Nonlinear Analysis: Modelling and Control, 2018, Vol. 23, No. 6, 866–888 ISSN 1392-5113
https://doi.org/10.15388/NA.2018.6.4

Time periodic boundary value Stokes problem in a domain with an outlet to infinity*

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Received: February 1, 2018 / **Revised:** October 2, 2018 / **Published online:** October 31, 2018

Abstract. We prove the existence of a unique weak solution to the time periodic nonhomogeneous boundary value Stokes problem in a domain having an outlet to infinity.

Keywords: nonhomogeneous boundary value, time periodic, Stokes problem, unbounded domain.

1 Introduction

The Stokes and stationary Navier–Stokes equations with homogeneous boundary condition were intensively studied in domains with outlets to infinity during the last 40 years (see [2, 3, 18, 19, 29, 30] and the literature cited there). In the last 10 years, the special attention was given to problems with nonhomogeneous boundary conditions (see [1, 4–6, 23–28]). Moreover, recently a big progress was obtained in Leray's problem in bounded and exterior domains [8–14]. On the other hand, the time periodic problem for the Navier–Stokes equations was mainly studied only in the case of homogeneous boundary conditions (see, for example, [15, 20, 21]). The time periodic problems with nonhomogeneous boundary conditions were essentially considered by H. Morimoto [22] and T. Kobayashi [7]. However, they investigated the problem only in domains with compact boundaries. A wide review and study of periodic problems could be found in the habilitation thesis of M. Kyed [16].

In this paper, we consider the time periodic Stokes system with nonhomogeneous boundary condition

$$\begin{aligned}
&\mathbf{u}(x,t) - \nu\Delta\mathbf{u}(x,t) + \nabla p(x,t) = \mathbf{f}(x,t), \quad (x,t) \in \Omega \times (0,2\pi),\\
&\operatorname{div}\mathbf{u}(x,t) = 0, \quad (x,t) \in \Omega \times (0,2\pi),\\
&\mathbf{u}(x,t) = \boldsymbol{\varphi}(x), \quad (x,t) \in \partial\Omega \times (0,2\pi),\\
&\mathbf{u}(x,0) = \mathbf{u}(x,2\pi), \quad x \in \Omega,
\end{aligned} \tag{1}$$

*This research was supported by grant No. S-MIP-17-68 from the Research Council of Lithuania.

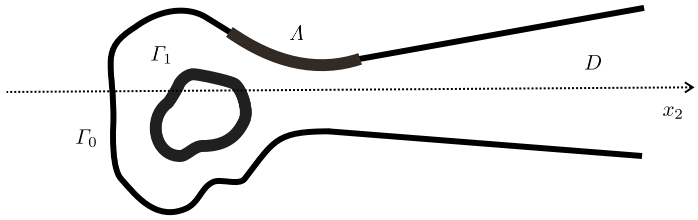

Figure 1. Domain Ω.

in a two dimensional multiply connected unbounded domain Ω. Here the vector valued function $\mathbf{u}(x,t)$ is the unknown velocity field, the scalar function $p(x,t)$ is the pressure of the fluid, while the vector valued functions $\boldsymbol{\varphi}(x)$ and $\mathbf{f}(x,t)$ denote the given boundary value and the external force, ν is the viscosity constant of the given fluid.

Let $\Omega \subset \mathbb{R}^2$ be a domain with an outlet to infinity (see Fig. 1). Then denote by $\Omega_0 = \Omega \cap B_{R_0}(0) = \Omega \cap \{x \in \mathbb{R}^2\colon |x| \leqslant R_0\}$ a bounded part of the domain Ω and by $D = \{x \in \mathbb{R}^2\colon |x_1| < g(x_2),\ x_2 > R_0\}$ an outlet to infinity. We suppose that function g satisfies the Lipschitz condition

$$\bigl|g(t_1) - g(t_2)\bigr| = L|t_1 - t_2|, \quad t_1, t_2 > R_0,\ g(t) \geqslant \mathrm{const} > 0$$

and $\partial\Omega \in C^2$. The boundary $\partial\Omega$ consists of the inner boundary Γ_1 and the outer boundary Γ_0. Notice that the inner boundary Γ_1 is compact, while the outer boundary Γ_0 is unbounded. We assume that boundary value $\boldsymbol{\varphi} \in W^{3/2,2}(\partial\Omega)$ has a compact support: $\operatorname{supp}\boldsymbol{\varphi} \subset \partial\Omega_0$. Denote $\Lambda = \operatorname{supp}\boldsymbol{\varphi} \cap \Gamma_0 \subset \Gamma_0 \cap B_{R_0}(0)$. Integrating by parts the divergence equation $\operatorname{div}\mathbf{u} = 0$ over the domain $\Omega \cap B_R(0)$ with sufficiently large R, we obtain

$$\begin{aligned} 0 &= \int\limits_{\Omega\cap B_R(0)} \operatorname{div}\mathbf{u}\,\mathrm{d}x = \int\limits_{\partial(\Omega\cap B_R(0))} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}x \\ &= \int\limits_{\Gamma_1} \boldsymbol{\varphi}\cdot\mathbf{n}\,\mathrm{d}S + \int\limits_{\Lambda} \boldsymbol{\varphi}\cdot\mathbf{n}\,\mathrm{d}S + \int\limits_{\sigma(R)} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S, \end{aligned}$$

where $\sigma(R) = (-g(R), g(R))$ is a cross-section of the outlet to infinity D with the vertical straight line parallel to x_1-axis and passing through the $(0,R)$-point.

Let $\mathcal{F}^{(\mathrm{inn})} = \int_{\Gamma_1} \boldsymbol{\varphi}\cdot\mathbf{n}\,\mathrm{d}S$ and $\mathcal{F}^{(\mathrm{out})} = \int_{\Lambda} \boldsymbol{\varphi}\cdot\mathbf{n}\,\mathrm{d}S$ be the fluxes of the boundary value $\boldsymbol{\varphi}$ over the inner and the outer boundary, respectively. Then

$$\int\limits_{\sigma(R)} \mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = -\bigl(\mathcal{F}^{(\mathrm{inn})} + \mathcal{F}^{(\mathrm{out})}\bigr).$$

This condition is natural, because we consider incompressible fluid.

In this paper, we prove the existence and uniqueness of a weak solution to problem (1) in a domain with an outlet to infinity Ω (see Fig. 1). Notice that this result is the first step to study the nonlinear time periodic Navier–Stokes problem in such domains.

2 Notation and preliminaries

Vector valued functions are denoted by bold letters, while function spaces for scalar and vector valued functions are denoted in the same way.

We use the symbols c, C, c_j, C_j, $j = 1, 2, \dots$, to denote constants whose numerical values are unessential to our considerations. In such case, c, C may have different values in single computations.

Let G be a domain in $\mathbb{R}^n$. As usual, $C^\infty(G)$ denotes the set of all infinitely differentiable functions defined on Ω, and $C_0^\infty(G)$ is the subset of all functions from $C^\infty(G)$ having compact supports in Ω. For a given nonnegative integer k and $q > 1$, $L^q(\Omega)$ and $W^{k,q}(\Omega)$ indicate the usual Lebesgue and Sobolev spaces, while $W^{k-1/q,q}(\partial\Omega)$ is the trace space on $\partial\Omega$ of functions from $W^{k,q}(\Omega)$. Denote by $J_0^\infty(\Omega)$ the set of all solenoidal ($\operatorname{div}\mathbf{u} = 0$) vector fields $\mathbf{u}$ from $C_0^\infty(\Omega)$. By $H(\Omega)$ we indicate the space formed as the closure of $J_0^\infty(\Omega)$ in the Dirichlet norm $\|\mathbf{u}\|_{H(\Omega)} = \|\nabla\mathbf{u}\|_{L^2(\Omega)}$ generated by the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v} \,\mathrm{d}x,$$

where $\nabla\mathbf{u} : \nabla\mathbf{v} = \sum_{j=1}^n \nabla u_j \cdot \nabla v_j = \sum_{j=1}^n \sum_{k=1}^n (\partial u_j/\partial x_k)(\partial v_j/\partial x_k)$.

Definition 1. By a weak solution of problem (1) we understand a solenoidal vector field $\mathbf{u}$ with $\nabla\mathbf{u}, \mathbf{u}_t \in L^2(0, 2\pi; L^2(\Omega))$ satisfying the boundary condition $\mathbf{u}|_{\partial\Omega} = \boldsymbol{\varphi}$, the time periodicity condition $\mathbf{u}(x, 0) = \mathbf{u}(x, 2\pi)$ and the integral identity

$$\int_0^{2\pi}\int_\Omega \mathbf{u}_t \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t + \nu \int_0^{2\pi}\int_\Omega \nabla\mathbf{u} : \nabla\boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t = \int_0^{2\pi}\int_\Omega \mathbf{f} \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t$$

for all $\boldsymbol{\eta} \in L^2(0, 2\pi; J_0^\infty(\Omega))$, where $J_0^\infty(\Omega) = \{\mathbf{w} \in C_0^\infty(\Omega)\colon \operatorname{div}\mathbf{w} = 0\}$.

Later, we will use the notion of the regularized distance.

Lemma 1. (*See* [31].) *Let $\mathcal{M}$ be a closed set in $\mathbb{R}^2$. Denote by $\Delta_\mathcal{M}(x)$ the regularized distance from the point x to the set $\mathcal{M}$. Function $\Delta_\mathcal{M}(x)$ is infinitely differentiable in $\mathbb{R}^2 \setminus \mathcal{M}$, and the following estimates*

$$a_1 d_\mathcal{M}(x) \leqslant \Delta_\mathcal{M}(x) \leqslant a_2 d_\mathcal{M}(x), \qquad \left|D^\alpha \Delta_\mathcal{M}(x)\right| \leqslant a_3 d_\mathcal{M}^{1-|\alpha|}(x), \tag{2}$$

hold, where $d_G(x) = \operatorname{dist}(x, G)$ is the distance from x to $\mathcal{M}$, positive constants a_1, a_2 and a_3 are independent of $\mathcal{M}$.

3 Construction of the extension of the boundary value

We start with the construction of a suitable extension $\mathbf{A}$ of the boundary value $\boldsymbol{\varphi}$. Then we can reduce a nonhomogeneous condition to the homogeneous one. Since the boundary value φ is independent of time, the extension of the boundary value could be constructed using the similar ideas as in [5]. Additionally, we need to estimate the term $\|\Delta\mathbf{A}\|$. We construct the extension $\mathbf{A}$ in the following form:

$$\mathbf{A}(x) = \mathbf{B}^{(\mathrm{inn})}(x) + \mathbf{B}^{(\mathrm{out})}(x),$$

where $\mathbf{B}^{(\mathrm{inn})}$ extends the boundary value φ from the inner boundary Γ_1, and $\mathbf{B}^{(\mathrm{out})}$ extends φ from the outer boundary Γ_0.

3.1 Construction of the extension $\mathbf{B}^{(\mathrm{inn})}$

First, we construct a vector field $\mathbf{b}^{(\mathrm{inn})}$ such that

$$\operatorname{div}\mathbf{b}^{(\mathrm{inn})} = 0, \qquad \mathbf{b}^{(\mathrm{inn})}\big|_{\partial D\cap\partial\Omega} = 0, \qquad \int\limits_{\sigma(R)} \mathbf{b}^{(\mathrm{inn})}\cdot\mathbf{n}\,\mathrm{d}S = \mathcal{F}^{(\mathrm{inn})}.$$

Let Δ_{γ_+} and $\Delta_{\partial D\cap\partial\Omega}$ be the regularized distances from a point $x \in D$ to the line $\gamma_+ = \{x\colon x_1 = 0,\, x_2 > R_0\}$ and the boundary $\partial D\cap\partial\Omega$, respectively. Define in D a Hopf's-type cut-off function

$$\xi(x) = \Psi\biggl(\ln\frac{\varrho(\Delta_{\gamma_+}(x))}{\Delta_{\partial D\cap\partial\Omega}(x)}\biggr),$$

where Ψ is a smooth monotone function, $0 \leqslant \Psi \leqslant 1$,

$$\Psi(t) = \begin{cases} 0, & t \leqslant 0,\\ 1, & t \geqslant 1, \end{cases} \tag{3}$$

$\varrho(\tau)$ is smooth monotone function

$$\varrho(\tau) = \begin{cases} \frac{a_1}{2}d_0, & \tau \leqslant \frac{a_2}{2}d_0,\\ \tau, & \tau \geqslant a_2 d_0, \end{cases} \tag{4}$$

where d_0 is a positive number such that $\operatorname{dist}(\gamma_+,\, \partial D\cap\partial\Omega) \geqslant d_0$, and a_1, a_2 are positive constants from the estimates of the regularized distance (see Lemma 1).

Lemma 2. *The function $\xi(x) = 0$ at those points of D where $\varrho(\Delta_{\gamma_+}(x)) \leqslant \Delta_{\partial D\cap\partial\Omega}(x)$, while the $d_0/2$-neighborhood of the line γ_+ is contained in this set*; *$\xi(x) = 1$ at those points of D where $\Delta_{\partial D\cap\partial\Omega}(x) \leqslant \mathrm{e}^{-1}\varrho(\Delta_{\gamma_+}(x))$. The following estimates hold*:

$$\biggl|\frac{\partial\xi(x)}{\partial x_k}\biggr| \leqslant \frac{c}{\Delta_{\partial D\cap\partial\Omega}(x)}, \qquad \biggl|\frac{\partial^2\xi(x)}{\partial x_k\partial x_l}\biggr| \leqslant \frac{c}{\Delta^2_{\partial D\cap\partial\Omega}(x)},$$
$$\biggl|\frac{\partial^3\xi(x)}{\partial^2 x_k\partial x_l}\biggr| \leqslant \frac{c}{\Delta^3_{\partial D\cap\partial\Omega}(x)}.$$

Proof. The proof of the lemma follows directly from the definition of the function $\xi(x)$, properties of the regularized distance and the fact that $\operatorname{supp}\nabla\xi(x)$ is contained in the set where $\Delta_{\partial D\cap\partial\Omega}(x) \leqslant \varrho(\Delta_{\gamma_+}(x))$. □

Let us define the vector field

$$\mathbf{b}_1^{(\mathrm{inn})}(x) = -\mathcal{F}^{(\mathrm{inn})}\left(\frac{\partial\tilde{\xi}(x)}{\partial x_2}; -\frac{\partial\tilde{\xi}(x)}{\partial x_1}\right), \quad x \in D^+ = \{x \in D\colon x_1 > 0\}, \tag{5}$$

where

$$\tilde{\xi}(x) = \begin{cases} \xi(x), & x \in D^+, \\ 0, & x \in D \setminus D^+. \end{cases}$$

Lemma 3. *The solenoidal vector field $\mathbf{b}_1^{(\mathrm{inn})}(x)$ is infinitely differentiable, vanishes near the boundary $\partial D \cap \partial\Omega$ and the contour γ_+, the support of $\mathbf{b}_1^{(\mathrm{inn})}(x)$ is contained in the set of points $x \in D^+$ satisfying the inequalities*

$$\varrho\big(\Delta_{\gamma_+}(x)\big)\mathrm{e}^{-1} \leqslant \Delta_{\partial D\cap\partial\Omega}(x) \leqslant \varrho\big(\Delta_{\gamma_+}(x)\big). \tag{6}$$

Moreover,

$$\int\limits_{\sigma(R)} \mathbf{b}_1^{(\mathrm{inn})} \cdot \mathbf{n}\,\mathrm{d}S = \mathcal{F}^{(\mathrm{inn})}, \tag{7}$$

and the following inequalities hold:

$$\big|\mathbf{b}_1^{(\mathrm{inn})}(x)\big| \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{d(x)}, \quad x \in D^+,\ d(x) = \operatorname{dist}(x,\, \partial D \cap \partial\Omega), \tag{8}$$

$$\big|\mathbf{b}_1^{(\mathrm{inn})}(x)\big| \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g(x_2)}, \quad x \in D, \tag{9}$$

$$\big|\nabla\mathbf{b}_1^{(\mathrm{inn})}(x)\big| \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g^2(x_2)}, \quad \big|\Delta\mathbf{b}_1^{(\mathrm{inn})}(x)\big| \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g^3(x_2)}, \quad x \in D. \tag{10}$$

Proof. Relation (6) follows directly from Lemma 2.

By the construction of $\mathbf{b}_1^{(\mathrm{inn})}$ we easily show (7):

$$\begin{aligned}
\int\limits_{\sigma(R)} \mathbf{b}_1^{(\mathrm{inn})} \cdot \mathbf{n}\,\mathrm{d}S &= \int\limits_{-g(R)}^{g(R)} \mathbf{b}_1^{(\mathrm{inn})} \cdot \mathbf{n}\,\mathrm{d}S = -\mathcal{F}^{(\mathrm{inn})} \int\limits_{-g(R)}^{g(R)} \left(\frac{\partial\tilde{\xi}(x)}{\partial x_2}, -\frac{\partial\tilde{\xi}(x)}{\partial x_1}\right) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}x_1 \\
&= -\mathcal{F}^{(\mathrm{inn})} \int\limits_{-g(R)}^{g(R)} \left(-\frac{\partial\tilde{\xi}(x)}{\partial x_1}\right) \mathrm{d}x_1 = \mathcal{F}^{(\mathrm{inn})} \int\limits_{-g(R)}^{g(R)} \frac{\partial\tilde{\xi}(x)}{\partial x_1}\,\mathrm{d}x_1 \\
&= \mathcal{F}^{(\mathrm{inn})}\big(\tilde{\xi}\big(g(R), R\big) - \tilde{\xi}\big(-g(R), R\big)\big) = \mathcal{F}^{(\mathrm{inn})}.
\end{aligned}$$

According to the definition of $\mathbf{b}_1^{(\mathrm{inn})}(x)$ and Lemma 2, we obtain the following estimates:

$$\left|\mathbf{b}_1^{(\mathrm{inn})}(x)\right| \leqslant \left|\mathcal{F}^{(\mathrm{inn})}\right| \sqrt{\left(\frac{\partial\tilde{\xi}(x)}{\partial x_2}\right)^2 + \left(\frac{\partial\tilde{\xi}(x)}{\partial x_1}\right)^2} \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{\Delta_{\partial D\cap\partial\Omega}(x)}; \tag{11}$$

$$\left|\nabla\mathbf{b}_1^{(\mathrm{inn})}(x)\right| \leqslant \left|\mathcal{F}^{(\mathrm{inn})}\right| \sqrt{\left(\frac{\partial^2\tilde{\xi}(x)}{\partial x_1\partial x_2}\right)^2 + \left(\frac{\partial^2\tilde{\xi}(x)}{\partial x_2\partial x_1}\right)^2} \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{\Delta^2_{\partial D\cap\partial\Omega}(x)}; \tag{12}$$

$$\left|\Delta\mathbf{b}_1^{(\mathrm{inn})}(x)\right| \leqslant \left|\mathcal{F}^{(\mathrm{inn})}\right| \sqrt{\left(\frac{\partial^3\tilde{\xi}(x)}{\partial^2 x_1\partial x_2}\right)^2 + \left(\frac{\partial^3\tilde{\xi}(x)}{\partial^2 x_2\partial x_1}\right)^2} \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{\Delta^3_{\partial D\cap\partial\Omega}(x)}. \tag{13}$$

Due to estimates for the regularized distance (2), estimate (8) follows from (11). Notice that for points $x \in \operatorname{supp}\mathbf{b}_1^{(\mathrm{inn})}$ the inequalities

$$c_1 g(x_2) \leqslant d(x) \leqslant c_2 g(x_2)$$

hold, where c_1, c_2 are positive constants (see [30] for details). Then estimates (9), (10) follow from inequalities (11)–(13). □

Let us define on $\partial\Omega$ another vector field

$$\mathbf{h}_1(x) = \begin{cases} 0, & x \in \Gamma_1, \\ \mathbf{b}_1^{(\mathrm{inn})} + \mathbf{b}_{\#}^{(\mathrm{inn})}, & x \in \partial\Omega_0 \cap \partial D, \\ \mathbf{b}_{\#}^{(\mathrm{inn})}, & x \in \partial\Omega_0 \setminus (\Gamma_1 \cup (\partial\Omega_0 \cap \partial D)), \end{cases}$$

with $\mathbf{b}_1^{(\mathrm{inn})}$ given by (5) and $\mathbf{b}_{\#}^{(\mathrm{inn})}$ defined as following:

$$\mathbf{b}_{\#}^{(\mathrm{inn})}(x) = \mathcal{F}^{(\mathrm{inn})}\nabla q(x),$$

where $q(x) = -1/(2\pi)\ln|x|$ is a fundamental solution of the Laplace operator in $\mathbb{R}^2$.

Notice that $\mathbf{b}_{\#}^{(\mathrm{inn})}(x)$ is a solenoidal vector field:

$$\operatorname{div}\mathbf{b}_{\#}^{(\mathrm{inn})} = \operatorname{div}\mathcal{F}^{(\mathrm{inn})}\nabla q(x) = \mathcal{F}^{(\mathrm{inn})}\operatorname{div}\nabla q(x) = \mathcal{F}^{(\mathrm{inn})}\Delta q(x) = 0.$$

Since

$$\int\limits_{\Gamma_1} \nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = 1, \qquad \int\limits_{\partial\Omega_0\setminus\Gamma_1} \nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = -1,$$

we have that

$$\int\limits_{\Gamma_1} \mathbf{b}_{\#}^{(\mathrm{inn})}\cdot\mathbf{n}\,\mathrm{d}S = \int\limits_{\Gamma_1} \mathcal{F}^{(\mathrm{inn})}\nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = \mathcal{F}^{(\mathrm{inn})}\int\limits_{\Gamma_1} \nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = \mathcal{F}^{(\mathrm{inn})},$$

$$\int\limits_{\partial\Omega_0\setminus\Gamma_1} \mathbf{b}_{\#}^{(\mathrm{inn})}\cdot\mathbf{n}\,\mathrm{d}S = \int\limits_{\partial\Omega_0\setminus\Gamma_1} \mathcal{F}^{(\mathrm{inn})}\nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = \mathcal{F}^{(\mathrm{inn})}\int\limits_{\partial\Omega_0\setminus\Gamma_1}\nabla q(x)\cdot\mathbf{n}\,\mathrm{d}S = -\mathcal{F}^{(\mathrm{inn})}.$$

Then according to the properties of the vector fields $\mathbf{b}_1^{(\mathrm{inn})}$ and $\mathbf{b}_{\#}^{(\mathrm{inn})}$, we get

$$\int\limits_{\partial\Omega_0}\mathbf{h}_1\cdot\mathbf{n}\,\mathrm{d}S = \int\limits_{\partial\Omega_0\cap\partial D}\mathbf{b}_1^{(\mathrm{inn})}\cdot\mathbf{n}\,\mathrm{d}S + \int\limits_{\partial\Omega_0\setminus\Gamma_1}\mathbf{b}_{\#}^{(\mathrm{inn})}\cdot\mathbf{n}\,\mathrm{d}S = \mathcal{F}^{(\mathrm{inn})} - \mathcal{F}^{(\mathrm{inn})} = 0.$$

In order to extend $\mathbf{h}_1$ into Ω_0, first, we define the solenoidal vector field

$$\tilde{\mathbf{b}}_{01}^{(\mathrm{inn})} = \left(\frac{\partial\mathbf{H}(x)}{\partial x_2}, -\frac{\partial\mathbf{H}(x)}{\partial x_1}\right),$$

where $\mathbf{H}\in W^{2,3}(\Omega_0)$ satisfies the following boundary conditions:

$$\begin{aligned}
\frac{\partial\mathbf{H}(x)}{\partial x_2}\bigg|_{\partial\Omega_0\cap\partial D} &= \left(b_{11}^{(\mathrm{inn})} + b_{\#1}^{(\mathrm{inn})}\right)\bigg|_{\partial\Omega_0\cap\partial D},\\
-\frac{\partial\mathbf{H}(x)}{\partial x_1}\big|_{\partial\Omega_0\cap\partial D} &= \left(b_{12}^{(\mathrm{inn})} + b_{\#2}^{(\mathrm{inn})}\right)\big|_{\partial\Omega_0\cap\partial D},\\
\frac{\partial^2\mathbf{H}(x)}{\partial x_2^2}\bigg|_{\partial\Omega_0\cap\partial D} &= \left(\frac{\partial b_{11}^{(\mathrm{inn})}}{\partial x_2} + \frac{\partial b_{\#1}^{(\mathrm{inn})}}{\partial x_2}\right)\bigg|_{\partial\Omega_0\cap\partial D},\\
\left(\frac{\partial\mathbf{H}(x)}{\partial x_2}, -\frac{\partial\mathbf{H}(x)}{\partial x_1}\right)\bigg|_{\partial\Omega_0\setminus\Gamma_1\cup(\partial\Omega_0\cap\partial D)} &= \mathbf{b}_{\#}^{(\mathrm{inn})}\big|_{\partial\Omega_0\setminus\Gamma_1\cup(\partial\Omega_0\cap\partial D)}.
\end{aligned}$$

Then we extend $\mathbf{h}_1$ into Ω_0 in the form

$$\mathbf{b}_{01}^{(\mathrm{inn})}(x) = \left(\frac{\partial(\kappa(x)\mathbf{H}(x))}{\partial x_2}, -\frac{\partial(\kappa(x)\mathbf{H}(x))}{\partial x_1}\right),$$

where the support of Hopf's-type smooth cut-off function κ is contained in the neighborhood of $\Omega_0\setminus\Gamma_1$. Moreover, $\mathbf{b}_{01}^{(\mathrm{inn})}\in W^{2,2}(\Omega_0)$ and satisfies the following estimate:

$$\begin{aligned}
\left\|\mathbf{b}_{01}^{(\mathrm{inn})}\right\|_{W^{2,2}(\Omega_0)} &\leqslant c\|\mathbf{h}_1\|_{W^{3/2,2}(\partial\Omega_0)}\\
&\leqslant c\left(\left\|\mathbf{b}_{\#}^{(\mathrm{inn})}\right\|_{W^{3/2,2}(\partial\Omega_0\setminus\Gamma_1)} + \left\|\mathbf{b}_1^{(\mathrm{inn})}\right\|_{W^{3/2,2}(\partial\Omega_0\cap\partial D)}\right)\\
&\leqslant c\left|\mathcal{F}^{(\mathrm{inn})}\right|,
\end{aligned}$$

where the constant c depends only on the domain Ω_0 (see [17]).

Next, we define the vector field, which "removes" the non-zero flux from the inner boundary Γ_1:

$$\mathbf{b}^{(\mathrm{inn})} = \begin{cases} \mathbf{b}_{\#}^{(\mathrm{inn})} - \mathbf{b}_{01}^{(\mathrm{inn})}, & x \in \Omega_0, \\ \mathbf{b}_1^{(\mathrm{inn})}, & x \in D. \end{cases}$$

Notice that by construction the function $\mathbf{b}^{(\mathrm{inn})}$ and its derivatives $\partial \mathbf{b}^{(\mathrm{inn})}/\partial x_1$, $\partial \mathbf{b}^{(\mathrm{inn})}/\partial x_2$ have no jump discontinuity passing from Ω_0 to D. Therefore, $\mathbf{b}^{(\mathrm{inn})} \in W^{2,2}(\Omega)$. Then we define a vector field

$$\mathbf{h}_0 = \begin{cases} \boldsymbol{\varphi} - \mathbf{b}_{\#}^{(\mathrm{inn})}, & x \in \Gamma_1, \\ 0, & x \in \partial\Omega_0 \setminus \Gamma_1, \end{cases}$$

which satisfies the following condition:

$$\int\limits_{\Gamma_1} \mathbf{h}_0 \cdot \mathbf{n} \, \mathrm{d}S = \int\limits_{\Gamma_1} \boldsymbol{\varphi} \cdot \mathbf{n} \, \mathrm{d}S - \int\limits_{\Gamma_1} \mathbf{b}_{\#}^{(\mathrm{inn})} \cdot \mathbf{n} \, \mathrm{d}S = \mathcal{F}^{(\mathrm{inn})} - \mathcal{F}^{(\mathrm{inn})} = 0.$$

Therefore, the function $\mathbf{h}_0$ can be extended inside Ω in the form

$$\mathbf{b}_0^{(\mathrm{inn})}(x) = \left(\frac{\partial(\chi(x)\mathbf{E}(x))}{\partial x_2}, -\frac{\partial(\chi(x)\mathbf{E}(x))}{\partial x_1} \right),$$

where $\mathbf{E}(x) \in W^{2,2}(\Omega_0)$, $(\partial \mathbf{E}(x)/\partial x_2, -\partial \mathbf{E}(x)/\partial x_1) = \mathbf{h}_0$, the support of Hopf's-type smooth cut-off function χ is contained in the neighborhood of Γ_1 (see [17]).

Finally, we put

$$\mathbf{B}^{(\mathrm{inn})}(x) = \mathbf{b}^{(\mathrm{inn})}(x) + \mathbf{b}_0^{(\mathrm{inn})}(x).$$

The properties of the extension $\mathbf{B}^{(\mathrm{inn})}$ we formulate in the following lemma.

Lemma 4. *The vector field* $\mathbf{B}^{(\mathrm{inn})}$ *is solenoidal,* $\mathbf{B}^{(\mathrm{inn})}|_{\Gamma_1} = \boldsymbol{\varphi}|_{\Gamma_1}$, $\mathbf{B}^{(\mathrm{inn})}|_{\partial\Omega \setminus \Gamma_1} = 0$, $\mathbf{B}^{(\mathrm{inn})} \in W^{2,2}(\overline{\Omega})$ *and satisfies the following estimates*:

$$\left|\mathbf{B}^{(\mathrm{inn})}(x)\right| \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g(x_2)}, \quad x \in D,$$

$$\left|\nabla \mathbf{B}^{(\mathrm{inn})}(x)\right| \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g^2(x_2)}, \quad \left|\Delta \mathbf{B}^{(\mathrm{inn})}(x)\right| \leqslant \frac{c|\mathcal{F}^{(\mathrm{inn})}|}{g^3(x_2)}, \quad x \in D,$$

$$\left|\mathbf{B}^{(\mathrm{inn})}(x)\right| + \left|\nabla \mathbf{B}^{(\mathrm{inn})}(x)\right| + \left|\Delta \mathbf{B}^{(\mathrm{inn})}(x)\right| \leqslant c\left|\mathcal{F}^{(\mathrm{inn})}\right|, \quad x \in \Omega \setminus D.$$

3.2 Construction of the extension $\mathbf{B}^{(\mathrm{out})}$

Take any point $x^{(1)} \in \Lambda \subset \Gamma_0$. Let γ be a smooth simple curve, which intersects $\partial\Omega$ at the point $x^{(1)}$, and

$$\gamma = \hat{\gamma} \cup \gamma_0,$$

where $\hat{\gamma}$ is a semi-infinite line lying in D, γ_0 is a finite simple curve connecting $\hat{\gamma}$ and the point $x^{(1)}$. Assume that $\inf_{x\in\gamma,\, y\in\partial\Omega\setminus\Lambda} |x-y| \geqslant d_0$.

Define a Hopf's-type cut-off function

$$\zeta(x) = \Psi\left(\ln\frac{\varrho(\Delta_\gamma(x))}{\Delta_{\partial\Omega\setminus\Lambda}(x)}\right),$$

where functions Ψ and ϱ are defined by (3) and (4), respectively.

Lemma 5. *Fuction $\zeta(x) = 0$ if $\varrho(\Delta_\gamma(x)) \leqslant \Delta_{\partial\Omega\setminus\Lambda}(x)$, while the $d_0/2$-neighborhood of the curve is contained in this set. Function $\zeta(x) = 1$ at those points where $\Delta_{\partial\Omega\setminus\Lambda}(x) \leqslant \mathrm{e}^{-1}\varrho(\Delta_\gamma(x))$. Moreover, the following estimates hold*:

$$\left|\frac{\partial\zeta(x)}{\partial x_k}\right| \leqslant \frac{c}{\Delta_{\partial\Omega\setminus\Lambda}(x)}, \qquad \left|\frac{\partial^2\zeta(x)}{\partial x_k\partial x_l}\right| \leqslant \frac{c}{\Delta^2_{\partial\Omega\setminus\Lambda}(x)},$$
$$\left|\frac{\partial^3\zeta(x)}{\partial^2 x_k\partial x_l}\right| \leqslant \frac{c}{\Delta^3_{\partial\Omega\setminus\Lambda}(x)}.$$

Proof. The proof follows directly from the definition of $\zeta(x)$, properties of the regularized distance and the fact that $\operatorname{supp}\nabla\zeta(x)$ is contained in the set where $\Delta_{\partial\Omega\setminus\Lambda}(x) \leqslant \varrho(\Delta_\gamma(x))$. □

Let us introduce the vector field

$$\mathbf{b}^{(\mathrm{out})}(x) = \mathcal{F}^{(\mathrm{out})}\left(\frac{\partial\tilde{\zeta}(x)}{\partial x_2}; -\frac{\partial\tilde{\zeta}(x)}{\partial x_1}\right),$$

where $\tilde{\zeta}(x) = \zeta(x)$ above the curve γ, and $\tilde{\zeta}(x) = 0$ under the curve γ.

Lemma 6. *The vector field $\mathbf{b}^{(\mathrm{out})}$ is infinitely differentiable and solenoidal, vanishes near the set $\partial\Omega\setminus\Lambda$ and in a small neighborhood of the curve γ. The following estimates hold*:

$$\left|\mathbf{b}^{(\mathrm{out})}(x)\right| \leqslant \frac{c}{d_{\partial\Omega\setminus\Lambda}}, \quad x \in D, \tag{14}$$

$$\left|\nabla\mathbf{b}^{(\mathrm{out})}(x)\right| \leqslant \frac{c}{d^2_{\partial\Omega\setminus\Lambda}}, \quad \left|\Delta\mathbf{b}^{(\mathrm{out})}(x)\right| \leqslant \frac{c}{d^3_{\partial\Omega\setminus\Lambda}}, \quad x \in D, \tag{15}$$

$$\left|\mathbf{b}^{(\mathrm{out})}(x)\right| \leqslant \frac{c|\mathcal{F}^{(\mathrm{out})}|}{g(x_2)}, \quad x \in D, \tag{16}$$

$$\left|\nabla\mathbf{b}^{(\mathrm{out})}(x)\right| \leqslant \frac{c|\mathcal{F}^{(\mathrm{out})}|}{g^2(x_2)}, \quad \left|\Delta\mathbf{b}^{(\mathrm{out})}(x)\right| \leqslant \frac{c|\mathcal{F}^{(\mathrm{out})}|}{g^3(x_2)}, \quad x \in D, \tag{17}$$

$$\int\limits_{\Lambda} \mathbf{b}^{(\mathrm{out})} \cdot \mathbf{n}\,\mathrm{d}S = \mathcal{F}^{(\mathrm{out})}. \tag{18}$$

Proof. Estimates (14)–(17) could be proved in the same way as in Lemma 3. Due to the construction of $\mathbf{b}^{(\mathrm{out})}$, we get (18):

$$\begin{aligned}
\int\limits_{\Lambda} \mathbf{b}^{(\mathrm{out})} \cdot \mathbf{n}\, \mathrm{d}S &= -\int\limits_{\sigma(R)} \mathbf{b} \cdot \mathbf{n}\, \mathrm{d}S = -\int\limits_{-g(R)}^{g(R)} \mathbf{b}^{(\mathrm{out})} \cdot \mathbf{n}\, \mathrm{d}S \\
&= -\mathcal{F}^{(\mathrm{out})} \int\limits_{-g(R)}^{g(R)} \left(\frac{\partial \tilde{\zeta}(x)}{\partial x_2}, -\frac{\partial \tilde{\zeta}(x)}{\partial x_1} \right) \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{d}x_1 \\
&= -\mathcal{F}^{(\mathrm{out})} \int\limits_{-g(R)}^{g(R)} \left(-\frac{\partial \tilde{\zeta}(x)}{\partial x_1} \right) \mathrm{d}x_1 = \mathcal{F}^{(\mathrm{out})} \int\limits_{-g(R)}^{g(R)} \frac{\partial \tilde{\zeta}(x)}{\partial x_1}\, \mathrm{d}x_1 \\
&= \mathcal{F}^{(\mathrm{out})} \big(\tilde{\zeta}\big(g(R), R\big) - \tilde{\zeta}\big(-g(R), R\big) \big) = \mathcal{F}^{(\mathrm{out})}. \qquad \square
\end{aligned}$$

Let us take

$$\mathbf{h}(x) = \boldsymbol{\varphi}(x)|_{\Lambda} - \mathbf{b}^{(\mathrm{out})}(x)|_{\Lambda}.$$

Then

$$\int\limits_{\Lambda} \mathbf{h}(x) \cdot \mathbf{n}\, \mathrm{d}S = \int\limits_{\Lambda} \boldsymbol{\varphi}(x) \cdot \mathbf{n}\, \mathrm{d}S - \int\limits_{\Lambda} \mathbf{b}^{(\mathrm{out})}(x) \cdot \mathbf{n}\, \mathrm{d}S = \mathcal{F}^{(\mathrm{out})} - \mathcal{F}^{(\mathrm{out})} = 0,$$

and $\mathbf{h}$ can be extended (see [17]) inside Ω in the form

$$\mathbf{b}_0^{(\mathrm{out})}(x) = \left(\frac{\partial(\chi(x)\mathbf{E}(x))}{\partial x_2}; -\frac{\partial(\chi(x)\mathbf{E}(x))}{\partial x_1} \right),$$

where $\mathbf{E}(x) \in W^{2,2}(\Omega_0)$, $(\partial \mathbf{E}(x)/\partial x_2; -\partial \mathbf{E}(x)/\partial x_1)|_{\Lambda} = \mathbf{h}$ and χ is a Hopf's cut-off function such that $\chi = 1$ on Λ, $\operatorname{supp} \chi$ is contained in a small neighborhood of Λ.

Finally, we put

$$\mathbf{B}^{(\mathrm{out})}(x) = \mathbf{b}^{(\mathrm{out})}(x) + \mathbf{b}_0^{(\mathrm{out})}(x).$$

The properties of the extension $\mathbf{B}^{(\mathrm{out})}$ are formulated in the following lemma.

Lemma 7. *The vector field* $\mathbf{B}^{(\mathrm{out})}(x)$ *is solenoidal,* $\mathbf{B}^{(\mathrm{out})}|_{\Lambda} = \boldsymbol{\varphi}|_{\Lambda}$, $\mathbf{B}^{(\mathrm{out})}|_{\partial\Omega \setminus \Lambda} = 0$, $\mathbf{B}^{(\mathrm{inn})} \in W^{2,2}(\overline{\Omega})$ *and satisfies the following estimates*:

$$\begin{gathered}
\big|\mathbf{B}^{(\mathrm{out})}(x)\big| \leqslant \frac{c|\mathcal{F}^{(\mathrm{out})}|}{g(x_2)}, \quad x \in D, \\
\big|\nabla \mathbf{B}^{(\mathrm{out})}(x)\big| \leqslant \frac{c|\mathcal{F}^{(\mathrm{out})}|}{g^2(x_2)}, \quad \big|\Delta \mathbf{B}^{(\mathrm{out})}(x)\big| \leqslant \frac{c|\mathcal{F}^{(\mathrm{out})}|}{g^3(x_2)}, \quad x \in D, \\
\big|\mathbf{B}^{(\mathrm{out})}(x)\big| + \big|\nabla \mathbf{B}^{(\mathrm{out})}(x)\big| + \big|\Delta \mathbf{B}^{(\mathrm{out})}(x)\big| \leqslant c\big|\mathcal{F}^{(\mathrm{out})}\big|, \quad x \in \Omega \setminus D.
\end{gathered}$$

Therefore, we have constructed the extension $\mathbf{A} = \mathbf{B}^{(\mathrm{inn})} + \mathbf{B}^{(\mathrm{out})}$ of the boundary value $\boldsymbol{\varphi}$. The properties of $\mathbf{A}$ are given in the following theorem.

Theorem 1. *The constructed extension $\mathbf{A} \in W^{2,2}(\Omega)$ is solenoidal, satisfies the boundary condition $\mathbf{A}|_{\partial\Omega} = \boldsymbol{\varphi}$ and the following estimates*:

$$\big|\mathbf{A}(x)\big| \leqslant \frac{c(|\mathcal{F}^{(\mathrm{inn})}| + |\mathcal{F}^{(\mathrm{out})}|)}{g(x_2)}, \quad x \in D, \tag{19}$$

$$\big|\nabla\mathbf{A}(x)\big| \leqslant \frac{c(|\mathcal{F}^{(\mathrm{inn})}| + |\mathcal{F}^{(\mathrm{out})}|)}{g^2(x_2)}, \quad x \in D, \tag{20}$$

$$\big|\Delta\mathbf{A}(x)\big| \leqslant \frac{c(|\mathcal{F}^{(\mathrm{inn})}| + |\mathcal{F}^{(\mathrm{out})}|)}{g^3(x_2)}, \quad x \in D, \tag{21}$$

$$\big|\mathbf{A}(x)\big| + \big|\nabla\mathbf{A}(x)\big| + \big|\Delta\mathbf{A}(x)\big| \leqslant c\big(\big|\mathcal{F}^{(\mathrm{inn})}\big| + \big|\mathcal{F}^{(\mathrm{out})}\big|\big), \quad x \in \Omega \setminus D. \tag{22}$$

Proof. Since $\mathbf{A}(x) = \mathbf{B}^{(\mathrm{inn})}(x) + \mathbf{B}^{(\mathrm{out})}(x)$, estimates (19)–(22) follows from Lemmas 4 and 7. □

4 Solvability of problem (1)

We look for the solution of (1) in the form

$$\mathbf{u}(x,t) = \mathbf{A}(x) + \mathbf{v}(x,t),$$

where $\mathbf{A}$ is the suitable extension of the boundary value $\boldsymbol{\varphi}$ constructed in the previous section. Then problem (1) is reduced to the problem with homogeneous boundary condition

$$\begin{aligned}
&\mathbf{v}_t(x,t) - \nu\Delta\mathbf{v}(x,t) + \nabla p(x,t) = \nu\Delta\mathbf{A}(x) + \mathbf{f}(x,t), \quad (x,t) \in \Omega \times (0,2\pi),\\
&\operatorname{div}\mathbf{v}(x,t) = 0, \quad (x,t) \in \Omega \times (0,2\pi),\\
&\mathbf{v}(x,t) = \mathbf{0}, \quad (x,t) \in \partial\Omega \times (0,2\pi),\\
&\mathbf{v}(x,0) = \mathbf{v}(x,2\pi), \quad x \in \Omega,
\end{aligned} \tag{23}$$

and now we look for the new unknown velocity field $\mathbf{v}$.

Let us denote the following space:

$$L^2_{\mathrm{per}}\big(0,2\pi; L^2_1(\Omega)\big) := \overline{C^\infty_{\mathrm{per}}\big(0,2\pi; L^2_1(\Omega)\big)}^{L^2(0,2\pi)},$$

where $L^2_1(\Omega)$ is weighted space with the norm

$$\|w\|_{L^2_1(\Omega)} = \sqrt{\int\limits_D |w|^2 g^2 \,\mathrm{d}x + \int\limits_{\Omega_0} |w|^2 \,\mathrm{d}x}.$$

Definition 2. Let $\mathbf{f} \in L^2_{\mathrm{per}}(0, 2\pi; L^2_1(\Omega))$. By a weak solution of problem (23) we understand a solenoidal vector field $\mathbf{v}$ with $\nabla\mathbf{v}$, $\mathbf{v}_t \in L^2(0, 2\pi; L^2(\Omega))$ satisfying the homogeneous boundary condition $\mathbf{v}|_{\partial\Omega} = \mathbf{0}$, the time periodicity condition $\mathbf{v}(x, 0) = \mathbf{v}(x, 2\pi)$ and the integral identity:

$$\int_0^{2\pi}\int_\Omega \mathbf{v}_t \cdot \boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t + \nu\int_0^{2\pi}\int_\Omega \nabla\mathbf{v} : \nabla\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t = -\nu\int_0^{2\pi}\int_\Omega \nabla\mathbf{A} : \nabla\boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t + \int_0^{2\pi}\int_\Omega \mathbf{f} \cdot \boldsymbol{\eta}\,\mathrm{d}x\,\mathrm{d}t \tag{24}$$

for all $\boldsymbol{\eta} \in L^2(0, 2\pi; J_0^\infty(\Omega))$.

Theorem 2. *Assume that the domain $\Omega \subset \mathbb{R}^2$ has one outlet to infinity, boundary value $\boldsymbol{\varphi} \in W^{3/2,2}(\partial\Omega)$ has a compact support, $\mathbf{f} \in L^2_{\mathrm{per}}(0, 2\pi; L^2_1(\Omega))$. If $\int_1^{+\infty} \mathrm{d}x_2/g^3(x_2) < +\infty$, then problem* (1) *has a unique weak solution $\mathbf{u} = \mathbf{A} + \mathbf{v}$ satisfying the following estimate*:

$$\|\mathbf{u}_t\|_{L^2(0,2\pi;L^2(\Omega))} + \|\nabla\mathbf{u}\|_{L^2(0,2\pi;L^2(\Omega))} \leqslant c\left(\left(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\left(1 + \int_1^{+\infty} \frac{1}{g^3(x_2)}\,\mathrm{d}x_2\right)\right)^{1/2} + \|\mathbf{f}\|_{L^2(0,2\pi;L^2_1(\Omega))}\right). \tag{25}$$

Proof. We start with the choosing a family of bounded domains Ω_k, i.e.,

$$\Omega_k = \Omega_0 \cup D_k,$$

where $\Omega_0 = \Omega \cap B_{R_0}$ and $D_k = \{x \in D\colon x_2 < R_k\}$ with $R_1 = 1$, $R_{k+1} = R_k + g(R_k)/(2L)$, $k \geqslant 1$.

The existence of a unique solution $\mathbf{v}$ satisfying the integral identity (24) could be proved by three following steps. Firstly, we prove the existence of the approximate solution $\mathbf{v}^{(k,N)}$ to the problem

$$\begin{aligned}
&\mathbf{v}_t^{(k,N)} - \nu\Delta\mathbf{v}^{(k,N)} + \nabla p^{(k,N)} = \nu\Delta\mathbf{A} + \mathbf{f}^{(N)}, \quad (x,t) \in \Omega_k \times (0, 2\pi),\\
&\operatorname{div}\mathbf{v}^{(k,N)} = 0, \quad (x,t) \in \Omega_k \times (0, 2\pi),\\
&\mathbf{v}^{(k,N)} = \mathbf{0}, \quad (x,t) \in \partial\Omega_k \times (0, 2\pi),\\
&\mathbf{v}^{(k,N)}(x, 0) = \mathbf{v}^{(k,N)}(x, 2\pi), \quad x \in \Omega_k.
\end{aligned} \tag{26}$$

Secondly, we show the convergence of the approximate weak solution $\mathbf{v}^{(k,N)}$ to the weak solution $\mathbf{v}^{(k)}$, which satisfies

$$\begin{aligned}
&\mathbf{v}_t^{(k)} - \nu\Delta\mathbf{v}^{(k)} + \nabla p^{(k)} = \nu\Delta\mathbf{A} + \mathbf{f}, \quad (x,t) \in \Omega_k \times (0, 2\pi),\\
&\operatorname{div}\mathbf{v}^{(k,N)} = 0, \quad (x,t) \in \Omega_k \times (0, 2\pi),\\
&\mathbf{v}^{(k)} = \mathbf{0}, \quad (x,t) \in \partial\Omega_k \times (0, 2\pi),\\
&\mathbf{v}^{(k)}(x, 0) = \mathbf{v}^{(k)}(x, 2\pi), \quad x \in \Omega_k.
\end{aligned} \tag{27}$$

Finally, passing to a limit as $k \to +\infty$, we get the existence of a weak solution $\mathbf{v}$ to problem (23).

Consider problem (26). It is well known that every 2π-periodic function in $L^2(0,2\pi)$ could be written as Fourier sieries:

$$\mathbf{f}(x,t) = \frac{\mathbf{f}_0^{(c)}(x)}{2} + \sum_{n=1}^{\infty} \big(\mathbf{f}_n^{(s)}(x)\sin(nt) + \mathbf{f}_n^{(c)}(x)\cos(nt)\big). \tag{28}$$

Let $\mathbf{f}^{(N)}$ be a partial sum of (28).

We look for the approximate solution $(\mathbf{v}^{(k,N)}, p^{(k,N)})$ in the form

$$\mathbf{v}^{(k,N)}(x,t) = \frac{\mathbf{b}_0^{(c)}(x)}{2} + \sum_{n=1}^{N} \big(\mathbf{a}_n^{(s)}(x)\sin(nt) + \mathbf{b}_n^{(c)}(x)\cos(nt)\big), \tag{29}$$

$$p^{(k,N)}(x,t) = \frac{p_0^{(c)}(x)}{2} + \sum_{n=1}^{N} \big(p_n^{(s)}(x)\sin(nt) + p_n^{(c)}(x)\cos(nt)\big). \tag{30}$$

In order to prove the existence of the approximate solution, we need to prove the existence of Fourier coefficients $\mathbf{a}_n^{(s)}$ and $\mathbf{b}_n^{(c)}$, $n = 0, 1, \dots, N$. To do this, we substitute (28)–(30) into problem (26), and by collecting the coefficients of $\sin$ and $\cos$ functions we obtain the following stationary problems:

$$\begin{aligned} &-\nu\Delta\mathbf{b}_0^{(c)}(x) + \nabla p_0^{(c)}(x) = 2\nu\Delta\mathbf{A}(x) + \mathbf{f}_0^{(c)}(x), \\ &\operatorname{div}\mathbf{b}_0^{(c)}(x) = 0, \qquad \mathbf{b}_0^{(c)}(x)|_{\partial\Omega_k} = \mathbf{0}, \end{aligned} \tag{31}$$

$$\begin{aligned} &n\mathbf{a}_n^{(s)}(x) - \nu\Delta\mathbf{b}_n^{(c)}(x) + \nabla p_0^{(c)}(x) = \mathbf{f}_n^{(c)}(x), \\ &-n\mathbf{b}_n^{(c)}(x) - \nu\Delta\mathbf{a}_n^{(s)}(x) + \nabla p_0^{(s)}(x) = \mathbf{f}_n^{(s)}(x), \\ &\operatorname{div}\mathbf{a}_n^{(s)}(x) = 0, \qquad \operatorname{div}\mathbf{b}_n^{(c)}(x) = 0, \\ &\mathbf{a}_n^{(s)}(x)|_{\partial\Omega_k} = \mathbf{0}, \quad \mathbf{b}_n^{(c)}(x)|_{\partial\Omega_k} = \mathbf{0}, \quad n = 1, 2, \dots, N. \end{aligned} \tag{32}$$

Notice that (31) is the Stokes system with homogeneous boundary condition and the existence of a weak solution of (31) is well known (see [17]).

In order to prove the existence of a unique solution to problem (32), we multiply $(32)_1$ by $\boldsymbol{\eta} \in H(\Omega_k)$ and $(32)_2$ by $\boldsymbol{\xi} \in H(\Omega_k)$. Then by integrating by parts over Ω_k we obtain the following system:

$$\begin{aligned} &n\int_{\Omega_k} \mathbf{a}_n^{(s)} \cdot \boldsymbol{\eta}\,\mathrm{d}x + \nu\int_{\Omega_k} \nabla\mathbf{b}_n^{(c)} : \nabla\boldsymbol{\eta}\,\mathrm{d}x = \int_{\Omega_k} \mathbf{f}_n^{(c)} \cdot \boldsymbol{\eta}\,\mathrm{d}x, \\ &-n\int_{\Omega_k} \mathbf{b}_n^{(c)} \cdot \boldsymbol{\xi}\,\mathrm{d}x + \nu\int_{\Omega_k} \nabla\mathbf{a}_n^{(s)} : \nabla\boldsymbol{\xi}\,\mathrm{d}x = \int_{\Omega_k} \mathbf{f}_n^{(s)} \cdot \boldsymbol{\xi}\,\mathrm{d}x. \end{aligned} \tag{33}$$

To prove the existence of the unique solution of (33), we use Fredholm alternative by reducing (33) to the system of operator equations

$$\begin{aligned} \mathcal{B}\mathbf{a}_n^{(s)} + \nu\mathbf{b}_n^{(c)} &= \mathbf{F}^{(c)} \quad \forall\boldsymbol{\eta} \in H(\Omega_k), \\ \mathcal{B}\mathbf{b}_n^{(c)} + \nu\mathbf{a}_n^{(s)} &= \mathbf{F}^{(s)} \quad \forall\boldsymbol{\xi} \in H(\Omega_k), \end{aligned}$$

where $\mathcal{B}$ is linear completely continuous operator.

Then we consider homogeneous operator equations

$$\begin{aligned} \mathcal{B}\mathbf{a}_n^{(s)} + \nu\mathbf{b}_n^{(c)} &= 0 \quad \forall\boldsymbol{\eta} \in H(\Omega_k), \\ \mathcal{B}\mathbf{b}_n^{(c)} + \nu\mathbf{a}_n^{(s)} &= 0 \quad \forall\boldsymbol{\xi} \in H(\Omega_k), \end{aligned}$$

i.e.,

$$\begin{aligned} n\int\limits_{\Omega_k} \mathbf{a}_n^{(s)} \cdot \boldsymbol{\eta}\,\mathrm{d}x + \nu\int\limits_{\Omega_k} \nabla\mathbf{b}_n^{(c)} : \nabla\boldsymbol{\eta}\,\mathrm{d}x = 0, \\ -n\int\limits_{\Omega_k} \mathbf{b}_n^{(c)} \cdot \boldsymbol{\xi}\,\mathrm{d}x + \nu\int\limits_{\Omega_k} \nabla\mathbf{a}_n^{(s)} : \nabla\boldsymbol{\xi}\,\mathrm{d}x = 0. \end{aligned}$$

After substituting $\boldsymbol{\eta}(x) = \mathbf{b}_n^{(c)}(x)$ and $\boldsymbol{\xi}(x) = \mathbf{a}_n^{(s)}(x)$ and summing up the equations, we obtain

$$\nu\int\limits_{\Omega_k} \left|\nabla\mathbf{b}_n^{(c)}(x)\right|^2 \mathrm{d}x + \nu\int\limits_{\Omega_k} \left|\nabla\mathbf{a}_n^{(s)}(x)\right|^2 \mathrm{d}x = 0.$$

Then it follows that

$$\mathbf{b}_n^{(c)}(x) = 0, \quad \mathbf{a}_n^{(s)}(x) = 0.$$

According to Fredholm alternative, we obtained that (32) has a unique solution. Therefore, the existence and uniqueness of the approximate solution $\mathbf{v}^{(k,N)}$ to problem (26) is proved.

In order to prove the convergence of an approximate solution $\mathbf{v}^{(k,N)}(x,t)$ to $\mathbf{v}^{(k)}(x,t)$ in bounded domains Ω_k, we need to obtain the estimates for the norms of $\mathbf{v}^{(k,N)}(x,t)$. To do this, we multiply equation $(26)_1$ by $\mathbf{v}^{(k,N)}(x,t)$, and after integrating by parts over Ω_k, we get

$$\begin{aligned} &\int\limits_{\Omega_k} \mathbf{v}_t^{(k,N)} \cdot \mathbf{v}^{(k,N)}\,\mathrm{d}x + \nu\int\limits_{\Omega_k} \left|\nabla\mathbf{v}^{(k,N)}\right|^2 \mathrm{d}x \\ &\quad = -\nu\int\limits_{\Omega_k} \nabla\mathbf{A} : \nabla\mathbf{v}^{(k,N)}\,\mathrm{d}x + \int\limits_{\Omega_k} \mathbf{f}^{(N)} \cdot \mathbf{v}^{(k,N)}\,\mathrm{d}x. \end{aligned} \tag{34}$$

Since

$$\mathbf{v}_t^{(k,N)} \cdot \mathbf{v}^{(k,N)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\mathbf{v}^{(k,N)}\right|^2,$$

from (34) it follows that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_k} |\mathbf{v}^{(k,N)}|^2\,\mathrm{d}x + \nu\int\limits_{\Omega_k}\left|\nabla\mathbf{v}^{(k,N)}\right|^2\mathrm{d}x\\ &\quad= -\nu\int\limits_{\Omega_k}\nabla\mathbf{A} : \nabla\mathbf{v}^{(k,N)}\,\mathrm{d}x + \int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot\mathbf{v}^{(k,N)}\,\mathrm{d}x.\end{aligned}$$

Integration with respect to time variable t from 0 till 2π yields

$$\begin{aligned}&\frac{1}{2}\int\limits_{\Omega_k}\left|\mathbf{v}^{(k,N)}(x,2\pi)\right|^2\mathrm{d}x - \frac{1}{2}\int\limits_{\Omega_k}\left|\mathbf{v}^{(k,N)}(x,0)\right|^2\mathrm{d}x + \nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\left|\nabla\mathbf{v}^{(k,N)}\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\quad= -\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\nabla\mathbf{A} : \nabla\mathbf{v}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_0^{2\pi}\int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot\mathbf{v}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using the periodicity condition $\mathbf{v}^{(k,N)}(x,0) = \mathbf{v}^{(k,N)}(x,2\pi)$, we derive

$$\begin{aligned}&\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\left|\nabla\mathbf{v}^{(k,N)}\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\quad= -\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\nabla\mathbf{A} : \nabla\mathbf{v}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_0^{2\pi}\int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot\mathbf{v}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{35}$$

Notice that we need to get estimates with the constant independent of the domain Ω_k. To do this, we rewrite equation (35) as follows:

$$\begin{aligned}&\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\left|\nabla\mathbf{v}^{(k,N)}\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\quad= -\nu\int\limits_0^{2\pi}\int\limits_{\Omega_k}\nabla\mathbf{A} : \nabla\mathbf{v}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_0^{2\pi}\int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot g\cdot g^{-1}\cdot\mathbf{v}^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

By Cauchy–Schwarz inequality,

$$
\begin{aligned}
&\nu \int_0^{2\pi}\int_{\Omega_k} \left|\nabla \mathbf{v}^{(k,N)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad = -\nu \int_0^{2\pi}\int_{\Omega_k} \nabla \mathbf{A}(x) : \nabla \mathbf{v}^{(k,N)}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^{2\pi}\int_{\Omega_k} \mathbf{f}^{(N)}(x,t)\cdot g(x_2)\cdot g^{-1}(x_2)\cdot \mathbf{v}^{(k,N)}(x,t)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \nu\left(\int_0^{2\pi}\int_{\Omega_k} \left|\nabla \mathbf{A}(x)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \left(\int_0^{2\pi}\int_{\Omega_k} \left|\nabla \mathbf{v}^{(k,N)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\
&\qquad + \left(\int_0^{2\pi}\int_{\Omega_k} \left|\mathbf{f}^{(N)}(x,t)\right|^2\cdot \left|g(x_2)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \left(\int_0^{2\pi}\int_{\Omega_k} \frac{|\mathbf{v}^{(k,N)}(x,t)|^2}{|g(x_2)|^2}\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}. \qquad (36)
\end{aligned}
$$

Since, due to Poincaré–Friedrichs inequality, we have that

$$
\int_0^{2\pi}\int_{\Omega_k} \frac{|\mathbf{v}^{(k,N)}(x,t)|^2}{|g(x_2)|^2}\,\mathrm{d}x\,\mathrm{d}t \leqslant c\int_0^{2\pi}\int_{\Omega_k} \left|\nabla \mathbf{v}^{(k,N)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t,
$$

from (36) we obtain

$$
\begin{aligned}
&\nu \int_0^{2\pi}\int_{\Omega_k} \left|\nabla \mathbf{v}^{(k,N)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t \\
&\quad \leqslant \nu\left(\int_0^{2\pi}\int_{\Omega_k} \left|\nabla \mathbf{A}(x)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \left(\int_0^{2\pi}\int_{\Omega_k} \left|\nabla \mathbf{v}^{(k,N)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\
&\qquad + c\left(\int_0^{2\pi}\int_{\Omega_k} \left|\mathbf{f}^{(N)}(x,t)\right|^2 \left|g(x_2)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \left(\int_0^{2\pi}\int_{\Omega_k} \left|\nabla \mathbf{v}^{(k,N)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\
&\quad \leqslant \left(\nu\sqrt{2\pi}\left(\int_{\Omega_k} \left|\nabla \mathbf{A}(x)\right|^2 \mathrm{d}x\right)^{1/2} + c\left(\int_0^{2\pi}\int_{\Omega_k} \left|\mathbf{f}^{(N)}(x,t)\right|^2\cdot \left|g(x_2)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2}\right) \\
&\qquad \times \left(\int_0^{2\pi}\int_{\Omega_k} \left|\nabla \mathbf{v}^{(k,N)}(x,t)\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2}.
\end{aligned}
$$

Dividing both sides by $\nu(\int_0^{2\pi}\int_{\Omega_k}|\nabla\mathbf{v}^{(k,N)}(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t)^{1/2}$, we rewrite the last estimate as follows:

$$\big\|\nabla\mathbf{v}^{(k,N)}\big\|_{L^2(0,2\pi;L^2(\Omega_k))} \leqslant C\big(\|\nabla\mathbf{A}\|_{L^2(\Omega_k)} + \big\|\mathbf{f}^{(N)}g\big\|_{L^2(0,2\pi;L^2(\Omega_k))}\big), \tag{37}$$

where the constant C is independent of the domain Ω_k.

Due to Theorem 1, we estimate the norm $\|\nabla\mathbf{A}\|^2_{L^2(\Omega_k)}$:

$$\begin{aligned}
\|\nabla\mathbf{A}\|^2_{L^2(\Omega_k)} &= \int\limits_{\Omega_k}|\nabla\mathbf{A}|^2\,\mathrm{d}x \leqslant \int\limits_{\Omega_k}\left(\frac{c(|\mathcal{F}^{(\mathrm{inn})}|+|\mathcal{F}^{(\mathrm{out})}|)}{g^2(x_2)}\right)^2\mathrm{d}x\\
&\leqslant c\big(\big|\mathcal{F}^{(\mathrm{inn})}\big|^2+\big|\mathcal{F}^{(\mathrm{out})}\big|^2\big)\left(1+\int\limits_1^{R_k}\int\limits_{-g(x_2)}^{g(x_2)}\frac{1}{g^4(x_2)}\,\mathrm{d}x_1\,\mathrm{d}x_2\right)\\
&\leqslant c\big(\big|\mathcal{F}^{(\mathrm{inn})}\big|^2+\big|\mathcal{F}^{(\mathrm{out})}\big|^2\big)\left(1+\int\limits_1^{R_k}\frac{1}{g^3(x_2)}\,\mathrm{d}x_2\right).
\end{aligned} \tag{38}$$

According to the fact that

$$\big|\mathcal{F}^{(\mathrm{inn})}\big|^2+\big|\mathcal{F}^{(\mathrm{out})}\big|^2 \leqslant c\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)},$$

from (37), using (38), we get

$$\begin{aligned}
&\big\|\nabla\mathbf{v}^{(k,N)}\big\|_{L^2(0,2\pi;L^2(\Omega_k))}\\
&\quad\leqslant C\left(\left(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\left(1+\int\limits_1^{R_k}\frac{1}{g^3(x_2)}\,\mathrm{d}x_2\right)\right)^{1/2} + \big\|\mathbf{f}^{(N)}\big\|_{L^2(0,2\pi;L^2_1(\Omega_k))}\right),
\end{aligned} \tag{39}$$

where C is independent of Ω_k.

Let us get the estimate for the norm of the term $\mathbf{v}_t^{(k,N)}$. Multiplying equation $(26)_1$ by $\mathbf{v}_t^{(k,N)}(x,t)$ and after integrating by parts over Ω_k, we arrive at

$$\begin{aligned}
&\int\limits_{\Omega_k}\big|\mathbf{v}_t^{(k,N)}\big|^2\,\mathrm{d}x + \nu\int\limits_{\Omega_k}\nabla\mathbf{v}^{(k,N)} : \nabla\mathbf{v}_t^{(k,N)}\,\mathrm{d}x\\
&\quad= \nu\int\limits_{\Omega_k}\Delta\mathbf{A}\cdot\mathbf{v}_t^{(k,N)}\,\mathrm{d}x + \int\limits_{\Omega_k}\mathbf{f}^{(N)}\cdot\mathbf{v}_t^{(k,N)}\,\mathrm{d}x.
\end{aligned} \tag{40}$$

Since

$$\nabla\mathbf{v}^{(k,N)} : \nabla\mathbf{v}_t^{(k,N)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big(\big|\nabla\mathbf{v}^{(k,N)}\big|^2\big),$$

from (40) it follows that

$$\int\limits_{\Omega_k} \left|\mathbf{v}_t^{(k,N)}\right|^2 \mathrm{d}x + \frac{\nu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Omega_k} \left(\left|\nabla\mathbf{v}^{(k,N)}\right|^2\right) \mathrm{d}x = \nu\int\limits_{\Omega_k} \Delta\mathbf{A}\cdot\mathbf{v}_t^{(k,N)}\,\mathrm{d}x + \int\limits_{\Omega_k} \mathbf{f}^{(N)}\cdot\mathbf{v}_t^{(k,N)}\,\mathrm{d}x.$$

Then integrating with respect to time variable t from 0 till 2π, we obtain

$$\int\limits_0^{2\pi}\int\limits_{\Omega_k} \left|\mathbf{v}_t^{(k,N)}\right|^2 \mathrm{d}x\,\mathrm{d}t + \frac{\nu}{2}\int\limits_{\Omega_k} \left|\nabla\mathbf{v}^{(k,N)}(x,2\pi)\right|^2 \mathrm{d}x - \frac{\nu}{2}\int\limits_{\Omega_k} \left|\nabla\mathbf{v}^{(k,N)}(x,0)\right|^2 \mathrm{d}x = \nu\int\limits_0^{2\pi}\int\limits_{\Omega_k} \Delta\mathbf{A}\cdot\mathbf{v}_t^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_0^{2\pi}\int\limits_{\Omega_k} \mathbf{f}^{(N)}\cdot\mathbf{v}_t^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t.$$

Using the periodicity condition $\nabla\mathbf{v}^{(k,N)}(x,0) = \nabla\mathbf{v}^{(k,N)}(x,2\pi)$, the last equality reduces to

$$\int\limits_0^{2\pi}\int\limits_{\Omega_k} \left|\mathbf{v}_t^{(k,N)}\right|^2 \mathrm{d}x\,\mathrm{d}t = \nu\int\limits_0^{2\pi}\int\limits_{\Omega_k} \Delta\mathbf{A}\cdot\mathbf{v}_t^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t + \int\limits_0^{2\pi}\int\limits_{\Omega_k} \mathbf{f}^{(N)}\cdot\mathbf{v}_t^{(k,N)}\,\mathrm{d}x\,\mathrm{d}t.$$

By Cauchy–Schwarz inequality,

$$\begin{aligned}\int\limits_0^{2\pi}\int\limits_{\Omega_k} \left|\mathbf{v}_t^{(k,N)}\right|^2 \mathrm{d}x\,\mathrm{d}t &\leqslant \nu\left(\int\limits_0^{2\pi}\int\limits_{\Omega_k} |\Delta\mathbf{A}|^2\,\mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int\limits_0^{2\pi}\int\limits_{\Omega_k} \left|\mathbf{v}_t^{(k,N)}\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\ &\quad + \left(\int\limits_0^{2\pi}\int\limits_{\Omega_k} \left|\mathbf{f}^{(N)}\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2}\left(\int\limits_0^{2\pi}\int\limits_{\Omega_k} \left|\mathbf{v}_t^{(k,N)}\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2} \\ &\leqslant \left(\nu\sqrt{2\pi}\left(\int\limits_{\Omega_k} |\Delta\mathbf{A}|^2\,\mathrm{d}x\right)^{1/2} + \left(\int\limits_0^{2\pi}\int\limits_{\Omega_k} \left|\mathbf{f}^{(N)}\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2}\right) \\ &\quad \times\left(\int\limits_0^{2\pi}\int\limits_{\Omega_k} \left|\mathbf{v}_t^{(k,N)}\right|^2 \mathrm{d}x\,\mathrm{d}t\right)^{1/2}.\end{aligned}$$

Then dividing both sides by $(\int_0^{2\pi}\int_{\Omega_k} |\mathbf{v}_t^{(k,N)}(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t)^{1/2}$, we rewrite the last estimate as follows:

$$\left\|\mathbf{v}_t^{(k,N)}\right\|_{L^2(0,2\pi;L^2(\Omega_k))} \leqslant C_1\left(\|\Delta\mathbf{A}\|_{L^2(\Omega_k)} + \left\|\mathbf{f}^{(N)}\right\|_{L^2(0,2\pi;L^2(\Omega_k))}\right), \tag{41}$$

where C_1 is independent of the domain Ω_k.

Due to Theorem 1, we estimate the norm $\|\Delta \mathbf{A}\|^2_{L^2(\Omega_k)}$:

$$\begin{aligned}\|\Delta \mathbf{A}\|^2_{L^2(\Omega_k)} &= \int\limits_{\Omega_k} |\Delta \mathbf{A}|^2 \,\mathrm{d}x \leqslant \int\limits_{\Omega_k} \left(\frac{c(|\mathcal{F}^{(\mathrm{inn})}| + |\mathcal{F}^{(\mathrm{out})}|)}{g^3(x_2)}\right)^2 \mathrm{d}x \\ &\leqslant c\big(\big|\mathcal{F}^{(\mathrm{inn})}\big|^2 + \big|\mathcal{F}^{(\mathrm{out})}\big|^2\big)\left(1 + \int\limits_1^{R_k} \int\limits_{-g(x_2)}^{g(x_2)} \frac{1}{g^6(x_2)} \,\mathrm{d}x_1 \,\mathrm{d}x_2\right) \\ &\leqslant c\big(\big|\mathcal{F}^{(\mathrm{inn})}\big|^2 + \big|\mathcal{F}^{(\mathrm{out})}\big|^2\big)\left(1 + \int\limits_1^{R_k} \frac{\mathrm{d}x_2}{g^5(x_2)}\right). \end{aligned} \tag{42}$$

According to the fact that

$$\big|\mathcal{F}^{(\mathrm{inn})}\big|^2 + \big|\mathcal{F}^{(\mathrm{out})}\big|^2 \leqslant c\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)},$$

it follows from (41) using (42) the following estimate:

$$\begin{aligned}&\big\|\mathbf{v}_t^{(k,N)}\big\|_{L^2(0,2\pi;L^2(\Omega_k))} \\ &\quad\leqslant C_1\big(\|\Delta \mathbf{A}\|_{L^2(\Omega_k)} + \big\|\mathbf{f}^{(N)}\big\|_{L^2(0,2\pi;L^2(\Omega_k))}\big) \\ &\quad\leqslant C_1\left(\left(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\left(1 + \int\limits_1^{R_k} \frac{1}{g^5(x_2)} \,\mathrm{d}x_2\right)\right)^{1/2} + \big\|\mathbf{f}^{(N)}\big\|_{L^2(0,2\pi;L^2(\Omega_k))}\right) \\ &\quad\leqslant C_1\left(\left(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\left(1 + \int\limits_1^{R_k} \frac{1}{g^3(x_2)} \,\mathrm{d}x_2\right)\right)^{1/2} + \big\|\mathbf{f}^{(N)}\big\|_{L^2(0,2\pi;L^2_1(\Omega_k))}\right), \end{aligned} \tag{43}$$

where C_1 is independent of Ω_k.

For the fixed k, from estimates (39), (43) we conclude that $\{\nabla \mathbf{v}^{(k,N)}\}$ and $\{\mathbf{v}_t^{(k,N)}\}$ are bounded sequences in the space $L^2(0, 2\pi; L^2(\Omega_k))$. Hence there exists a subsequence $\{\mathbf{v}^{(k,N_m)}\}$ such that $\{\nabla \mathbf{v}^{(k,N_m)}\}$ and $\{\mathbf{v}_t^{(k,N_m)}\}$ are converging weakly to $\{\nabla \mathbf{v}^{(k)}\}$ and $\{\mathbf{v}_t^{(k)}\}$ in the space $L^2(0, 2\pi; L^2(\Omega_k))$. Moreover, $\{\mathbf{f}^{(N)}\}$ converges to $\{\mathbf{f}\}$ in the space $L^2(0, 2\pi, L^2(\Omega_k))$. For the approximate solution, the following integral identity holds:

$$\begin{aligned}&\int\limits_0^{2\pi} \int\limits_{\Omega_k} \mathbf{v}_t^{(k,N_m)} \cdot \boldsymbol{\eta} \,\mathrm{d}x \,\mathrm{d}t + \nu \int\limits_0^{2\pi} \int\limits_{\Omega_k} \nabla \mathbf{v}^{(k,N_m)} : \nabla \boldsymbol{\eta} \,\mathrm{d}x \,\mathrm{d}t \\ &\quad= -\nu \int\limits_0^{2\pi} \int\limits_{\Omega_k} \nabla \mathbf{A} : \nabla \boldsymbol{\eta} \,\mathrm{d}x \,\mathrm{d}t + \int\limits_0^{2\pi} \int\limits_{\Omega_k} \mathbf{f}^{(N_m)} \cdot \boldsymbol{\eta} \,\mathrm{d}x \,\mathrm{d}t\end{aligned}$$

for $\boldsymbol{\eta} \in L^2(0, 2\pi; W^{1,2}(\Omega_k))$. Passing to the limit as $N_m \to +\infty$, we get

$$\int_0^{2\pi}\int_{\Omega_k} \mathbf{v}_t^{(k)} \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t + \nu \int_0^{2\pi}\int_{\Omega_k} \nabla \mathbf{v}^{(k)} : \nabla \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t = -\nu \int_0^{2\pi}\int_{\Omega_k} \nabla \mathbf{A} : \nabla \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t + \int_0^{2\pi}\int_{\Omega_k} \mathbf{f} \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t. \tag{44}$$

Thus, $\mathbf{v}^{(k)}$ are weak solutions of problem (27) in bounded domains Ω_k.

Finally, we will get the solution in whole domain Ω. Since the estimates we got for the approximate solution $\mathbf{v}^{(k,N)}$ remain valid for the limit solution $\mathbf{v}^{(k)}$, using estimates (39) and (43), we have:

$$\begin{aligned}&\big\|\mathbf{v}_t^{(k)}\big\|_{L^2(0,2\pi;L^2(\Omega_k))} + \big\|\nabla \mathbf{v}^{(k)}\big\|_{L^2(0,2\pi;L^2(\Omega_k))} \\ &\quad\leqslant c\left(\left(\|\boldsymbol{\varphi}\|^2_{W^{3/2,2}(\partial\Omega)}\left(1 + \int_1^{R_k} \frac{1}{g^3(x_2)}\,\mathrm{d}x_2\right)\right)^{1/2} + \|\mathbf{f}\|_{L^2(0,2\pi;L_1^2(\Omega_k))}\right),\end{aligned} \tag{45}$$

where constant c is independent of domain Ω_k.

Since $\int_1^{+\infty} 1/g^3(x_2)\,\mathrm{d}x_2 < +\infty$, the right-hand side of estimate (45) is bounded by a constant independent of k. So $\{\nabla \mathbf{v}^{(k)}\}$ and $\{\mathbf{v}_t^{(k)}\}$ are bounded sequences in the space $L^2(0, 2\pi; L^2(\Omega_k))$. Therefore, there exists a subsequence $\{\mathbf{v}^{(k_m)}\}$ such that $\{\nabla \mathbf{v}^{(k_m)}\}$ and $\{\mathbf{v}_t^{(k_m)}\}$ converge weakly to $\{\nabla \mathbf{v}\}$ and $\{\mathbf{v}_t\}$ as $k_m \to +\infty$ in the space $L^2(0, 2\pi; L^2(\Omega))$. Taking in integral identity (44) an arbitrary test function $\boldsymbol{\eta}$ with a compact support, we can pass to a limit as $k \to +\infty$. As a result, we get for the limit function $\mathbf{v}$ integral identity (24).

The uniqueness is obtained by standard way assuming that (23) has two weak solutions $\mathbf{w}_1$ and $\mathbf{w}_2$, which satisfy the integral identities

$$\int_0^{2\pi}\int_{\Omega} \frac{\partial}{\partial t}\mathbf{w}_i \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t + \nu \int_0^{2\pi}\int_{\Omega} \nabla \mathbf{w}_i : \nabla \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t = -\nu \int_0^{2\pi}\int_{\Omega} \nabla \mathbf{A} :! \nabla \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t + \int_0^{2\pi}\int_{\Omega} \mathbf{f} \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t, \quad i = 1, 2.$$

Making a difference of the last two integral identities, we get

$$\int_0^{2\pi}\int_{\Omega} \frac{\partial}{\partial t}(\mathbf{w}_1 - \mathbf{w}_2) \cdot \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t + \nu \int_0^{2\pi}\int_{\Omega} \nabla (\mathbf{w}_1 - \mathbf{w}_2) : \nabla \boldsymbol{\eta} \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Taking $\boldsymbol{\eta} = \mathbf{w}_1 - \mathbf{w}_2$, we have

$$\int_0^{2\pi}\int_\Omega \frac{\partial}{\partial t}(\mathbf{w}_1 - \mathbf{w}_2)\cdot(\mathbf{w}_1 - \mathbf{w}_2)\,\mathrm{d}x\,\mathrm{d}t + \nu\int_0^{2\pi}\int_\Omega \nabla(\mathbf{w}_1 - \mathbf{w}_2) : \nabla(\mathbf{w}_1 - \mathbf{w}_2)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Since $\partial(\mathbf{w}_1 - \mathbf{w}_2)\cdot(\mathbf{w}_1 - \mathbf{w}_2)/\partial t = (1/2)\,\partial|\mathbf{w}_1 - \mathbf{w}_2|^2/\partial t$, it follows that

$$\frac{1}{2}\int_\Omega |\mathbf{w}_1 - \mathbf{w}_2|^2\,\mathrm{d}x + \nu\int_0^{2\pi}\int_\Omega \bigl|\nabla(\mathbf{w}_1 - \mathbf{w}_2)\bigr|^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Notice that both terms are positive. Therefore, we have

$$\nu\int_0^{2\pi}\int_\Omega \bigl|\nabla(\mathbf{w}_1 - \mathbf{w}_2)\bigr|^2\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Then $\mathbf{w}_1 - \mathbf{w}_2 = \mathrm{const} = 0$ a.e. in Ω since $\mathbf{w}_1|_{\partial\Omega} = 0$ and $\mathbf{w}_2|_{\partial\Omega} = 0$.

Therefore, we have proved that $\mathbf{u} = \mathbf{A} + \mathbf{v}$ is a unique weak solution of problem (1). Estimate (25) for $\mathbf{v}$ follows from (45). Since, for $\mathbf{A}$, the analogues to (25) is also valid, we obtain (25) for the sum $\mathbf{u} = \mathbf{A} + \mathbf{v}$. □

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