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Nonlocal q-fractional boundary value problem with Stieltjes integral conditions*

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Abstract. In this paper, we are dedicated to investigating a new class of one-dimensional lower-order fractional q-differential equations involving integral boundary conditions supplemented with Stieltjes integral. This condition is more general as it contains an arbitrary order derivative. It should be pointed out that the problem discussed in the current setting provides further insight into the research on nonlocal and integral boundary value problems. We first give the Green's functions of the boundary value problem and then develop some properties of the Green's functions that are conductive to our main results. Our main aim is to present two results: one considering the uniqueness of nontrivial solutions is given by virtue of contraction mapping principle associated with properties of u_0-positive linear operator in which Lipschitz constant is associated with the first eigenvalue corresponding to related linear operator, while the other one aims to obtain the existence of multiple positive solutions under some appropriate conditions via standard fixed point theorems due to Krasnoselskii and Leggett–Williams. Finally, we give an example to illustrate the main results.

Keywords: fractional q-difference equations, existence and uniqueness, integral boundary conditions.

1 Introduction

In this paper, we discuss the existence of unique solution and multiple positive solutions for the following fractional q-differential equation:

$$D_q^\alpha u(t) + f\bigl(t, u(t)\bigr) = 0, \quad t \in [0,1], \tag{1}$$

with nonlocal boundary conditions

$$u(0) = D_q^{\alpha-2} u(0) = 0, \qquad D_q^{\alpha-1} u(1) = \alpha[u] + \int_0^\xi \phi(t) D_q^\beta u(t)\,\mathrm{d}_q t, \tag{2}$$

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where D_q^α is the standard Riemann–Liouville fractional q-derivative of order α, $2<\alpha\leqslant 3$, $\alpha-1-\beta>0$, $0<q<1$, $\phi\in L^1[0,1]$ is nonnegative, $\alpha[u]$ is a linear functional given by

$$\alpha[u]=\int_0^1 u(t)\,\mathrm{d}A(t)$$

involving the Stieltjes integral with respect to the function $A:[0,1]\to\mathbb{R}$, $A(t)$ is right-continuous on $[0,1)$, left-continuous at $t=1$. Particularly, A is nondecreasing function with $A(0)=0$, then $\mathrm{d}A$ is positive Stieltjes measure.

Fractional calculus occurs naturally in various fields of technical sciences and physical engineering, details of recent development can be found in [15], whereas for the background and relevant theory of fractional calculus, we refer to [15, 21]. Since then, in order to describe and simulate the pragmatic phenomena with mathematic tools more accurately, the fractional q-integrals and q-derivatives arise at the historic moment.

Fractional q-calculus, initially proposed by Jackson [12, 13], is regarded as the fractional analogue of q-calculus. Soon afterwards, it has further promotion by Al-Salam [4] and Agarwal [1], where many outstanding theoretical results were given. Its emergence and development extended the application of interdisciplinary to be further and aroused widespread attention of the scholars; see [2, 19, 22, 23] and references therein.

More recently, in connection with broad research on the mathematical modeling of systems, the description of hereditary properties of various materials and the optimal control theory, it has become necessary to investigate boundary value problems of fractional differential equations as the nonlocal characteristics of the corresponding fractional-order operators [5, 6, 8–10, 18, 20, 24–27, 29, 31, 32]. Moreover, these equations are always completely controllable, meanwhile the research on fractional q-differential equation boundary value problems is in a stage of rapid development; one can see [2, 3, 7, 14, 18, 23, 29, 30, 33] and the references cited therein.

In consideration of the fact that the existence of unique solutions or multiple solutions of integral boundary value problem have been widely considered in recent decades as it focused on building diversiform fractional-order models, for more recent results, we can see [3, 10, 19, 20, 23, 24, 26–28, 33]. For example, Zhao et al. [33] discussed equation (1) under integral boundary condition

$$u(0)=D_q u(0)=0,\qquad u(1)=\mu\int_0^1 u(s)\,\mathrm{d}_q s,$$

here $f:[0,1]\times\mathbb{R}^+\to\mathbb{R}$ is semipositone and may be singular at $u=0$. By using fixed point index theorem, sufficient conditions for the existence of at least two and three positive solutions are obtained. As is known to all, initial values play a key role in the study of the classical boundary value problems (for example, Dirichlet boundary value problem, Neumann boundary value problem, periodic boundary value problem). In simulating special features of physical mathematics, biochemistry or some complicated and changeable process happening in diverse positions, nonlocal conditions are of great important. However, this cannot be possible implemented by initial/boundary value conditions. For remarkable results, we refer to [2, 7, 10, 33].

Owing to the limitations of end-point conditions, in the study of computational fluid dynamics associated with the problem of blood flow, it is not always reasonable to assume that the cross section of blood vessel is circular. To this end, integral conditions are found to be more suitable for analysing this problem and attract more attention. Ahmad et al. [3] considered a sub-strip-type boundary condition of the form

$$x(\xi) = b \int_{\eta}^{1} x(s)\,\mathrm{d}_q s, \quad 0 < \xi < \eta < 1,$$

and the flux sub-strip condition

$$D_q x(\xi) = b \int_{\eta}^{1} x(s)\,\mathrm{d}_q s, \quad 0 < \xi < \eta < 1.$$

Such conditions are more plausible to explain some physical phenomena, the uniqueness of solution is established by a fixed point theorem due to O'Regan in [34]. Interestingly, Liu et al. [19] considered the Riemann–Stieltjes integral boundary conditions involving fractional derivatives

$$D_{0^+}^{\beta} u(1) = \int_{0}^{1} l(t) D_{0^+}^{\beta_n - 1} u(t)\,\mathrm{d}A(t), \quad 0 < t < 1,$$

here $\alpha \in (n-1, n]$, $\beta_i \in (i-1, i]$, $i = 1, 2, \dots, n-1$, $l \in L^1(0,1)$, f may be singular at $x_0 = x_1 = \cdots = x_{n-1} = 0$. By using the spectral analysis of the relevant linear operator and Gelfand's formula, the existence of positive solutions for boundary conditions involving fractional derivatives is obtained. Min et al. [20] concerned with integral boundary conditions

$$D_{0^+}^{\beta_1} u(1) = \int_{0}^{\eta} h(s) D_{0^+}^{\beta_2} u(s)\,\mathrm{d}A(s) + \int_{0}^{1} a(s) D_{0^+}^{\beta_3} u(s)\,\mathrm{d}A(s), \quad 0 < t < 1,$$

here $D_{0^+}^{\beta_i} u$ denote the standard Riemann–Liouville derivatives, and $\beta_1 \geqslant \beta_2$, $\beta_1 \geqslant \beta_3$, $\alpha - \beta_i > 1$, $a, h \in C(0,1)$. By using fixed point theorem of mixed monotone operators, the uniqueness of positive solutions is derived.

Inspired by the previous works, different from [19, 20], we consider problem (1)–(2) in which the boundary conditions involve the Stieltjes integral, β is an arbitrary order derivative, hence (2) is more general boundary condition. Up to now, this case of nonlocal boundary conditions for fractional q-differential equations is relatively rare to be done. In addition, the main difficulty in studying fractional q-differential equations is the calculation of Green's functions, which produce more complexities than in integer order case. Therefore, in Section 2, we obtain the Green's function and relevant properties that are propitious to our main results. In Section 3, relying on the Green's functions and properties of u_0-positive linear operator, Lipschitz constant is associated with the first eigenvalues corresponding to related linear operator that guarantee the uniqueness of solutions of problem (1)–(2). Moreover, the existence of multiple positive solutions is also enunciated by employing fixed point theorems due to Krasnoselskii and Leggett–Williams in Section 4. The main results are well applied with the aid of an example.

Throughout this paper, we always assume that

(H1) $f : [0,1] \times \mathbb{R} \to \mathbb{R}^+$ is continuous, and $f(t,0) \not\equiv 0$, $t \in [0,1]$;

(H2) $\Delta := \Gamma_q(\alpha-\beta) - \int_0^\xi \phi(t) t^{\alpha-1-\beta}\,\mathrm{d}_q t - (\Gamma_q(\alpha-\beta)/\Gamma_q(\alpha))\alpha[t^{\alpha-1}] > 0$, where $\alpha[t^{\alpha-1}] > 0$, and $\alpha[1] > 0$.

2 Auxiliary results

For the convenience of readers, we recall some useful definitions and lemmas, which can be found in [1, 4, 12, 13, 22].

Definition 1. (See [22].) Let $\alpha \geqslant 0$ and f be a function defined on $[0,1]$. The fractional q-integral of Riemann–Liouville type is $I_q^0 f(t) = f(t)$, and

$$I_q^\alpha f(t) = \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} f(s)\,\mathrm{d}_q s, \quad \alpha > 0.$$

Note that $I_q^\alpha f(t) = I_q f(t)$ when $\alpha = 1$.

Definition 2. (See [22].) The fractional q-derivative of Riemann–Liouville type of order $\alpha \geqslant 0$ is defined by

$$D_q^\alpha f(t) = D_q^{\lceil\alpha\rceil} I_q^{\lceil\alpha\rceil-\alpha} f(t), \quad t \in [0,1],$$

where $\lceil\alpha\rceil$ is the smallest integer greater than or equal to α.

Further analysis showed that

$$I_q^\alpha D_q^p f(t) = D_q^p I_q^\alpha f(t) - \sum_{n=0}^{p-1} \frac{t^{\alpha-p+n}}{\Gamma_q(\alpha-p+n+1)} D_q^n f(0), \quad p \in \mathbb{N}. \tag{3}$$

Lemma 1. (*See* [30].) *Suppose that $f(t)$ is a continuous function on $[0,1]$, and there exists $t_0 \in (0,1)$ such that $f(t_0) \neq 0$. If $f(t) \geqslant 0$, then $\int_0^1 f(t)\,\mathrm{d}_q t > 0, t \in [0,1]$, where*

$$\int_0^1 f(t)\,\mathrm{d}_q t = (1-q)\sum_{n=0}^\infty q^n f(q^n), \quad 0 < q < 1.$$

Lemma 2. *Assume $f \in C[0,1]$ and $\Delta > 0$, then the fractional q-differential equation*

$$D_q^\alpha u(t) + g(t) = 0, \quad \alpha \in (2,3],\ t \in [0,1],$$

$$u(0) = D_q^{\alpha-2} u(0) = 0, \qquad D_q^{\alpha-1} u(1) = \int_0^1 u(t)\,\mathrm{d}A(t) + \int_0^\xi \phi(t) D_q^\beta u(t)\,\mathrm{d}_q t$$

has a solution

$$u(t) = \int_0^1 G(t,qs) g(s)\,\mathrm{d}_q s, \quad t \in [0,1],$$

where

$$G(t,qs) = G_1(t,qs) + G_2(t,qs) + G_3(t,qs),$$

$$G_1(t,qs) = \begin{cases} \frac{t^{\alpha-1} - (t-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)}, & 0 \leqslant qs \leqslant t \leqslant 1, \\ \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}, & 0 \leqslant t \leqslant qs \leqslant 1, \end{cases}$$

$$G_2(t,qs) = \frac{t^{\alpha-1}}{\Delta} \int_0^\xi \phi(t) H(t,qs)\,\mathrm{d}_q t,$$

$$G_3(t,qs) = \frac{\Gamma_q(\alpha-\beta) t^{\alpha-1}}{\Delta \Gamma_q(\alpha)} \int_0^1 G_1(t,qs)\,\mathrm{d}A(t),$$

$$H(t,qs) = \begin{cases} \frac{t^{\alpha-1-\beta} - (t-qs)^{(\alpha-1-\beta)}}{\Gamma_q(\alpha)}, & 0 \leqslant qs \leqslant t \leqslant 1, \\ \frac{t^{\alpha-1-\beta}}{\Gamma_q(\alpha)}, & 0 \leqslant t \leqslant qs \leqslant 1. \end{cases}$$

Proof. Let us consider $p = 3$, then from Definitions 1, 2 and (3), the above problem can be changed into

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} - I_q^\alpha g(t)$$

for some $c_1, c_2, c_3 \in \mathbb{R}$. By the condition $u(0) = 0$, we find that $c_3 = 0$. Now considering the condition $D_q^{\alpha-2} u(0) = 0$, we get

$$\begin{aligned} D_q^{\alpha-2} u(t) &= c_1 D_q^{\alpha-2} t^{\alpha-1} + c_2 D_q^{\alpha-2} t^{\alpha-2} - D_q^{\alpha-2} I_q^\alpha g(t) \\ &= c_1 \frac{\Gamma_q(\alpha)}{\Gamma_q(2)} t + c_2 \frac{\Gamma_q(\alpha-1)}{\Gamma_q(1)} - I_q^2 g(t), \end{aligned}$$

in view of $I_q^2 g(t) \to 0$ as $t \to 0$, we must set $c_2 = 0$. Then we have

$$u(t) = c_1 t^{\alpha-1} - \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} g(s)\,\mathrm{d}_q s. \tag{4}$$

Further, we obtain

$$D_q^{\alpha-1}u(t) = c_1 D_q^{\alpha-1} t^{\alpha-1} - D_q^{\alpha-1} I_q^{\alpha} g(t) = c_1\Gamma_q(\alpha) - \int_0^t g(s)\,\mathrm{d}_q s \tag{5}$$

and

$$D_q^{\beta}u(t) = c_1\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}t^{\alpha-1-\beta} - \frac{1}{\Gamma_q(\alpha-\beta)}\int_0^t (t-qs)^{(\alpha-1-\beta)}g(s)\,\mathrm{d}_q s. \tag{6}$$

Then we obtain

$$\begin{aligned}
&\int_0^{\xi} \phi(t) D_q^{\beta}u(t)\,\mathrm{d}_q t + \alpha[u] \\
&\quad = c_1\frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}\int_0^{\xi}\phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t + c_1\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t) \\
&\qquad - \frac{1}{\Gamma_q(\alpha-\beta)}\int_0^{\xi}\phi(t)\int_0^t (t-qs)^{(\alpha-1-\beta)}g(s)\,\mathrm{d}_q s\,\mathrm{d}_q t \\
&\qquad - \frac{1}{\Gamma_q(\alpha)}\int_0^1\int_0^t (t-qs)^{(\alpha-1)}g(s)\,\mathrm{d}_q s\,\mathrm{d}A(t).
\end{aligned}$$

From (5) and (6) we have

$$\begin{aligned}
c_1 &= \frac{\int_0^1 g(s)\,\mathrm{d}_q s}{\Gamma_q(\alpha) - \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha-\beta)}\int_0^{\xi}\phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)} \\
&\quad - \frac{\int_0^{\xi}\phi(t)\int_0^t (t-qs)^{(\alpha-1-\beta)}g(s)\,\mathrm{d}_q s\,\mathrm{d}_q t}{\Gamma_q(\alpha)\Gamma_q(\alpha-\beta) - \Gamma_q(\alpha)\int_0^{\xi}\phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \Gamma_q(\alpha-\beta)\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)} \\
&\quad - \frac{\int_0^1\int_0^t (t-qs)^{(\alpha-1)}g(s)\,\mathrm{d}_q s\,\mathrm{d}A(t)}{\Gamma_q^2(\alpha) - \frac{\Gamma_q^2(\alpha)}{\Gamma_q(\alpha-\beta)}\int_0^{\xi}\phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \Gamma_q(\alpha)\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)}.
\end{aligned} \tag{7}$$

Thus, substituting (7) into (4), we deduce that

$$\begin{aligned}
u(t) &= -\frac{t^{\alpha-1}\int_0^{\xi}\phi(t)\int_0^t (t-qs)^{(\alpha-1-\beta)}g(s)\,\mathrm{d}_q s\,\mathrm{d}_q t}{\Gamma_q(\alpha)\Gamma_q(\alpha-\beta) - \Gamma_q(\alpha)\int_0^{\xi}\phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \Gamma_q(\alpha-\beta)\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)} \\
&\quad - \frac{t^{\alpha-1}\int_0^1\int_0^t (t-qs)^{(\alpha-1)}g(s)\,\mathrm{d}_q s\,\mathrm{d}A(t)}{\Gamma_q^2(\alpha) - \frac{\Gamma_q^2(\alpha)}{\Gamma_q(\alpha-\beta)}\int_0^{\xi}\phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \Gamma_q(\alpha)\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{t^{\alpha-1}\int_0^1 g(s)\,\mathrm{d}_q s}{\Gamma_q(\alpha)\Gamma_q(\alpha-\beta) - \Gamma_q(\alpha)\int_0^\xi \phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \Gamma_q(\alpha-\beta)\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)} \\
&\times \Biggl(\Gamma_q(\alpha-\beta) - \int_0^\xi \phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t) \\
&+ \int_0^\xi \phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t + \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)\Biggr) - \frac{\int_0^t (t-qs)^{(\alpha-1)}g(s)\,\mathrm{d}_q s}{\Gamma_q(\alpha)} \\
= {}& \frac{t^{\alpha-1}}{\Gamma_q(\alpha-\beta) - \int_0^\xi \phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)} \\
&\times \frac{1}{\Gamma_q(\alpha)}\Biggl[\int_0^\xi\int_0^1 \phi(t)t^{\alpha-1-\beta}g(s)\,\mathrm{d}_q s\,\mathrm{d}_q t - \int_0^\xi\int_0^t \phi(t)(t-qs)^{(\alpha-1-\beta)}g(s)\,\mathrm{d}_q s\,\mathrm{d}_q t\Biggr] \\
&+ \frac{\Gamma_q(\alpha-\beta)t^{\alpha-1}}{\Gamma_q(\alpha)\Gamma_q(\alpha-\beta) - \Gamma_q(\alpha)\int_0^\xi \phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \Gamma_q(\alpha-\beta)\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)} \\
&\times \frac{1}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}\int_0^1 g(s)\,\mathrm{d}_q s\,\mathrm{d}A(t) \\
&- \frac{\Gamma_q(\alpha-\beta)t^{\alpha-1}}{\Gamma_q^2(\alpha)\Gamma_q(\alpha-\beta) - \Gamma_q^2(\alpha)\int_0^\xi \phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \Gamma_q(\alpha)\Gamma_q(\alpha-\beta)\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)} \\
&\times \int_0^1\int_0^t (t-qs)^{(\alpha-1)}g(s)\,\mathrm{d}_q s\,\mathrm{d}A(t) + \int_0^1 G_1(t,qs)g(s)\,\mathrm{d}_q s \\
= {}& \frac{t^{\alpha-1}}{\Gamma_q(\alpha-\beta) - \int_0^\xi \phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)} \\
&\times \frac{1}{\Gamma_q(\alpha)}\Biggl[\int_0^\xi\int_0^t \phi(t)\bigl[t^{\alpha-1-\beta} - (t-qs)^{(\alpha-1-\beta)}\bigr]g(s)\,\mathrm{d}_q s\,\mathrm{d}_q t \\
&+ \int_0^\xi\int_t^1 \phi(t)t^{\alpha-1-\beta}g(s)\,\mathrm{d}_q s\,\mathrm{d}_q t\Biggr] \\
&+ \frac{\Gamma_q(\alpha-\beta)t^{\alpha-1}}{\Gamma_q(\alpha)\Gamma_q(\alpha-\beta) - \Gamma_q(\alpha)\int_0^\xi \phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \Gamma_q(\alpha-\beta)\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t)} \\
&\times \frac{1}{\Gamma_q(\alpha)}\Biggl[\int_0^1\int_0^t \bigl(t^{\alpha-1} - (t-qs)^{(\alpha-1)}\bigr)g(s)\,\mathrm{d}_q s\,\mathrm{d}A(t)
\end{aligned}
$$

$$+\int_0^1\int_t^1 t^{\alpha-1}g(s)\,\mathrm{d}_q s\,\mathrm{d}A(t)\Bigg]+\int_0^1 G_1(t,qs)g(s)\,\mathrm{d}_q s$$
$$=\frac{t^{\alpha-1}}{\Delta}\int_0^\xi \phi(t)\int_0^1 H(t,qs)g(s)\,\mathrm{d}_q s\,\mathrm{d}_q t+\int_0^1 G_1(t,qs)g(s)\,\mathrm{d}_q s$$
$$+\frac{\Gamma_q(\alpha-\beta)t^{\alpha-1}}{\Delta\Gamma_q(\alpha)}\int_0^1\int_0^1 G_1(t,qs)g(s)\,\mathrm{d}_q s\,\mathrm{d}A(t)$$
$$=\int_0^1\big(G_1(t,qs)+G_2(t,qs)+G_3(t,qs)\big)g(s)\,\mathrm{d}_q s=\int_0^1 G(t,qs)g(s)\,\mathrm{d}_q s.$$

The proof of this lemma is finished. □

Lemma 3. *The Green's functions $G_1(t,qs)$, $H(t,qs)$ and $G(t,qs)$ have the following properties*:

(i) $G_1(t,qs)>0\quad \forall t,s\in(0,1)$,
$$G_1(t,qs)\leqslant\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\leqslant\frac{1}{\Gamma_q(\alpha)}\quad \forall t,s\in[0,1];$$

(ii) $t^{\alpha-1}G_1(1,qs)\leqslant G_1(t,qs)\leqslant G_1(1,qs)\quad \forall t,s\in[0,1]$,
$t^{\alpha-1-\beta}H(1,qs)\leqslant H(t,qs)\leqslant H(1,qs)\quad \forall t,s\in[0,1]$;

(iii) $$\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}K(s)\leqslant G(t,qs)\leqslant\frac{t^{\alpha-1}}{\Gamma_q(\alpha)}(K_1+K_2)\quad \forall t,s\in[0,1],$$
here $K(s)=\big[1-(1-qs)^{(\alpha-1)}\big]K_1+\big[1-(1-qs)^{(\alpha-1-\beta)}\big]K_2$,
$$K_1=1+\frac{\Gamma_q(\alpha-\beta)\alpha[t^{\alpha-1}]}{\Delta\Gamma_q(\alpha)},\qquad K_2=\frac{\int_0^\xi\phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t}{\Delta};$$

(iv) $\Gamma_q(\alpha)G(t,qs)\leqslant\widehat{K}(s)+K_2\quad \forall t,s\in[0,1]$,
$$\textit{here } \widehat{K}(s)=\big[(\alpha-1)s+1-(1-qs)^{(\alpha-1)}\big]\left(1+\frac{\Gamma_q(\alpha-\beta)\alpha[1]}{\Delta\Gamma_q(\alpha)}\right).$$

Proof. Obviously, condition (i) holds.

(ii) We start by defining two functions
$$g_1(t,qs)=t^{\alpha-1}-(t-qs)^{(\alpha-1)},\quad 0\leqslant qs\leqslant t\leqslant 1,$$
$$g_2(t,qs)=t^{\alpha-1},\quad 0\leqslant t\leqslant qs\leqslant 1.$$

For fixed $s\in[0,1]$,
$${}_tD_q g_1(t,qs)=[\alpha-1]_q t^{\alpha-2}-[\alpha-1]_q t^{\alpha-2}\left(1-\frac{qs}{t}\right)^{(\alpha-2)}\geqslant 0,$$

that is, $g_1(t,qs)$ is an increasing function in t. Similarly, $g_2(t,qs)$ is also increasing in t. Then we have, $G_1(t,qs) \leqslant G_1(1,qs)$. For $qs \leqslant t$, one has

$$\begin{aligned} G_1(t,qs) &= \frac{1}{\Gamma_q(\alpha)}\left[t^{\alpha-1} - t^{\alpha-1}\left(1-\frac{qs}{t}\right)^{(\alpha-1)}\right] \\ &\geqslant \frac{1}{\Gamma_q(\alpha)}\left[t^{\alpha-1}\left(1-(1-qs)^{(\alpha-1)}\right)\right] = t^{\alpha-1}G_1(1,qs). \end{aligned}$$

For $qs \geqslant t$, one has

$$G_1(t,qs) = \frac{1}{\Gamma_q(\alpha)}t^{\alpha-1} \geqslant t^{\alpha-1}G_1(1,qs).$$

Similarly, for $t,s \in [0,1]$, the result $t^{\alpha-1-\beta}H(1,qs) \leqslant H(t,qs) \leqslant H(1,qs)$ is also true.

(iii) By means of (i), (ii), we obtain

$$\begin{aligned} G(t,qs) &= G_1(t,qs) + G_2(t,qs) + G_3(t,qs) \\ &\leqslant \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} + \frac{t^{\alpha-1}}{\Delta\Gamma_q(\alpha)}\int_0^{\xi} \phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t + \frac{\Gamma_q(\alpha-\beta)t^{\alpha-1}}{\Delta\Gamma_q^2(\alpha)}\int_0^1 t^{\alpha-1}\,\mathrm{d}A(t) \\ &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\left(1 + \frac{\Gamma_q(\alpha-\beta)\alpha[t^{\alpha-1}]}{\Delta\Gamma_q(\alpha)} + \frac{1}{\Delta}\int_0^{\xi}\phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t\right) \\ &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}(K_1+K_2). \end{aligned}$$

Moreover, we have

$$\begin{aligned} G(t,qs) &= G_1(t,qs) + G_2(t,qs) + G_3(t,qs) \\ &\geqslant t^{\alpha-1}G_1(1,qs) + \frac{t^{\alpha-1}}{\Delta}\int_0^{\xi}\phi(t)t^{\alpha-1-\beta}H(1,qs)\,\mathrm{d}_q t \\ &\quad + \frac{\Gamma_q(\alpha-\beta)t^{\alpha-1}}{\Delta\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}G_1(1,qs)\,\mathrm{d}A(t) \\ &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}\left[\left(1-(1-qs)^{(\alpha-1)}\right) + \frac{1-(1-qs)^{(\alpha-1-\beta)}}{\Delta}\int_0^{\xi}\phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t \right. \\ &\quad \left. + \frac{\Gamma_q(\alpha-\beta)}{\Delta\Gamma_q(\alpha)}\int_0^1 t^{\alpha-1}\left(1-(1-qs)^{(\alpha-1)}\right)\mathrm{d}A(t)\right] \\ &= \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}K(s). \end{aligned}$$

(iv) From (iii) we have

$$\Gamma_q(\alpha)G(t,qs) \leqslant (\alpha-1)s+1-(1-qs)^{(\alpha-1)} + \frac{1}{\Delta}\int_0^{\xi} \phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t + \frac{\Gamma_q(\alpha-\beta)}{\Delta\Gamma_q(\alpha)}\int_0^1 \bigl[(\alpha-1)s+1-(1-qs)^{(\alpha-1)}\bigr]\,\mathrm{d}A(t)$$
$$= \bigl[(\alpha-1)s+1-(1-qs)^{(\alpha-1)}\bigr]\biggl(1+\frac{\Gamma_q(\alpha-\beta)\alpha[1]}{\Delta\Gamma_q(\alpha)}\biggr) + K_2$$
$$= \widehat{K}(s) + K_2.$$

The proof of this lemma is finished. □

Definition 3. (See [16].) Let E be a real Banach space and $P \subset E$ be a cone. A bounded linear operator $S : E \to E$ is called u_0-positive on P if for every nonzero $u \in P$, a natural number $n = n(x)$ and two positive number α_0, β_0 can be found such that $\alpha_0 u_0 \leqslant S^n u \leqslant \beta_0 u_0$.

Lemma 4. (*See* [16].) *Suppose that $S : C[0,1] \to C[0,1]$ is a completely continuous linear operator and $SP \subset P$. If there exist $\psi \in C[0,1] \setminus (-P)$ and a constant $c > 0$ such that $cS\psi \geqslant \psi$, then the spectral radius $r(S) \neq 0$ and S has a positive eigenfunction φ corresponding to its first eigenvalue λ_1, i.e. $\varphi = \lambda_1 S\varphi$.*

In this paper, we work in $E = C[0,1]$, the Banach space endowed with the norm $\|u\| = \max\{|u(t)|\colon t \in [0,1]\}$. We consider the standard cone $P = \{u \in C[0,1]\colon u(t) \geqslant 0,\ t \in [0,1]\}$. Define operators $T, S : E \to E$ by

$$Tu(t) = \int_0^1 G(t,qs) f\bigl(s,u(s)\bigr)\,\mathrm{d}_q s, \quad t \in [0,1],$$
$$Su(t) = \int_0^1 G(t,qs) u(s)\,\mathrm{d}_q s, \quad t \in [0,1].$$

Lemma 5. *If* (H1), (H2) *hold, then $T : P \to P$ is a completely continuous operator.*

Proof. By standard method, it can be easily admitted, so we omit it. □

Remark 1. It is not hard to verify that $S : P \to P$ is a completely continuous linear operator. The existence of solutions of problem (1)–(2) is equivalent to the existence of fixed points of S on E. In reality, a real number λ is an eigenvalue of the operator S if there exists $u \in E \setminus \{\theta\}$ such that $Su = \lambda u$. Now, let $r(S)$ be the spectral radius of the operator S, $\lambda_1 = (r(S))^{-1}$ be the first eigenvalue of S.

Lemma 6. *Suppose that* (H1), (H2) *hold, the operator S is a u_0-positive linear operator with $u_0(t) = t^{\alpha-1}$. For the operator S, we have $r(S) \neq 0$, and S has a positive eigenfunction φ_0 corresponding to its first eigenvalue λ_1.*

Proof. From (iii) of Lemma 1, for each $u \in P \setminus \{\theta\}$, we can obtain

$$Su(t) = \int_0^1 G(t, qs)u(s)\, \mathrm{d}_q s \leqslant \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (K_1 + K_2)u(s)\, \mathrm{d}_q s$$

and

$$Su(t) = \int_0^1 G(t, qs)u(s)\, \mathrm{d}_q s \geqslant \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 K(s)u(s)\, \mathrm{d}_q s.$$

From Definition 3 we know that S is a u_0-positive linear operator in which $u_0(t) = t^{\alpha-1}$. Next, we can choose

$$c = \Gamma_q(\alpha)\left(\int_0^1 K(s)s^{\alpha-1}\, \mathrm{d}_q s\right)^{-1} > 0$$

such that $cS\psi \geqslant \psi$, here $\psi = t^{\alpha-1}$. By Lemma 4, we know that $r(S) \neq 0$ and S has a positive eigenfunction φ_0 corresponding to its first eigenvalue λ_1. □

Remark 2. Let S be a u_0-positive linear operator, and φ_0 be a positive eigenfunction of S. Then S is also a φ_0-positive operator [11]. Without loss of generality, in this paper, we assume $\|\varphi_0\| = 1$. Moreover, if φ_0 be a positive eigenfunction corresponding to λ_1, then $S\varphi_0 = r(S)\varphi_0$. From Lemma 6 and Definition 3 we have

$$\alpha_{\varphi_0} t^{\alpha-1} = \alpha_{\varphi_0} u_0 \leqslant S\varphi_0(t) = r(S)\varphi_0(t), \quad \text{i.e.} \quad t^{\alpha-1} \leqslant \frac{r(S)}{\alpha_{\varphi_0}}\varphi_0(t).$$

3 Unique solution

In this section, we are in a position to present the existence of unique solution for problem (1)–(2).

Theorem 1. *Suppose that conditions* (H1), (H2) *hold and $f(t, u)$ satisfies*

$$\bigl|f(t, u) - f(t, v)\bigr| \leqslant a(t)|u - v| \quad \forall t \in [0, 1],\ u, v \in \mathbb{R}, \tag{8}$$

with

$$\int_0^1 (K_1 + K_2)a(s)s^{\alpha-1}\, \mathrm{d}_q s < \Gamma_q(\alpha). \tag{9}$$

Then problem (1)–(2) *has a unique nontrivial solution.*

Proof. First, for n large enough, we shall show that S^n is a contraction operator. By means of Lemma 3 and (8), for any $u, v \in \mathbb{R}$, we have

$$\begin{aligned}
\big|Su(t) - Sv(t)\big| &\leqslant \int_0^1 G(t, qs)\big|f\big(s, u(s)\big) - f\big(s, v(s)\big)\big| \,\mathrm{d}_q s \\
&\leqslant \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (K_1 + K_2)a(s)\big|u(s) - v(s)\big| \,\mathrm{d}_q s \\
&\leqslant \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 (K_1 + K_2)a(s) \,\mathrm{d}_q s \cdot \|u - v\|.
\end{aligned}$$

Then we have

$$\big|Su(t) - Sv(t)\big| \leqslant \frac{t^{\alpha-1}}{\Gamma_q(\alpha)}(K_1 + K_2)\Lambda\|u - v\|, \quad t \in [0, 1], \tag{10}$$

here $\Lambda = \int_0^1 a(s) \,\mathrm{d}_q s$. Next, considering (8) and (10), one has

$$\begin{aligned}
\big|S^2u(t) - S^2v(t)\big| &\leqslant \int_0^1 G(t, qs)\big|f\big(s, Su(s)\big) - f\big(s, Sv(s)\big)\big| \,\mathrm{d}_q s \\
&\leqslant \frac{(K_1 + K_2)t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 a(s)\big|Su(s) - Sv(s)\big| \,\mathrm{d}_q s \\
&\leqslant \frac{(K_1 + K_2)^2 t^{\alpha-1}}{\Gamma_q^2(\alpha)} \int_0^1 s^{\alpha-1}a(s)\Lambda \,\mathrm{d}_q s \cdot \|u - v\| \\
&= \left(\frac{K_1 + K_2}{\Gamma_q(\alpha)}\right)^2 \Lambda^* \Lambda t^{\alpha-1}\|u - v\|,
\end{aligned}$$

here $\Lambda^* = \int_0^1 s^{\alpha-1}a(s) \,\mathrm{d}_q s$. Furthermore, by induction, we obtain

$$\big|S^n u(t) - S^n v(t)\big| \leqslant \frac{(K_1 + K_2)}{\Gamma_q(\alpha)} \cdot \left[\frac{(K_1 + K_2)\Lambda^*}{\Gamma_q(\alpha)}\right]^{n-1} \Lambda t^{\alpha-1}\|u - v\|, \quad t \in [0, 1],$$

which implies

$$\big\|S^n u(t) - S^n v(t)\big\| \leqslant \left[\frac{(K_1 + K_2)\Lambda^*}{\Gamma_q(\alpha)}\right]^{n-1} \cdot \frac{(K_1 + K_2)\Lambda t^{\alpha-1}}{\Gamma_q(\alpha)}\|u - v\|.$$

According to (9), we have

$$\lim_{n\to+\infty} \left[\frac{(K_1 + K_2)\Lambda^*}{\Gamma_q(\alpha)}\right]^{n-1} = 0,$$

it shows that there exists a sufficiently large $N > 0$ such that

$$\left\|S^N u - S^N v\right\| \leqslant \frac{1}{8}\|u - v\|,$$

thus S^N has one fixed point in E, and further, we can easily get that S has one fixed point in E. From condition (H1) we know that problem (1)–(2) has a unique nontrivial solution by using Banach's contraction principle. □

If $a(t) \equiv A$, then condition (8) reduces to the Lipschitz condition. Then we obtain unique results as follows:

Theorem 2. *Suppose that conditions* (H1), (H2) *hold, and there exists* $A > 0$ *such that*

$$\left|f(t,u) - f(t,v)\right| \leqslant A|u - v| \quad \forall t \in [0,1],\ u, v \in \mathbb{R}, \tag{11}$$

with $A < \lambda_1$*, then problem* (1)–(2) *has a unique nontrivial solution* u^**. Moreover, the iterative sequence* $u_n = Tu_{n-1}$ $(n = 1, 2, \dots)$ *converges to* u^* *with initial* $u_0 \in E$*. In addition, there exists a constant* M *satisfies the error estimate*

$$\|u_n - u^*\| \leqslant M\lambda_1 \frac{(\frac{A}{\lambda_1})^n}{1 - \frac{A}{\lambda_1}}. \tag{12}$$

Proof. Firstly, by Lemma 5, $T : P \to P$ is completely continuous. We know that problem (1)–(2) has a unique solution if and only if T has a unique fixed point in E. First, for any given $u_0 \in E$, we construct a sequence $u_n = Tu_{n-1}$, $n = 1, 2, \dots$. We see that the iterative sequence $\{u_n\} \subset E$. If $u_1 = u_0$, i.e. $Tu_0 = u_0$, then u_0 is a solution of problem (1)–(2). If $u_1 \neq u_0$, then $|u_1 - u_0| \in P \setminus \{\theta\}$. From Lemma 6 and Remark 2 there exists $M = M(|u_1 - u_0|)$ such that

$$S\big(|u_1 - u_0|\big)(t) \leqslant (M\varphi_0)(t), \quad t \in [0,1]. \tag{13}$$

By induction, for any $n \in N$, $t \in [0,1]$, it follows from (11) and (13) that

$$\begin{aligned}
\left|u_{n+1}(t) - u_n(t)\right| &= \left|Tu_n(t) - Tu_{n-1}(t)\right| \\
&\leqslant \int_0^1 G(t,qs)\left|f\big(s,u_n(s)\big) - f\big(s,u_{n-1}(s)\big)\right| \mathrm{d}_q s \\
&\leqslant AS\big(|u_n - u_{n-1}|\big)(t) \leqslant \cdots \leqslant A^n S^n\big(|u_1 - u_0|\big)(t) \\
&\leqslant A^n S^{n-1}(M\varphi_0)(t) = MA^n S^{n-1}\varphi_0(t) \\
&= MA^n \lambda_1^{1-n}\varphi_0(t).
\end{aligned}$$

Thus for $n, p \in N$, one has

$$\begin{aligned}
\left|u_{n+p}(t) - u_n(t)\right| \leqslant{}& \left|u_{n+p}(t) - u_{n+p-1}(t)\right| + \left|u_{n+p-1}(t) - u_{n+p-2}(t)\right| \\
&+ \cdots + \left|u_{n+1}(t) - u_n(t)\right|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant M\left(A^{n+p-1}\lambda_1^{2-n-p} + A^{n+p-2}\lambda_1^{3-n-p} + \cdots + A^n\lambda_1^{1-n}\right)\varphi_0(t)\\
&= M\varphi_0(t)\frac{A^n\lambda_1^{1-n}[1-(\frac{A}{\lambda_1})^p]}{1-\frac{A}{\lambda_1}} = M\lambda_1\varphi_0(t)\frac{(\frac{A}{\lambda_1})^n[1-(\frac{A}{\lambda_1})^p]}{1-\frac{A}{\lambda_1}}.
\end{aligned}
$$

Noting that $\|\varphi_0\| = 1$, we obtain

$$
\|u_{n+p} - u_n\| \leqslant M\lambda_1\varphi_0(t)\frac{(\frac{A}{\lambda_1})^n[1-(\frac{A}{\lambda_1})^p]}{1-\frac{A}{\lambda_1}} \to 0, \quad n, p \to +\infty.
$$

By the completeness of E and $A/\lambda_1 \in (0,1)$, we know that there exists $u^* \in E$ such that $\lim_{n\to+\infty} u_n = u^*$. Taking limit on $u_n = Tu_{n-1}$, it shows that u^* is a fixed point of T.

Next, we prove uniqueness. Let u_* be other fixed point of T, that is, $Tu_* = u_*$. From Lemma 3 there exists $M(u^*, u_*) = M(|u^* - u_*|) > 0$ such that $S(|u^* - u_*|) \leqslant M(|u^* - u_*|)\varphi_0$. Similarly, we have

$$
\left|u^*(t) - u_*(t)\right| \leqslant M\lambda_1\left(\frac{A}{\lambda_1}\right)^n\varphi_0(t) \to 0 \quad \text{as } n \to +\infty. \tag{14}
$$

Then problem (1)–(2) has a unique nontrivial solution. Taking limit on (14) as $p \to \infty$, the error estimate formula (12) holds. □

Remark 3. Condition (11) can be changed into $|f(t,u) - f(t,v)| \leqslant A\lambda_1|u - v|$, $A \in [0,1)$, λ_1 is the first eigenvalue of S, then the result of Theorem 2 is still hold, and we also get the error estimate

$$
\|u_n - u^*\| \leqslant M\lambda_1\frac{A^n}{1-A}.
$$

Corollary 1. *Suppose that conditions* (H1), (H2) *and* (11) *hold. If* $Ar(S) < 1$*, then problem* (1)–(2) *has a unique solution in* E_1*, where*

$$
E_1 = \left\{u \in E\colon \sup_{t\in[0,1]}\frac{|u(t)|}{\varphi_0(t)} < +\infty\right\} \quad \textit{with the norm } \|u\|_1 = \sup_{t\in[0,1]}\frac{|u(t)|}{\varphi_0(t)}. \tag{15}
$$

Proof. For $u \in E$, by Lemma 3 and Remark2, we get

$$
\begin{aligned}
\left|Tu(t)\right| &\leqslant \int_0^1 G(t,qs)\left|f\left(s,u(s)\right) - f(s,0)\right|\mathrm{d}_q s + \int_0^1 G(t,qs)f(s,0)\,\mathrm{d}_q s\\
&\leqslant \frac{(K_1+K_2)t^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^1 A\left|u(s)\right|\mathrm{d}_q s\\
&\leqslant \frac{(K_1+K_2)r(S)}{\alpha_0\Gamma_q(\alpha)}\int_0^1 A\left|u(s)\right|\mathrm{d}_q s\cdot\varphi_0(t).
\end{aligned} \tag{16}
$$

In view of (15), (16), T maps E into E_1. Now we consider fixed points of T in E_1. Notice that $S\varphi_0 = r(S)\varphi_0$, namely,

$$r(S)\varphi_0(t) = \int_0^1 G(t, qs)\varphi_0(s)\,\mathrm{d}_q s.$$

Let $u_1, u_2 \in E$, we have

$$\begin{aligned}
\left|Tu_1(t) - Tu_2(t)\right| &\leqslant \int_0^1 G(t, qs)\left|f\left(s, u_1(s)\right) - f\left(s, u_2(s)\right)\right| \mathrm{d}_q s \\
&\leqslant \int_0^1 G(t, qs)A\left|u_1(s) - u_2(s)\right| \mathrm{d}_q s \\
&\leqslant A\int_0^1 G(t, qs)A\varphi_0(s)\,\mathrm{d}_q s \cdot \|u_1 - u_2\|_1 \\
&= A\|u_1 - u_2\|_1 r(S)\varphi_0(t).
\end{aligned} \tag{17}$$

Inequality (17) implies that

$$\|Tu_1 - Tu_2\|_1 \leqslant Ar(S)\|u_1 - u_2\|_1,$$

since $Ar(S) < 1$, the operator T is a contraction. Then from (H1) problem (1)–(2) has a unique nontrivial solution in E_1. $\square$

Remark 4. Under the Lipschitz condition (11), the Lipschitz constant is closely associated with the first eigenvalue of the relevant u_0-positive linear operator. Moreover, we know that the basic space considered in Corollary 1 is E_1, not E. So we only need to replace the restrict $r(S) < 1$ by $\|S\| < 1$, here $\|S\| = sup_{u\in E}\|Su\|/\|u\|$. By means of Gelfand's Formula, we have

$$r(S) = \lim_{n\to\infty} \sqrt[n]{\|S^n\|} \leqslant \|S\|,$$

then Corollary 1 follows directly in E rely on the Banach's contraction principle.

4 Multiple positive solutions

In this section, we first give two lemmas that will be used to show the existence of multiple positive solutions. For the forthcoming analysis, given $\delta \in (0, 1/2)$, denote

$$l = \min_{t\in[\delta,1-\delta]} t^{\alpha-1}, \qquad \Theta_1 = \frac{\Gamma_q(\alpha)}{\int_0^1 (\widehat{K}(s) + K_2)\,\mathrm{d}_q s}, \qquad \Theta_2 = \frac{\Gamma_q(\alpha)}{l\int_\delta^{1-\delta} K(s)\,\mathrm{d}_q s}.$$

Our results rely on fixed point theorems due to Krasnoselskii's [16] and Leggett–Williams [17]. Now we give the following assumption:

(H1′) $f \in C([0,1] \times \mathbb{R}^+, \mathbb{R}^+)$, $f(t, 0) \not\equiv 0$, $t \in [0, 1]$.

Lemma 7. *Let E be a Banach space, $P \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open disks contained in E with $\Omega_1 \subset \Omega_2$. Let $T : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(i) $\|Tx\| \leqslant \|x\|$ *for any* $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \geqslant \|x\|$ *for any* $x \in P \cap \partial\Omega_2$, *or*
(ii) $\|Tx\| \geqslant \|x\|$ *for any* $x \in P \cap \partial\Omega_1$ *and* $\|Tx\| \leqslant \|x\|$ *for any* $x \in P \cap \partial\Omega_2$.

Then T has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Lemma 8. *Let P be a cone of a Banach space E, $P_c = \{x \in P\colon \|x\| \leqslant c\}$, θ is a nonnegative continuous concave function on P such that $\theta(x) \leqslant \|x\|$ for any $x \in \bar{P}_c$ and $P(\theta, b, d) = \{x \in P\colon b \leqslant \theta(x),\ \|x\| \leqslant d\}$. Assume that $T : \bar{P}_c \to \bar{P}_c$ is completely continuous, and there exist constants $a < b < d \leqslant c$ such that*

(i) $\{x \in P(\theta, b, d)\colon \theta(x) > b\} \neq \emptyset$ *and* $\theta(Tx) > b$, $x \in P(\theta, b, d)$;
(ii) $\|Tx\| < a$ *for* $x \in \bar{P}_a$;
(iii) $\theta(Tx) > b$ *for any* $x \in P(\theta, b, c)$ *with* $\|Tx\| > d$.

Then T has at least three fixed points x_1, x_2, x_3 with $\|x_1\| < a, b < \theta(x_2)$, $\|x_3\| > a$ and $\theta(x_3) < b$.

Theorem 3. *Suppose that* (H1′), (H2) *hold, and there exist two constants $r_2 > r_1 > 0$ such that*

(H3) $f(t, u) \geqslant \Theta_2 r_1$, $(t, u) \in [0, 1] \times [0, r_1]$;
(H4) $f(t, u) \leqslant \Theta_1 r_2$, $(t, u) \in [0, 1] \times [0, r_2]$.

Then problem (1)–(2) *has at least one positive solution u with $r_1 \leqslant \|u\| \leqslant r_2$.*

Proof. From Lemma 5, $T : P \to P$ is completely continuous. Let $\Omega_1 = \{u \in P\colon \|u\| < r_1\}$. For $u \in P \cap \partial\Omega_1$, $t \in [0, 1]$, we have $0 \leqslant u(t) \leqslant r_1$. Then from Lemma 3 and (H3) we obtain

$$\begin{aligned} Tu(t) &= \int_0^1 G(t, qs) f\big(s, u(s)\big)\, \mathrm{d}_q s \geqslant \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 K(s) f\big(s, u(s)\big)\, \mathrm{d}_q s \\ &\geqslant \frac{\min_{t\in[\delta,1-\delta]} t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 \Theta_2 r_1 K(s)\, \mathrm{d}_q s \geqslant \frac{\Theta_2 r_1 l}{\Gamma_q(\alpha)} \int_\delta^{1-\delta} K(s)\, \mathrm{d}_q s \\ &= r_1 = \|u\|, \end{aligned}$$

which means that $\|Tu\| \geqslant \|u\|$ for $u \in P \cap \partial\Omega_1$.

Let $\Omega_2 = \{u \in P\colon \|u\| < r_2\}$. For $u \in P \cap \partial\Omega_2$, $t \in [0, 1]$, one has $0 \leqslant u(t) \leqslant r_2$. Then from Lemma 3 and (H4) we obtain

$$\begin{aligned} Tu(t) &\leqslant \frac{1}{\Gamma_q(\alpha)} \int_0^1 \big(\widehat{K}(s) + K_2\big) f\big(s, u(s)\big)\, \mathrm{d}_q s \leqslant \frac{\Theta_1 r_2}{\Gamma_q(\alpha)} \int_0^1 \big(\widehat{K}(s) + K_2\big)\, \mathrm{d}_q s \\ &= r_2 = \|u\|, \end{aligned}$$

which means that $\|Tu\| \leqslant \|u\|$ for $u \in P \cap \partial\Omega_2$. By means of Lemma 7 and (H1′), we know that problem (1)–(2) has at least one positive solution. □

Theorem 4. *Suppose that* (H1′), (H2) *hold, and there exist four positive constants* a, b, c, d *with* $0 < a < b < (lK/(K_1 + K_2))d < d < c$ *such that*

(H5) $f(t,u) < \Theta_1 a,\ (t,u) \in [0,1] \times [0,a]$;
(H6) $f(t,u) > \Theta_2 b,\ (t,u) \in [\delta, 1-\delta] \times [b,d]$;
(H7) $f(t,u) \leqslant \Theta_1 c,\ (t,u) \in [0,1] \times [0,c]$.

Then problem (1)–(2) *has at least three positive solutions* u_1, u_2, u_3 *with*

$$\max_{t\in[0,1]} u_1(t) < a, b < \min_{t\in[\delta,1-\delta]} u_2(t) < \max_{t\in[0,1]} u_2(t) \leqslant c,$$

$$a < \max_{t\in[0,1]} u_3(t) \leqslant c, \min_{t\in[\delta,1-\delta]} u_3(t) < b.$$

Proof. First, we show that operator $T : \bar{P}_c \to \bar{P}_c$. For any $u \in \bar{P}_c$, we have $\|u\| \leqslant c$. By using Lemma 3(iv) and (H7), one has

$$\begin{aligned}\|Tu\| &\leqslant \frac{1}{\Gamma_q(\alpha)} \int_0^1 \big(\widehat{K}(s) + K_2\big) f\big(s, u(s)\big)\, \mathrm{d}_q s \\ &\leqslant \frac{\Theta_1 c}{\Gamma_q(\alpha)} \int_0^1 \big(\widehat{K}(s) + K_2\big)\, \mathrm{d}_q s = c,\end{aligned}$$

thus $T(\bar{P}_c) \subset \bar{P}_c$. From Lemma 5 it is easily to show that $T : \bar{P}_c \to \bar{P}_c$ is completely continuous. Similarly, let $u \in \bar{P}_a$, it follows from (H5) that $\|Tu\| < a$, then condition (ii) of Lemma 8 is fulfilled.

In order to justify condition (i), define a nonnegative continuous concave function $\theta(u) = \min_{t\in[\delta,1-\delta]} u(t)$ and let $u(t) = (b + d)/2$, $K = \min_{t\in[0,1]} K(s)$. We can easily get $u \in P(\theta, b, d)$ and $\{u \in P(\theta, b, d) | \theta(u) > b\} \neq \emptyset$. If $u \in P(\theta, b, d)$, then $u(t) \in [b, d]$ for any $t \in [\delta, 1-\delta]$. By condition (H6), we have

$$\begin{aligned}\theta(Tu) = \min_{t\in[\delta,1-\delta]} Tu(t) &= \min_{t\in[\delta,1-\delta]} \int_0^1 G(t, qs) f\big(s, u(s)\big)\, \mathrm{d}_q s \\ &\geqslant \frac{\min_{t\in[\delta,1-\delta]} t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 K(s) f\big(s, u(s)\big)\, \mathrm{d}_q s \\ &> \frac{lb\Theta_2}{\Gamma_q(\alpha)} \int_\delta^{1-\delta} K(s)\, \mathrm{d}_q s = b,\end{aligned}$$

then condition (i) is satisfied.

Next, for $u \in P(\theta, b, c)$ with $\|Tu\| > d$, we get

$$\|Tu\| \leqslant \frac{K_1 + K_2}{\Gamma_q(\alpha)} \int_0^1 f\big(s, u(s)\big)\, \mathrm{d}_q s,$$

thus

$$\begin{aligned} Tu(t) &\geqslant \frac{t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 K(s) f\big(s, u(s)\big)\, \mathrm{d}_q s \geqslant \frac{K t^{\alpha-1}}{\Gamma_q(\alpha)} \int_0^1 f\big(s, u(s)\big)\, \mathrm{d}_q s \\ &\geqslant \frac{K t^{\alpha-1}}{K_1 + K_2} \|Tu\|. \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \theta(Tu) &= \min_{t \in [\delta, 1-\delta]} Tu(t) \geqslant \min_{t \in [\delta, 1-\delta]} t^{\alpha-1} \frac{K}{K_1 + K_2} \|Tu\| \\ &\geqslant \frac{lK}{K_1 + K_2} d > b, \end{aligned}$$

then condition (iii) is satisfied. From Lemma 8 we know that T has at least three fixed points, which means that problem (1)–(2) has at least three positive solutions u_1, u_2, u_3 such that

$$\max_{t \in [0,1]} u_1(t) < a, b < \min_{t \in [\delta, 1-\delta]} u_2(t) < \max_{t \in [0,1]} u_2(t) \leqslant c,$$
$$a < \max_{t \in [0,1]} u_3(t) \leqslant c, \min_{t \in [\delta, 1-\delta]} u_3(t) < b. \qquad \square$$

5 An example

Consider the boundary value problem

$$\begin{aligned} &D_q^{5/2} u(t) + \frac{\ln(1+u)}{1 + t^{3/2}} + h(t) = 0, \quad t \in [0,1], \\ &D_q u(0) = D_q^{1/2} u(0) = 0, \qquad D_q^{3/2} u(1) = \alpha[u] + \int_0^1 t^{5/2} D_q^{1/2} u(t)\, \mathrm{d}_q t, \end{aligned} \tag{18}$$

where $q = 1/2$, $\alpha = 5/2$, $\beta = 1/2$, $\xi = 1$, $\phi(t) = t^{5/2} \geqslant 0$, $h \in C[0,1]$, $h(t) \geqslant 0$, $h(t) \not\equiv 0$ for $t \in [0,1]$. Moreover, $a(t) = 1/(1 + t^{3/2})$, $\alpha[u] = u(1/4)/23$, and

$$f(t, u) = \frac{\ln(1+u)}{1 + t^{3/2}} + h(t), \quad t \in [0,1].$$

By simple computations, we find that

$$\alpha[1] = \frac{1}{23} > 0, \qquad \alpha\big[t^{\alpha-1}\big] = \alpha\big[t^{3/2}\big] = \frac{1}{184} > 0, \quad t \in [0,1],$$

$$
\begin{aligned}
\Delta &= \Gamma_q(\alpha-\beta) - \int_0^{\xi} \phi(t)t^{\alpha-1-\beta}\,\mathrm{d}_q t - \frac{\Gamma_q(\alpha-\beta)}{\Gamma_q(\alpha)}\alpha[t^{\alpha-1}] \\
&= 1 - \int_0^1 t^{5/2}\cdot t d_q t - \frac{\alpha[t^{3/2}]}{\Gamma_q(\frac{5}{2})} = \frac{8\sqrt{2}-1}{16\sqrt{2}-1} - \frac{1}{184\Gamma_q(\frac{5}{2})} \\
&\approx 0.4728 > 0.
\end{aligned}
$$

Then assumptions (H1), (H2) hold. Moreover, we get

$$
\begin{aligned}
\bigl|f(t,u)-f(t,v)\bigr| &= \left|\frac{\ln(1+u)}{1+t^{3/2}} - \frac{\ln(1+v)}{1+t^{3/2}}\right| \\
&\leqslant \frac{1}{1+t^{3/2}}|u-v|, \quad t\in[0,1],
\end{aligned}
$$

$$
f(t,0) = h(t) \not\equiv 0, \quad t\in[0,1],
$$

$$
\begin{aligned}
K_1 &= 1 + \frac{\Gamma_q(\alpha-\beta)\alpha[t^{\alpha-1}]}{\Delta\Gamma_q(\alpha)} = 1 + \frac{\alpha[t^{3/2}]}{\Delta\Gamma_q(5/2)} \\
&= 1 + \frac{1}{184\Gamma_q(\frac{5}{2})\frac{8\sqrt{2}-1}{16\sqrt{2}-1} - 1} \approx 1.0086,
\end{aligned}
$$

and

$$
K_2 = \frac{1}{\Delta}\int_0^1 t^{\frac{7}{2}}\,\mathrm{d}_q t = \frac{8\sqrt{2}}{\Delta(16\sqrt{2}-1)} \approx 0.0978.
$$

Further, we can obtain

$$
\begin{aligned}
&\int_0^1 (K_1+K_2)a(s)s^{\alpha-1}\,\mathrm{d}_q s \\
&\quad = (K_1+K_2)\int_0^1 \frac{s^{3/2}}{1+\frac{3}{2}}\,\mathrm{d}_q s \leqslant (K_1+K_2)\int_0^1 1\,\mathrm{d}_q s \approx 1.106 < \Gamma_q\left(\frac{5}{2}\right),
\end{aligned}
$$

and thus all the conditions of Theorem 1 are satisfied, so problem (18) has a unique nontrivial solution.

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