl(h_2(\tau)\bigr)\bigr)\,\mathrm{d}\tau\right) \mathrm{d}s \end{aligned}$$

$$+\int_0^t R_q(t-s)Bx_\lambda(s)\,\mathrm{d}s+\int_0^t R_q(t-s)f_\lambda(s)\,\mathrm{d}s$$
$$+\int_0^t R_q(t-s)\rho_\lambda(s)\,\mathrm{d}s,\quad t\in[0,T],$$

where $f_\lambda \in S_{F,u_\lambda} = \{f_\lambda \in L^1([0,T],E)\colon\ f_\lambda(t)\in F(t,u_\lambda(t)),\ \text{for a.e. } t\in[0,T]\}$, $\rho_\lambda \in S_{H,u_\lambda} = \{\rho_\lambda \in L^1([0,T],E)\colon\ \rho_\lambda(t)\in H(t,u_\lambda(t)),\ \text{for a.e. } t\in[0,T]\}$ and satisfies $u_\lambda(T)=u_T-\lambda R(\lambda,\Gamma_0^T)p(u_\lambda)$, where

$$\begin{aligned}p(u_\lambda)&=u_T-S_q(T)\big[u_0+g(u_\lambda)\big]-K_q(T)\big[u_1+h(u_\lambda)\big]\\&\quad-\int_0^T K_q(T-s)G\left(s,\,u_\lambda\big(h_1(s)\big),\int_0^s a_1\big(s,\tau,u_\lambda\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\right)\mathrm{d}s\\&\quad-\int_0^T R_q(T-s)f_\lambda(s)\,\mathrm{d}s-\int_0^T R_q(T-s)\rho_\lambda(s)\,\mathrm{d}s.\end{aligned}$$

We have that F, G and H are uniformly bounded on $[0,T]$. Then there are subsequences, denoted by $\{G(t,u^\lambda(h_1(t)),\int_0^t a_1(t,\zeta,u^\lambda(h_2(\zeta)))\,\mathrm{d}\zeta)\}$, $\{F(t,u^\lambda(t))\}$ and $\{H(t,u^\lambda(t))\}$, which converge weakly to, say, $G(s)$, $F(s)$ and $H(s)$. Define

$$\begin{aligned}w(u)&=u_T-S_q(T)\big[u_0+g(u)\big]-K_q(T)\big[u_1+h(u)\big]\\&\quad-\int_0^T K_q(T-s)G(s)\,\mathrm{d}s-\int_0^T R_q(T-s)f(s)\,\mathrm{d}s\\&\quad-\int_0^T R_q(T-s)\rho(s)\,\mathrm{d}s,\end{aligned}$$

with $f\in S_{F,u}$ and $\rho\in S_{H,u}$. For $t\in[0,T]$, it follows that

$$\begin{aligned}&\big\|p(u_\lambda)-w(u)\big\|\\&\quad\leqslant\big\|S_q(t)\big[g(u_\lambda)-g(u)\big]\big\|+\big\|K_q(t)\big[h(u_\lambda)-h(u)\big]\big\|\\&\qquad+\left\|\int_0^T K_q(T-s)\left[G\left(s,\,u_\lambda\big(h_1(s)\big),\int_0^s a_1\big(s,\tau,u_\lambda\big(h_2(\tau)\big)\big)\,\mathrm{d}\tau\right)-G(s)\right]\mathrm{d}s\right\|\\&\qquad+\left\|\int_0^T R_q(T-s)\big[f_\lambda(s)-f(s)\big]\,\mathrm{d}s\right\|+\left\|\int_0^T R_q(T-s)\big[\rho_\lambda(s)-\rho(s)\big]\,\mathrm{d}s\right\|.\end{aligned}$$

Using (A1)–(A4) and strongly continuity of $S_q(t)$, $K_q(t)$ and $R_q(t)$, we have that $p(u_\lambda) \to w(u)$ as $\lambda \to 0^+$ and

$$\begin{aligned}\|u_\lambda(T) - u_T\| &\leqslant \|\lambda R(\lambda, \Gamma_0^T) w(u)\| + \|\lambda R(\lambda, \Gamma_0^T)\| \cdot \|p(u_\lambda) - w(u)\| \\ &\leqslant \|\lambda R(\lambda, \Gamma_0^T) w(u)\| + \|p(u_\lambda) - w(u)\|.\end{aligned}$$

Then $u_\lambda(T) \to u_T$ as $\lambda \to 0^+$. Therefore, the nonlocal fractional control system (1)–(2) is approximately controllable on $[0, T]$. Hence, the proof of the theorem is completed. □

4 Conclusion

Very few works are available in the literature, which deal with solvability and approximate controllability of nonlocal differential inclusion involving fractional derivative utilizing measure of noncompactness. Evolution inclusions of fractional order are committed to a quickly developing area of the examination for inclusions and their applications to control theory. Both linear and nonlinear differential inclusions can describe many phenomena investigated in hybrid systems with dry friction, processes of controlled heat transfer, obstacle problems and others. This work studied the approximate controllability of nonlocal neutral differential inclusion of fractional order $q \in (1, 2)$ utilizing sectorial operator, multivalued fixed point strategy and measure of noncompactness under the assumption that the corresponding linear system is approximately controllable. The approximate controllability of neutral fractional differential inclusion with nonlocal and impulsive conditions will be investigated in our future work.

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