Nonlinear Analysis: Modelling and Control, Vol. 23, No. 5, 749–770 ISSN 1392-5113
https://doi.org/10.15388/NA.2018.5.7

The effect of delayed feedback on the dynamics of an autocatalysis reaction–diffusion system*

Xin Wei[a,b], **Junjie Wei**[a]

[a]Department of Mathematics, Harbin Institute of Technology,
Harbin, Heilongjiang, 150001, China
weijj@hit.edu.cn

[b]School of Mathematical Sciences, Heilongjiang University,
74 Xuefu Street, Harbin, Heilongjiang, 150080, China

Received: February 27, 2018 / **Revised:** May 30, 2018 / **Published online:** September 3, 2018

Abstract. This paper deals with an arbitrary-order autocatalysis model with delayed feedback subject to Neumann boundary conditions. We perform a detailed analysis about the effect of the delayed feedback on the stability of the positive equilibrium of the system. By analyzing the distribution of eigenvalues, the existence of Hopf bifurcation is obtained. Then we derive an algorithm for determining the direction and stability of the bifurcation by computing the normal form on the center manifold. Moreover, some numerical simulations are given to illustrate the analytical results. Our studies show that the delayed feedback not only breaks the stability of the positive equilibrium of the system and results in the occurrence of Hopf bifurcation, but also breaks the stability of the spatial inhomogeneous periodic solutions. In addition, the delayed feedback also makes the unstable equilibrium become stable under certain conditions.

Keywords: autocatalysis model, delayed feedback control, diffusion, stability switch, Hopf bifurcation.

1 Introduction

Autocatalysis is the process whereby a chemical is involved in its own production. In recent years, the diffusive autocatalysis reaction models, which attracts much attention, have been extensively used in the studies of Turing instability or Turing pattern. For example, see [8, 21, 29] for the Brusselator model, see [4, 20, 23] for the Sel'kov model, see [14, 15, 27] for the Lengyel–Epstein model, see [17, 26] for the Schnakenberg model.

When the reaction rates are the same and the reactor is assumed to be closed, we obtain an arbitrary-order autocatalysis model with Neumann boundary conditions, which

*This research was supported by National Natural Science Foundation of China (grant No. 11771109) and Natural Science Foundation of Shandong Province, China (grant No. ZR2015AQ005).

takes the following form:

$$
\begin{aligned}
&\frac{\partial u}{\partial t} - d_1 \Delta u = a - uv^p, \quad x \in \Omega,\ t > 0, \\
&\frac{\partial v}{\partial t} - d_2 \Delta v = uv^p - v, \quad x \in \Omega,\ t > 0, \\
&\partial_\nu u = \partial_\nu v = 0, \quad x \in \partial\Omega,\ t > 0, \\
&u(x,0) = u_0(x) \geqslant 0, \quad v(x,0) = v_0(x) \geqslant 0, \quad x \in \partial\Omega.
\end{aligned} \tag{1}
$$

Here u and v describe the dimensionless concentrations of the reactant and autocatalyst, respectively, and Ω is an open bounded domain in $\mathbb{R}^N$ ($N \geqslant 1$) with a smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$. Moreover, a denotes the initial concentration of the reaction precursor, p is the order of the reaction with respect to autocatalytic species, and d_1, d_2 are the diffusion coefficients of reactant and autocatalyst, respectively. The parameters a, p, d_1 and d_2 are assumed to be positive constants. The derivation of the model and more details can be found in [6, 18]. In [11], Guo et al. proved the existence of Hopf bifurcation and steady state bifurcation of system (1) by taking a as a parameter. In addition, the authors also derived the conditions for the occurrence of Turing instability. In [10], Guo et al. supplemented and improved the results in [11] and further established the Turing instability region determined by diffusion coefficients. In addition, the authors discussed the effect of diffusion coefficients on the existence of Hopf bifurcation.

It is well known that time delay is universal in ecological and chemical systems, and its influence on the dynamics of systems is crucial and instrumental. In [19], Ott et al. first proposed delay to control system by utilizing the input signals adjusted to the temporal states of the system, and then delayed feedback and its modifications are widely applied to control chaos and to stabilize unstable oscillations. Motivated by the idea of Ott and Grebogi, many investigators have studied the effect of time delay in ecological and chemical models (see [1, 2, 7, 9, 13, 16, 24, 28, 30]). Although many researches have been devoted to the experiments about the suppression of the delayed feedback on the chemical turbulent, there is few analysis of the effect of delayed feedback on the dynamics of chemical reaction models theoretically. Based on the previous work, we consider the following autocatalysis model with delayed feedback:

$$
\begin{aligned}
&\frac{\partial u(x,t)}{\partial t} - d_1 \Delta u(x,t) = a - u(x,t)v^p(x,t) + g\bigl(u(x,\, t-\tau) - u(x,t)\bigr), \\
&\quad x \in \Omega,\ t > 0, \\
&\frac{\partial v(x,t)}{\partial t} - d_2 \Delta v(x,t) = u(x,t)v^p(x,t) - v(x,t), \quad x \in \Omega,\ t > 0, \\
&\frac{\partial u(x,t)}{\partial x} = \frac{\partial v(x,t)}{\partial x} = 0, \quad t \geqslant 0,\ x \in \partial\Omega, \\
&u(x,t) = u_0(x,t) \geqslant 0\ (\not\equiv 0), \quad v(x,t) = v_0(x,t) \geqslant 0\ (\not\equiv 0), \\
&\quad x \in \Omega,\ t \in [-\tau, 0],
\end{aligned} \tag{2}
$$

where the last term in the first equation $g(u(x,t-\tau)-u(x,t))$ denotes the local delayed feedback control, g denotes the feedback intensity, and τ the time delay.

In this paper, we perform a detailed analysis of delayed feedback on the dynamics of system (2). The rest of the paper is organized as follows. In Section 2, by analyzing the distribution of the roots of the associated characteristic equation, we study the stability of the positive equilibrium E^*. As time delay varies, we study its effect on the stability of the positive equilibrium E^* and prove the existence of Hopf bifurcation. In Section 3, by applying the normal form theory and center manifold reduction for partial differential systems, the explicit formulas, which determine the stability and the direction of the bifurcating periodic solutions, are given. In Section 4, some simulations are given to illustrate our theoretical results. The simulations show that the delayed feedback not only breaks the stability of E^* and results in the occurrence of the Hopf bifurcation, but also effects the stability of the spatial inhomogeneous periodic solutions.

2 Stability analysis

It is easy to see that system (2) has a unique positive equilibrium $E^*(u^*,v^*)=(a^{1-p},a)$. In this paper, we consider system (2) on the domain $\Omega=(0,l\pi)$, $l\in\mathbb{R}^+$. Now, we devote our attention to the study of the effect of delayed feedback on the stability of E^*.

Denote $X=L^2([0,l\pi],\mathbb{R}^2)$. Setting $u_1(t)=u(x,t)$, $u_2(t)=v(x,t)$ and $U(t)=(u_1(t),u_2(t))^{\mathrm{T}}$, then system (2) can be written as an abstract differential equation in the phase space $\mathcal{C}=C([-\tau,0],X)$ as follows:

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t}=D\Delta U(t)+G(U_t), \tag{3}$$

where $D=\mathrm{diag}(d_1,d_2)$, $U_t(\cdot)=U(t+\cdot)$ and $G:\mathcal{C}\to X$ is given by

$$G(U_t)=\begin{pmatrix} a-u_1(t)u_2^p(t)+g(u_1(t-\tau)-u_1(t)) \\ u_1(t)u_2^p(t)-u_2(t)\end{pmatrix}.$$

Linearizing system (3) at the positive equilibrium E^*, we have

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t}=D\Delta U(t)+L(U_t), \tag{4}$$

where $L:\mathcal{C}\to X$ is given by

$$L(\phi_t)=L_1\phi(0)+L_2\phi(-\tau)$$

and

$$L_1=\begin{pmatrix} -a^p-g & -p \\ a^p & p-1\end{pmatrix},\qquad L_2=\begin{pmatrix} g & 0 \\ 0 & 0\end{pmatrix},$$
$$\phi(t)=\bigl(\phi_1(t),\phi_2(t)\bigr)^{\mathrm{T}},\qquad \phi_t(\cdot)=\bigl(\phi_1(t+\cdot),\phi_2(t+\cdot)\bigr).$$

From Wu [25] the corresponding characteristic equation of (4) can be written as

$$\lambda y - D\Delta y - L\big(\mathrm{e}^{\lambda\cdot} y\big) = 0, \tag{5}$$

where $y \in \operatorname{dom}(\Delta) \setminus \{0\}$, $\operatorname{dom}(\Delta) \subset X$, and

$$\big(\mathrm{e}^{\lambda\cdot} y\big)(\theta) = \mathrm{e}^{\lambda\theta} y \quad \text{for } \theta \in [-\tau, 0].$$

It is well known that the following eigenvalue problem

$$-\Delta\varphi = \mu\varphi, \quad x \in (0, l\pi), \qquad \varphi_x|_{x=0,\, l\pi} = 0$$

has eigenvalues $\mu_n = n^2/l^2$ ($n \in \mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$) with corresponding eigenfunctions $\varphi_n = \cos(n/l)x$, $n \in \mathbb{N}_0$. Substituting the Fourier expansion

$$y = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos\frac{n}{l}x, \quad a_n, b_n \in \mathbb{C},$$

into the characteristic equation (5), we have

$$\det\begin{pmatrix} \lambda + \frac{d_1 n^2}{l^2} + a^p + g - g\mathrm{e}^{-\lambda\tau} & p \\ -a^p & \lambda + \frac{d_2 n^2}{l^2} - p + 1 \end{pmatrix} = 0, \quad n \in \mathbb{N}_0.$$

Therefore, the characteristic equation (5) is equivalent to

$$\lambda^2 + A_n\lambda + B_n + (C_n\lambda + D_n)\mathrm{e}^{-\lambda\tau} = 0, \quad n \in \mathbb{N}_0, \tag{6}$$

where

$$\begin{aligned} A_n &= \frac{(d_1 + d_2)n^2}{l^2} + a^p + 1 - p + g, \\ B_n &= \frac{d_1 d_2 n^4}{l^4} + \frac{[(1-p)d_1 + (a^p + g)d_2]n^2}{l^2} + a^p + (1-p)g, \\ C_n &= -g, \qquad D_n = -g\left(\frac{d_2 n^2}{l^2} + 1 - p\right). \end{aligned}$$

When $\tau = 0$, equation (6) becomes

$$\lambda^2 + P_n\lambda + Q_n = 0, \quad n \in \mathbb{N}_0,$$

where

$$\begin{aligned} P_n &= A_n + C_n = \frac{(d_1 + d_2)n^2}{l^2} + a^p + 1 - p, \\ Q_n &= B_n + D_n = \frac{d_1 d_2 n^4}{l^4} + \frac{[(1-p)d_1 + a^p d_2]n^2}{l^2} + a^p. \end{aligned} \tag{7}$$

From Theorem 2.1 in [11] and Theorem 3.1 in [10] we obtain the following results on system (2) without delayed feedback ($\tau = 0$).

Lemma 1.

(i) *If* $0 < p \leqslant 1$, *then the equilibrium* E^* *of system* (2) *is locally asymptotically stable.*

(ii) *Assume that* $p > 1$ *and* $a > (p-1)^{1/p}$.

(a) *If* $0 < d_1/d_2 < a^p/(\sqrt{p}-1)^2$, *then the equilibrium* E^* *of system* (2) *is locally asymptotically stable.*

(b) *If* $d_1/d_2 > a^p/(\sqrt{p}-1)^2$, *then the positive equilibrium* E^* *of system* (2) *in the absence of diffusion is locally asymptotically stable, and unstable whenever the diffusion is present.*

Now, we investigate the effect of time delay on the stability of E^* when $p > 1$. From Lemma (1) we can see that $E^*(u^*, v^*)$ of (2) is locally asymptotically stable under the assumption:

(H1) $p > 1$, $a > (p-1)^{1/p}$ and $0 < d_1/d_2 < a^p/(\sqrt{p}-1)^2$

when $\tau = 0$. In fact, if $a < (p-1)^{1/p}$, then (H1) is false. In this case, we can still use the following techniques to analyze the effect of time delay on the stability of E^*. According to Corollary 2.4 in Ruan and Wei [22], we have that the stability of the equilibrium E^* of system (2) changes only if the characteristic equation (6) has a root appears on or crosses the imaginary axis. It can be verified that 0 is not a root of the characteristic equation (6) for any $n \in \mathbb{N}_0$. Therefore, we only need to check whether the characteristic equation (6) has purely imaginary roots. Let $\pm\mathrm{i}\omega$ ($\omega > 0$) be the roots of (6), then we have

$$-\omega^2 + A_n\mathrm{i}\omega + B_n + (C_n\mathrm{i}\omega + D_n)\mathrm{e}^{-\mathrm{i}\omega\tau} = 0.$$

Separating the real and imaginary parts, it yields to

$$\begin{aligned} \omega^2 - B_n &= D_n\cos\omega\tau + C_n\omega\sin\omega\tau, \\ A_n\omega &= D_n\sin\omega\tau - C_n\omega\cos\omega\tau. \end{aligned} \tag{8}$$

It follows that ω should satisfy

$$\omega^4 + \big(A_n^2 - C_n^2 - 2B_n\big)\omega^2 + B_n^2 - D_n^2 = 0, \tag{9}$$

with

$$\begin{aligned} A_n^2 - C_n^2 - 2B_n &= \big(d_1^2 + d_2^2\big)\frac{n^4}{l^4} + 2\big[\big(a^p + g\big)d_1 + (1-p)d_2\big]\frac{n^2}{l^2} \\ &\quad + 2a^p(g-p) + a^{2p} + (1-p)^2, \\ B_n^2 - D_n^2 &= Q_n\left\{d_1d_2\frac{n^4}{l^4} + \big[(1-p)d_1 + \big(a^p + 2g\big)d_2\big]\frac{n^2}{l^2} + a^p + 2g(1-p)\right\}, \end{aligned} \tag{10}$$

where Q_n is defined as in (7).

Denote $z = \omega^2$, then we can rewrite equation (9) as follows:

$$z^2 + \big(A_n^2 - C_n^2 - 2B_n\big)z + B_n^2 - D_n^2 = 0. \tag{11}$$

From the first equation of (10) we know that the graph of $A_n^2 - C_n^2 - 2B_n$ is a parabola. Therefore, $A_n^2 - C_n^2 - 2B_n > 0$ holds for any $(n/l)^2 \in \mathbb{R}^+$ just as

$$\big[\big(a^p + g\big)d_1 + (1-p)d_2\big]^2 - \big(d_1^2 + d_2^2\big)\big[2a^p(g-p) + a^{2p} + (1-p)^2\big] < 0,$$

or

$$a^p d_1 + (1-p)d_2 > 0 \quad \text{and} \quad A_0^2 - C_0^2 - 2B_0 > 0.$$

Since $A_0^2 - C_0^2 - 2B_0 = 2a^p(g-p) + a^{2p} + (1-p)^2 > 0$ when $g > (2a^p p - a^{2p} - (1-p)^2)/(2a^p)$, so, the sufficient conditions for $A_n^2 - C_n^2 - 2B_n > 0$ can be given by

$$\frac{d_1}{d_2} > \frac{p-1}{a^p}, \qquad g > \max\left\{0, \frac{2a^p p - a^{2p} - (1-p)^2}{2a^p}\right\}, \tag{12}$$

or

$$\big[\big(a^p + g\big)d_1 + (1-p)d_2\big]^2 - \big(d_1^2 + d_2^2\big)\big[2a^p(g-p) + a^{2p} + (1-p)^2\big] < 0. \tag{13}$$

Note that $Q_n > 0$ for any $n \in \mathbb{N}_0$ when (H1) holds. We know that the sign of $B_n^2 - D_n^2$ is determined by the sign of the following variable:

$$T_n \stackrel{\text{def}}{=} \frac{d_1 d_2 n^4}{l^4} + \frac{[(1-p)d_1 + (a^p + 2g)d_2]n^2}{l^2} + a^p + 2g(1-p)$$

when (H1) holds. Similar to the above analysis, we obtain that the sufficient condition for $B_n^2 - D_n^2 > 0$ can be given by

$$0 < \frac{d_1}{d_2} < \frac{a^p}{p-1}, \qquad g < \frac{a^p}{2(p-1)}, \tag{14}$$

or

$$\big[(1-p)d_1 + \big(a^p + 2g\big)d_2\big]^2 - 4d_1 d_2\big[a^p + 2g(1-p)\big] < 0. \tag{15}$$

Hence, we have that

$$A_n^2 - C_n^2 - 2B_n > 0 \quad \text{and} \quad B_n^2 - C_n^2 > 0 \quad \text{for any } n \in \mathbb{N}_0$$

if (12) and (14), or (12) and (15), or (13) and (14), or (13) and (15) hold. Recall that $z = \omega^2$, we deduce that characteristic equation (6) does not have purely imaginary roots if equation (11) does not have positive roots. We know that z_n^+ and z_n^- are negative when either of the following assumptions holds:

(H2) $A_n^2 - C_n^2 - 2B_n > 0$ and $B_n^2 - D_n^2 > 0$,
(H3) $(A_n^2 - C_n^2 - 2B_n)^2 - 4(B_n^2 - D_n^2) < 0$.

Hence, we can summarize the above discission as follows:

Lemma 2. *Suppose that* (H1) *holds. If* (12) *and* (14)*, or* (12) *and* (15)*, or* (13) *and* (14)*, or* (13) *and* (15)*, or* (H3) *holds, then all the roots of the characteristic equation* (6) *have negative real parts.*

From the above analysis we know that if equation (11) has a positive root, then the characteristic equation (6) has a pair of simply imaginary roots. It implies that there exists a $n \in \mathbb{N}_0$ such that $B_n^2 - D_n^2 < 0$. Since $B_n^2 - D_n^2 \to \infty$ as $n \to \infty$ and $B_0^2 - D_0^2 = a^p[a^p + 2g(1-p)] < 0$ when $g > a^p/(2(p-1))$, there exists a minimal integer $N_0 \in \mathbb{N}_0$ such that $B_n^2 - D_n^2 < 0$ at most for $0 \leqslant n \leqslant N_0$, and $B_n^2 - D_n^2 > 0$ for $n > N_0$. This means that equation (11) has a positive root z_n satisfying

$$z_n^+ = \frac{-(A_n^2 - C_n^2 - 2B_n) + \sqrt{(A_n^2 - C_n^2 - 2B_n)^2 - 4(B_n^2 - D_n^2)}}{2}$$

for $0 \leqslant n \leqslant N_0$. Let $\omega_n^+ = \sqrt{z_n^+}$. Then equation (6) has a pair of simply imaginary roots $\pm \mathrm{i}\omega_n^+$ as long as $\tau = \tau_{n,j}^+$ for $0 \leqslant n \leqslant N_0$, where

$$\tau_{n,j}^+ = \begin{cases} \frac{1}{\omega_n^+}[\arccos \frac{{\omega_n^+}^2(D_n - A_nC_n) - B_nD_n}{C_n^2{\omega_n^+}^2 + D_n^2} + 2j\pi], & \sin\omega_n^+\tau > 0, \\ \frac{1}{\omega_n^+}[2\pi - \arccos \frac{{\omega_n^+}^2(D_n - A_nC_n) - B_nD_n}{C_n^2{\omega_n^+}^2 + D_n^2} + 2j\pi], & \sin\omega_n^+\tau < 0, \end{cases} \tag{16}$$

for $j \in \mathbb{N}_0$, $0 \leqslant n \leqslant N_0$.

Summarizing the above discussion, we have the following result.

Lemma 3. *Suppose that* (H1) *holds. If $g > a^p/(2(p-1))$, then there exists a minimal integer $N_0 \in \mathbb{N}_0$ such that for $0 \leqslant n \leqslant N_0$, the characteristic equation* (6) *has a pair of imaginary roots $\pm \mathrm{i}\omega_n^+$ as long as $\tau = \tau_{n,j}^+$, and all the other roots of equation* (6)*, except $\pm \mathrm{i}\omega^+$, have negative roots.*

Assume that $g > a^p/(2(p-1))$. For $n > N_0$, we further assume that

$$A_n^2 - 2B_n - C_n^2 < 0 \quad \text{and} \quad \left(A_n^2 - C_n^2 - 2B_n\right)^2 - 4\left(B_n^2 - D_n^2\right) \geqslant 0,$$

then equation (11) has two positive roots, which can be given by

$$z_n^\pm = \frac{-(A_n^2 - 2B_n - C_n^2) \pm \sqrt{(A_n^2 - 2B_n - C_n^2)^2 - 4(B_n^2 - D_n^2)}}{2}.$$

From the first formula of (10) we obtain that $A_n^2 - C_n^2 - 2B_n \to \infty$ as $n \to \infty$, so there are at most infinite terms in $A_n^2 - C_n^2 - 2B_n$ less than zero, that is, there exists a minimal integer $N \in \mathbb{N}_0$ such that $A_n^2 - C_n^2 - 2B_n < 0$. Let $\omega_n^\pm = \sqrt{z_n^\pm}$. By equation (8), we have

$$\tau_{n,j}^\pm = \begin{cases} \frac{1}{\omega_n^\pm}[\arccos \frac{{\omega_n^\pm}^2(D_n - A_nC_n) - B_nD_n}{C_n^2{\omega_n^\pm}^2 + D_n^2} + 2j\pi], & \sin\omega_n^\pm\tau > 0, \\ \frac{1}{\omega_n^\pm}[2\pi - \arccos \frac{{\omega_n^\pm}^2(D_n - A_nC_n) - B_nD_n}{C_n^2{\omega_n^\pm}^2 + D_n^2} + 2j\pi], & \sin\omega_n^\pm\tau < 0, \end{cases} \tag{17}$$

for $j \in \mathbb{N}_0$, $N_0 < n \leqslant N$.

From the above analysis and according to Lemma 3, we have the following results.

Proposition 1. *Suppose that* (H1) *holds.*

(i) *Assume further* $(p-1)/a^p \leqslant d_1/d_2 \leqslant a^p/(p-1)$. *If*

$$g > \max\left\{\frac{2a^p p - a^{2p} - (1-p)^2}{2a^p}, \qquad \frac{a^p}{2(p-1)}\right\}$$

and

$$d_1 d_2 + \left[(1-p)d_1 + \left(a^p + 2g\right)d_2\right]l^2 + \left[a^p + 2g(1-p)\right]l^4 > 0$$

hold, then (6) *only has a pair of simply imaginary roots just for* $n = 0$.

(ii) *Assume that* $T_1 < 0$ *is satisfied. Then there must exist an integer* $n \in \mathbb{N}$ *such that* (6) *has a pair of simply imaginary roots.*

(iii) *If*

$$\text{(H4)} \qquad \begin{cases} \frac{d_1}{d_2} < \frac{p-1}{a^p}, \\ 0 < g < \min\left\{\frac{a^p}{2(p-1)}, \ \frac{(p-1)d_2 - a^p d_1}{d_1}, \ \frac{2a^p p - a^{2p} - (p-1)^2}{2a^p}\right\}, \\ \left(A_1^2 - C_1^2 - 2B_1\right)^2 - 4(B_1^2 - D_1^2) > 0 \end{cases}$$

holds, then the characteristic equation (6) *has two pairs of simply imaginary roots.*

Remark 1. Assume that $a < (p-1)^{1/p}$ and $p > 1$. If (H4) holds, conclusion (iii) in Proposition 1 is still true.

Lemma 4. *Suppose that* (H1) *holds.*

(i) *If* $B_n^2 - D_n^2 < 0$ *holds for some* $n \in \mathbb{N}_0$, *then* $\operatorname{Re}\lambda'(\tau_{n,j}^+) > 0$ *for* $j \in \mathbb{N}_0$.

(ii) *If* $A_n^2 - C_n^2 - 2B_n < 0$ *and* $B_n^2 - D_n^2 > 0$ *hold for some* $n \in \mathbb{N}_0$, *then for* $j \in \mathbb{N}_0$:

(a) $\operatorname{Re}\lambda'(\tau_{n,j}^\pm) = 0$ *as* $(A_n^2 - C_n^2 - 2B_n)^2 - 4(B_n^2 - D_n^2) = 0$;

(b) $\operatorname{Re}\lambda'(\tau_{n,j}^+) > 0$, $\operatorname{Re}\lambda'(\tau_{n,j}^-) < 0$ *as* $(A_n^2 - C_n^2 - 2B_n)^2 - 4(B_n^2 - D_n^2) > 0$.

Proof. Differentiating the two sides of equation (6) with respect to τ, it follows that

$$\left(2\lambda + A_n + C_n e^{-\lambda\tau}\right)\frac{d\lambda}{d\tau} - (C_n\lambda + D_n)\left(\lambda + \tau\frac{d\lambda}{d\tau}\right)e^{-\lambda\tau} = 0.$$

Thus,

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda + A_n + C_n e^{-\lambda\tau}}{\lambda(C_n\lambda + D_n)e^{-\lambda\tau}} - \frac{\tau}{\lambda}.$$

From (6) we obtain that

$$e^{-\lambda\tau} = -\frac{\lambda^2 + A_n\lambda + B_n}{C_n\lambda + D_n}.$$

Then

$$\begin{aligned}\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\bigg|_{\tau=\tau_{n,j}^{\pm}} &= \operatorname{Re}\left[\frac{C_n}{\lambda(C_n\lambda+D_n)} - \frac{2\lambda+A_n}{\lambda(\lambda^2+A_n\lambda+B_n)} - \frac{\tau}{\lambda}\right]_{\tau=\tau_{n,j}^{\pm}} \\ &= \operatorname{Re}\left[\frac{C_n}{\lambda(C_n\lambda+D_n)}\right]_{\lambda=\mathrm{i}\omega_n^{\pm}} - \operatorname{Re}\left[\frac{2\lambda+A_n}{\lambda(\lambda^2+A_n\lambda+B_n)}\right]_{\lambda=\mathrm{i}\omega_n^{\pm}} \\ &= -\frac{C_n^2}{C_n^2{\omega_n^{\pm}}^2+D_n^2} + \frac{A_n^2+2{\omega_n^{\pm}}^2-2B_n}{A_n^2{\omega_n^{\pm}}^2+(B_n-{\omega_n^{\pm}}^2)^2}.\end{aligned}$$

According to (8) and (9), then we have

$$\begin{aligned}\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\bigg|_{\tau=\tau_{n,j}^{\pm}} &= \frac{A_n^2-2B_n-C_n^2+2{\omega_n^{\pm}}^2}{C_n^2{\omega_n^{\pm}}^2+D_n^2}\bigg|_{\lambda=\mathrm{i}\omega_n^{\pm}} \\ &= \pm\frac{\sqrt{(A_n^2-C_n^2-2B_n)^2-4(B_n^2-D_n^2)}}{C_n^2{\omega_n^{\pm}}^2+D_n^2}.\end{aligned}$$

Since

$$\operatorname{sign}\left\{\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)\bigg|_{\tau=\tau_{n,j}^{\pm}}\right\} = \operatorname{sign}\left\{\operatorname{Re}\left(\frac{\mathrm{d}\lambda}{\mathrm{d}\tau}\right)^{-1}\bigg|_{\tau=\tau_{n,j}^{\pm}}\right\},$$

then the conclusion is claimed. □

Denote

$$\mathcal{D}_1 = \left\{n\in\mathbb{N}_0\colon\ B_n^2-D_n^2<0\right\}$$

and

$$\begin{aligned}\mathcal{D}_2 = \big\{n\in\mathbb{N}_0\colon\ &B_n^2-D_n^2>0,\ A_n^2-C_n^2-2B_n<0 \\ &\text{and } \left(A_n^2-C_n^2-2B_n\right)^2 > 4\left(B_n^2-D_n^2\right)\big\}.\end{aligned}$$

It is obvious from (17) that $\{\tau_{n,j}^{\pm}\}|_{j=0}^{\infty}$ is increasing on j for the fixed $n\in\mathcal{D}_2$, so, for the fixed n, $\tau_{n,0}^{\pm}=\min_{j\in\mathbb{N}_0}\{\tau_{n,j}^{\pm}\}$. Recall that E^* is locally asymptotically stable when $\tau=0$ under the assumption (H1), then necessarily $\tau_{n,0}^{+}\leqslant\tau_{n,0}^{-}$ ($n\in\mathbb{N}_0$) when (H1) holds. Hence, for all $n\in\mathcal{D}_1\cup\mathcal{D}_2$, we can define the smallest critical value such that the stability of $E^*(u^*,v^*)$ will change, which is given by

$$\tau^* \overset{\text{def}}{=} \tau_{n_0,0}^{+} = \min\left\{\tau_{n,0}^{+}\right\}, \quad \text{if } n\in\mathcal{D}_1\cup\mathcal{D}_2.$$

From Lemmas (2)–(4) and Proposition (1) we obtain the following conclusion.

Theorem 1. *Assume that* (H1) *holds.*

(i) *If* (12) *and* (14)*, or* (12) *and* (15)*, or* (13) *and* (14)*, or* (13) *and* (15)*, or* (H3) *holds, then the equilibrium* E^* *of system* (2) *is locally asymptotically stable for any* $\tau\geqslant 0$.

(ii) *If* $g > a^p/(2(p-1))$ *or* $T_1 < 0$, *then there exists a minimal integer* $N_0 \in \mathbb{N}_0$ *such that, for* $0 \leqslant n \leqslant N_0$:

(a) *the equilibrium* E^* *is locally asymptotically stable for* $\tau \in [0, \tau^*)$, *unstable for* $\tau > \tau^*$;

(b) *system* (2) *undergoes a Hopf bifurcation at the equilibrium* E^* *as* $\tau = \tau_{n,j}^+$ *for* $j \in \mathbb{N}_0$.

(iii) *If* (H4) *or*

$$\text{(H5)} \qquad \begin{cases} g > \frac{a^p}{2(p-1)}, \\ A_n^2 - C_n^2 - 2B_n < 0 \ \text{ and } \ \big(A_n^2 - C_n^2 - 2B_n\big)^2 - 4(B_n^2 - D_n^2) > 0 \end{cases}$$

holds for some $N \in \mathbb{N}$, *then the stability switch may exist. Moreover, system* (2) *undergoes Hopf bifurcation at the equilibrium* E^* *as* $\tau = \tau_{n,j}^{\pm}$ *for* $j \in \mathbb{N}_0$, $0 \leqslant n \leqslant N$.

Remark 2. Assume that $a < (p-1)^{1/p}$ and other conditions in (H1) still hold. According to the discussion in [3], we have that the conclusion in Theorem 1 is still true just as (H4) or (H5) holds. In this case, the stability switches may occur.

3 Stability and direction of Hopf bifurcation

In the previous section, we have verified that system (2) undergoes Hopf bifurcation at the positive equilibrium $E^*(u^*, v^*) = (a^{1-p}, a)$ as $\tau = \tau_{n,j}^+(\tau_{n,j}^-)$. In this section, we will investigate the direction and stability of period solutions bifurcating from the positive equilibrium E^* by applying the center manifold theorem and normal form theory for the partial differential equations presented in Faria [5] and Wu [25].

Setting $u_1(x,t) = u(x,t\tau) - a^{1-p}$, $u_2(x,t) = v(x,t\tau) - a$, $U(t) = (u_1(x,t), u_2(x,t))$, then system (2) can be written in the following form:

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \tau D\Delta U(t) + L(\tau)(U_t) + f(U_t, \tau)$$

in the space $\mathcal{C} = C([-1,0], X)$, where $D = \operatorname{diag}(d_1, d_2)$, $L(\tau)(\cdot) : \mathcal{C} \to X$ and $f : \mathcal{C} \times \mathbb{R} \to X$ are given, respectively, by

$$L(\tau)(\varphi) = \tau L_1\varphi(0) + \tau L_2\varphi(-1), \qquad f(\varphi, \tau) = \tau\big(f_1(\varphi,\tau), f_2(\varphi,\tau)\big)^{\mathrm{T}},$$

where

$$f_1(\varphi,\tau) = a_1\varphi_1(0)\varphi_2(0) + a_2\varphi_2^2(0) + a_3\varphi_1(0)\varphi_2^2(0) + a_4\varphi_2^3(0) + \mathcal{O}(4),$$
$$f_2(\varphi,\tau) = -a_1\varphi_1(0)\varphi_2(0) - a_2\varphi_2^2(0) - a_3\varphi_1(0)\varphi_2^2(0) - a_4\varphi_2^3(0) + \mathcal{O}(4),$$

with

$$a_1 = -pa^{p-1}, \quad a_2 = -\frac{p(p-1)}{2a}, \quad a_3 = -\frac{p(p-1)a^{p-2}}{2}, \quad a_4 = -\frac{p(p-1)(p-2)}{6a^2},$$

for $\varphi = (\varphi_1, \varphi_2)^{\mathrm{T}} \in \mathcal{C}$.

Setting $\tau = \tau^* + \alpha$, then system (2) can be rewritten in an abstract form in the space $\mathcal{C} = C([-1,0], X)$ as

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = \tau^* D\Delta U(t) + L(\tau^*)(U_t) + F(U_t, \alpha), \tag{18}$$

where

$$F(\varphi, \alpha) = \alpha D\Delta\varphi(0) + L(\alpha)(\varphi) + f(\varphi, \tau^* + \alpha),$$

for $\varphi \in \mathcal{C}$. From the previous section it is easy to see that system (18) undergoes Hopf bifurcation at the equilibrium $(0,0)$ when $\alpha = 0$. We also know that $\pm\mathrm{i}\omega^*\tau^*$ are the simply purely imaginary characteristic values of linearised system of (18) at $(0,0)$

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = (\tau^* + \alpha) D\Delta U(t) + L(\tau^* + \alpha)(U_t) \tag{19}$$

as $\alpha = 0$ and all the other characteristic values have negative real parts as $\alpha = 0$.

From previous analysis we have that the characteristic values of $\tau D\Delta$ on X are $-\tau d_1 n^2/l^2$ and $-\tau d_2 n^2/l^2$, $n \in \mathbb{N}_0$, with corresponding eigenfunctions $\beta_n^1(x) = (b_n(x), 0)^{\mathrm{T}}$ and $\beta_n^2(x) = (0, b_n(x))^{\mathrm{T}}$, where

$$b_n(x) = \frac{\cos\frac{nx}{l}}{\sqrt{\int_0^{l\pi} \cos^2\frac{nx}{l}\,\mathrm{d}x}}.$$

Denote $M_n = \operatorname{span}\{\langle\varphi, \beta_n^i\rangle\beta_n^i\colon \varphi \in \mathcal{C},\ i = 1, 2\}$, $n \in \mathbb{N}_0$, where the inner product is given by

$$\langle u, v\rangle = \int_0^{l\pi} u^{\mathrm{T}} v\,\mathrm{d}x \quad \text{for } u, v \in X.$$

So, on M_n, (19) is equivalent to the following equation on $\mathbb{R}^2$:

$$\frac{\mathrm{d}U(t)}{\mathrm{d}t} = -(\tau^* + \alpha)\frac{n^2}{l^2} DU(t) + L(\tau^* + \alpha)(U_t). \tag{20}$$

Using the Riesz representation theorem, we know that there exists a bounded variation $\eta_n(\alpha, \theta)(\theta \in [-1, 0])$ such that

$$-(\tau^* + \alpha)\frac{n^2}{l^2} D\varphi(0) + L(\tau^* + \alpha)(\varphi) = \int_{-1}^{0} \mathrm{d}\eta_n(\alpha, \theta)\varphi(\theta) \tag{21}$$

for $\varphi \in C([-1, 0], \mathbb{R}^2)$. In fact, we can choose

$$\eta_n(\alpha, \theta) = \begin{cases} -(\tau^* + \alpha)L_2, & \theta = -1, \\ 0, & \theta \in (-1, 0), \\ (\tau^* + \alpha)(L_1 - \frac{n^2}{l^2} D), & \theta = 0. \end{cases}$$

Let A denote the infinitesimal generator of the semigroup defined by (20) with $\alpha = 0$ and $n = n_0$, and A^* denote the formal adjoint of A under the bilinear form

$$(\psi_n, \varphi_n)_n = \psi_n(0)\varphi_n(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi_n(\xi - \theta)\,\mathrm{d}\eta_n(\mu, \theta)\varphi_n(\xi)\,\mathrm{d}\xi$$

for $\varphi_n \in C([-1, 0], \mathbb{R}^2)$, $\psi_n \in C([0, 1], \mathbb{R}^{2^{\mathrm{T}}})$. According to the above discussion, we see that $\pm\mathrm{i}\omega^*\tau^*$ are simply imaginary eigenvalues of A and A^*. In the following, we let $n = n_0$. Let

$$q(\theta) = q(0)\mathrm{e}^{\mathrm{i}\omega^*\tau^*\theta}, \quad \theta \in [-1, 0], \qquad q^*(s) = q^*(0)\mathrm{e}^{-\mathrm{i}\omega^*\tau^* s}, \quad s \in [0, 1],$$

be the eigenvectors of A and A^* corresponding to the eigenvalue $\mathrm{i}\omega^*\tau^*$. By calculation, we get

$$q(0) = (1, q_1)^{\mathrm{T}}, \qquad q^*(0) = (1, q_2),$$

where

$$q_1 = \frac{a^p}{(\mathrm{i}\omega^* + \frac{d_2 n_0^2}{l^2} + 1 - p)}, \qquad q_2 = \frac{-p}{(\mathrm{i}\omega^* + \frac{d_2 n_0^2}{l^2} + 1 - p)}.$$

Let $\Phi = (\Phi_1, \Phi_2) = (\operatorname{Re} q, \operatorname{Im} q)$ and $\Psi^* = (\Psi_1^*, \Psi_2^*)^{\mathrm{T}} = (\operatorname{Re} q^*, \operatorname{Im} q^*)^{\mathrm{T}}$. By direct calculation with use of (21), we can obtain

$$(\Psi^*, \Phi)_{n_0} = \begin{pmatrix} (\Psi_1^*, \Phi_1)_{n_0} & (\Psi_1^*, \Phi_2)_{n_0} \\ (\Psi_2^*, \Phi_1)_{n_0} & (\Psi_2^*, \Phi_2)_{n_0} \end{pmatrix},$$

where

$$\begin{aligned}
(\Psi_1^*, \Phi_1)_{n_0} &= 1 + \operatorname{Re} q_1 \operatorname{Re} q_2 - \frac{\tau^* g}{2}\left(\cos\omega^*\tau^* + \frac{\sin\omega^*\tau^*}{\omega^*\tau^*}\right), \\
(\Psi_1^*, \Phi_2)_{n_0} &= \operatorname{Re} q_2 \operatorname{Im} q_1 + \frac{\tau^* g}{2}\sin\omega^*\tau^* = (\Psi_2^*, \Phi_1)_{n_0}, \\
(\Psi_2^*, \Phi_2)_{n_0} &= \operatorname{Im} q_1 \operatorname{Im} q_2 + \frac{\tau^* g}{2}\left(\cos\omega^*\tau^* - \frac{\sin\omega^*\tau^*}{\omega^*\tau^*}\right).
\end{aligned}$$

We choose $\Psi = (\Psi_1, \Psi_2)^{\mathrm{T}} = (\Psi^*, \Phi)_{n_0}^{-1}\Psi^*$ such that $(\Psi, \Phi)_{n_0} = I_2$, where I_2 is a 2×2 identity matrix. Then the center subspace of the linear equation (19) with $\alpha = 0$ is given by $P_{CN}\mathcal{C}$, where

$$P_{CN}\varphi = \varphi\big(\Psi, \langle\varphi, \beta_{n_0}\rangle\big) \cdot \beta_{n_0}$$

for $\varphi \in \mathcal{C}$, here $\beta_n = (\beta_n^1, \beta_n^2)$ and $c \cdot \beta_n = c_1\beta_n^1 + c_2\beta_n^2$ for any $c = (c_1, c_2)^{\mathrm{T}} \in \mathcal{C}$. Let $P_S\mathcal{C}$ be the stable subspace of the linear equation (19) with $\alpha = 0$, then $\mathcal{C} = P_{CN}\mathcal{C} \bigoplus P_S\mathcal{C}$. From Wu [25] we know that the flow of system (18) with $\alpha = 0$ in the center manifold is given by the following formula:

$$\begin{aligned}
\big(x_1(t), x_2(t)\big)^{\mathrm{T}} &= \big(\Psi, \langle U_t, \beta_{n_0}\rangle\big)_{n_0}, \\
U_t &= \Phi\big(x_1(t), x_2(t)\big)^{\mathrm{T}} \cdot \beta_{n_0} + h(x_1, x_2, 0),
\end{aligned} \tag{22}$$

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix}0 & \omega^*\tau^*\\ -\omega^*\tau^* & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \Psi(0)\big\langle F(U_t,0),\beta_{n_0}\big\rangle \tag{23}$$

with $h(0,0,0)=0$ and $Dh(0,0,0)=0$. Let $z=x_1-\mathrm{i}x_2$ and $\Psi(0)=(\Psi_1(0),\Psi_2(0))^{\mathrm{T}}$. Notice that $q=\Phi_1+\mathrm{i}\Phi_2$, then (22) can be transformed into

$$U_t = \frac{1}{2}(qz+\overline{qz})\cdot\beta_{n_0} + W(z,\bar{z}), \tag{24}$$

where

$$W(z,\bar{z}) = h\left(\frac{z+\bar{z}}{2},\frac{\mathrm{i}(z-\bar{z})}{2},0\right).$$

According to (23) and (24), we know that z should satisfy

$$\dot{z} = \mathrm{i}\omega^*\tau^* z + g(z,\bar{z}), \tag{25}$$

where

$$\begin{aligned} g(z,\bar{z}) &= \big(\Psi_1(0)-\mathrm{i}\Psi_2(0)\big)\big\langle F(U_t,0),\beta_{n_0}\big\rangle \\ &= \big(\Psi_1(0)-\mathrm{i}\Psi_2(0)\big)\big\langle f(U_t,\tau^*),\beta_{n_0}\big\rangle. \end{aligned}$$

Let

$$\begin{aligned} g(z,\overline{z}) &= g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots, \\ W(z,\overline{z}) &= W_{20}\frac{z^2}{2} + W_{11}z\overline{z} + W_{02}\frac{\overline{z}^2}{2} + \cdots. \end{aligned} \tag{26}$$

Denote $(\psi_1,\psi_2)=\Psi_1(0)-i\Psi_2(0)$. From (22), (24) and (25) we have

$$g_{20} = \frac{\tau^*}{2}\int_0^{l\pi} b_{n_0}^3\,\mathrm{d}x\big[\big(a_1q_1+a_2q_1^2\big)\psi_1 - \big(a_1q_1+a_2q_1^2\big)\psi_2\big] = \overline{g_{02}},$$

$$g_{11} = \frac{\tau^*}{4}\int_0^{l\pi} b_{n_0}^3\,\mathrm{d}x\big[\big(a_1(q_1+\overline{q_1})+2a_2q_1\overline{q_1}\big)\psi_1 - \big(a_1(q_1+\overline{q_1})+2a_2q_1\overline{q_1}\big)\psi_2\big]$$

and

$$\begin{aligned} g_{21} = {}& \frac{\tau^*}{4}\int_0^{l\pi} b_{n_0}^4\,\mathrm{d}x\big[\big(a_3q_1(q_1+2\overline{q_1})+3a_4q_1^2\overline{q_1}\big)\psi_1 - \big(a_3q_1(q_1+2\overline{q_1})+3a_4q_1^2\overline{q_1}\big)\psi_2\big] \\ & + \frac{\tau^*}{2}\big\langle\big[a_3\big(W_{20}^{(1)}(0)\overline{q_1} + W_{20}^{(2)}(0) + 2W_{11}^2(0) + 2W_{11}^1(0)q_1\big) \\ & + 2a_2\big(2W_{11}^2(0)q_1 + W_{20}^2(0)\overline{q_1}\big)\big]b_{n_0},b_{n_0}\big\rangle\psi_1 \\ & - \frac{\tau^*}{2}\big\langle\big[a_3\big(W_{20}^{(1)}(0)\overline{q_1} + W_{20}^{(2)}(0) + 2W_{11}^2(0) + 2W_{11}^1(0)q_1\big) \\ & + 2a_2\big(2W_{11}^2(0)q_1 + W_{20}^2(0)\overline{q_1}\big)\big]b_{n_0},b_{n_0}\big\rangle\psi_2. \end{aligned}$$

To obtain g_{21}, we need to calculate $W_{20}(\theta)$ and $W_{11}(\theta)(\theta \in [-1,0])$. Following the notation presented in Wu [25], we also let A_U denote the generator of the semigroup generated by the linear system (19) with $\alpha = 0$. Combining (24) with (25) and following the idea of Wu [25] and Zhao [28], we know that $W(z,\overline{z})$ satisfies

$$\begin{aligned} \dot{W} &= \dot{U}_t - \frac{1}{2}(q\dot{z} + \overline{q}\dot{\overline{z}}) \cdot \beta_{n_0} \\ &= \begin{cases} A_U W - \frac{1}{2}(q(\theta)g(z,\overline{z}) + \overline{q}(\theta)\overline{g}(z,\overline{z})) \cdot \beta_{n_0}, & \theta \in [-1,0), \\ A_U W - \frac{1}{2}(q(\theta)g(z,\overline{z}) + \overline{q}(\theta)\overline{g}(z,\overline{z})) \cdot \beta_{n_0} \\ \quad + f(\frac{1}{2}(qz + \overline{q}\overline{z}) \cdot \beta_{n_0} + W, \tau^*), & \theta = 0, \end{cases} \\ &= A_U W + H(z,\overline{z},\theta), \end{aligned} \tag{27}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots.$$

Denote

$$f\left(\frac{1}{2}(qz + \overline{q}\overline{z}) \cdot \beta_{n_0} + W, \tau^*\right) = f_{z^2}\frac{z^2}{2} + f_{z\overline{z}}z\overline{z} + f_{\overline{z}^2}\frac{\overline{z}^2}{2} + \cdots.$$

Thus, it is easy to see that

$$\begin{aligned} H_{20}(\theta) &= \begin{cases} -\frac{1}{2}(q(\theta)g_{20} + \overline{q}(\theta)\overline{g}_{02}) \cdot \beta_{n_0}, & \theta \in [-1,0), \\ -\frac{1}{2}(q(\theta)g_{20} + \overline{q}(\theta)\overline{g}_{02}) \cdot \beta_{n_0} + f_{z^2}, & \theta = 0, \end{cases} \\ H_{11}(\theta) &= \begin{cases} -\frac{1}{2}(q(\theta)g_{11} + \overline{q}(\theta)\overline{g}_{11}) \cdot \beta_{n_0}, & \theta \in [-1,0), \\ -\frac{1}{2}(q(\theta)g_{11} + \overline{q}(\theta)\overline{g}_{11}) \cdot \beta_{n_0} + f_{z\overline{z}}, & \theta = 0. \end{cases} \end{aligned} \tag{28}$$

Note that

$$\dot{W} = \frac{\partial W(z,\overline{z})}{\partial z}\dot{z} + \frac{\partial W(z,\overline{z})}{\partial \overline{z}}\dot{\overline{z}},$$

we obtain from (26) and (27) that

$$H_{20} = (2\mathrm{i}\omega^*\tau^* - A_U)W_{20}, \qquad H_{11} = -A_U W_{11}. \tag{29}$$

Since $2\mathrm{i}\omega^*\tau^*$ and 0 are not eigenvalues of (19), (29) has unique solutions W_{20} and W_{11} in $P_S\mathcal{C}$, which are given by

$$W_{20} = (2\mathrm{i}\omega^*\tau^* - A_U)^{-1}H_{20}, \qquad W_{11} = -A_U^{-1}H_{11}.$$

Combining the first equation of (28) with (29) and using the definition of A_U, we have

$$\dot{W}_{20} = 2\mathrm{i}\omega^*\tau^* W_{20}(\theta) + \frac{1}{2}\big(q(\theta)g_{20} + \overline{q}(\theta)\overline{g}_{02}\big) \cdot \beta_{n_0} \quad \text{for } \theta \in [-1,0].$$

Since $q(\theta) = q(0)\mathrm{e}^{\mathrm{i}\omega^*\tau^*\theta}$ for $\theta \in [-1, 0]$, we obtain

$$W_{20}(\theta) = \frac{1}{2}\left[\frac{\mathrm{i}g_{20}}{\omega^*\tau^*}q(\theta) + \frac{\mathrm{i}\overline{g}_{02}}{3\omega^*\tau^*}\overline{q}(\theta)\right] \cdot \beta_{n_0} + E\mathrm{e}^{2\mathrm{i}\omega^*\tau^*\theta},$$

where E is 2-dimensional vectors in X. With β_n^i $(i = 1, 2)$ and $q(\theta)$ $(\theta \in [-1, 0])$ defined as above, we obtain that

$$\begin{aligned}
\tau^* D\Delta q(0) \cdot \beta_{n_0} + L(\tau^*)\big(q(\theta) \cdot \beta_{n_0}\big) &= \mathrm{i}\omega^* q(0) \cdot \beta_{n_0},\\
\tau^* D\Delta \overline{q}(0) \cdot \beta_{n_0} + L(\tau^*)\big(\overline{q}(\theta) \cdot \beta_{n_0}\big) &= -\mathrm{i}\omega^* \overline{q}(0) \cdot \beta_{n_0}.
\end{aligned}$$

From (29) we have that

$$2\mathrm{i}\omega^*\tau^* E - \tau^* D\Delta E - L(\tau^*)\big(E\mathrm{e}^{2\mathrm{i}\omega^*\tau^*\theta}\big) = f_{z^2}. \qquad (30)$$

Representation E and f_{z^2} by series: $E = \sum_{n=0}^{\infty} E_n \cdot \beta_n = \sum_{n=0}^{\infty} E_n b_n$ $(E_n \in \mathbb{R}^2)$ and $f_{z^2} = \sum_{n=0}^{\infty}\langle f_{z^2}, \beta_n\rangle \cdot \beta_n = \sum_{n=0}^{\infty}\langle f_{z^2}, \beta_n\rangle b_n$. Then we get from (30) that

$$2\mathrm{i}\omega^*\tau^* E_n + \tau^*\frac{n^2}{l^2} D E_n - L(\tau^*)\big(E_n \mathrm{e}^{2\mathrm{i}\omega^*\tau^*\cdot}\big) = \big\langle f_{z^2}, \beta_n\big\rangle, \quad n \in \mathbb{N}_0.$$

Thus, by calculation, E_n can be expressed as

$$E_n = \tilde{E}_n^{-1}\big\langle f_{z^2}, \beta_n\big\rangle,$$

where

$$\tilde{E}_n = \tau^*\begin{pmatrix} 2\mathrm{i}\omega^* + \frac{d_1 n^2}{l^2} + a^p + g - g\mathrm{e}^{-2\mathrm{i}\omega^*\tau^*} & p \\ -a^p & 2\mathrm{i}\omega^* + \frac{d_2 n^2}{l^2} - p + 1 \end{pmatrix},$$

$$\big\langle f_{z^2}, \beta_n\big\rangle = \begin{cases} \frac{1}{\sqrt{l\pi}}\tilde{f}_{z^2}, & n_0 \neq 0,\ n = 0,\\ \frac{1}{\sqrt{2l\pi}}\tilde{f}_{z^2}, & n_0 \neq 0,\ n = 2n_0,\\ \frac{1}{\sqrt{l\pi}}\tilde{f}_{z^2}, & n_0 = 0,\ n = 0,\\ 0, & \text{other}, \end{cases}$$

with

$$\tilde{f}_{z^2} = \frac{\tau^*}{2}\begin{pmatrix} a_1 q_1 + a_2 q_1^2 \\ -a_1 q_1 - a_2 q_1^2 \end{pmatrix}.$$

Similarly, from the second equation of (28) and (29) we have

$$\begin{aligned}
W_{11}(\theta) &= \frac{1}{2}\left[\frac{-\mathrm{i}g_{11}}{\omega^*\tau^*}q(\theta) + \frac{\mathrm{i}\overline{g}_{11}}{\omega^*\tau^*}\overline{q}(\theta)\right] \cdot \beta_{n_0} + F,\\
F &= \sum_{n=0}^{\infty} F_n b_n, \quad F_n \in \mathbb{R}^2,\ F_n = \tilde{F}_n^{-1}\langle f_{z\overline{z}}, \beta_n\rangle,
\end{aligned}$$

where

$$\tilde{F}_n = \tau^* \begin{pmatrix} \frac{d_1 n^2}{l^2} + a^p & p \\ -a^p & \frac{d_2 n^2}{l^2} - p + 1 \end{pmatrix},$$

$$\langle f_{z\bar{z}}, \beta_n \rangle = \begin{cases} \frac{1}{\sqrt{l\pi}} \tilde{f}_{z\bar{z}}, & n_0 \neq 0,\ n = 0, \\ \frac{1}{\sqrt{2l\pi}} \tilde{f}_{z\bar{z}}, & n_0 \neq 0,\ n = 2n_0, \\ \frac{1}{\sqrt{l\pi}} \tilde{f}_{z\bar{z}}, & n_0 = 0,\ n = 0, \\ 0, & \text{other}, \end{cases}$$

with

$$\tilde{f}_{z\bar{z}} = \frac{\tau^*}{4} \begin{pmatrix} a_1(q_1 + \overline{q}_1) + 2a_2 q_1 \overline{q}_1 \\ -a_1(q_1 + \overline{q}_1) - 2a_2 q_1 \overline{q}_1 \end{pmatrix}.$$

Then g_{21} can be determined. Consequently, we can compute the following quantities:

$$c_1(0) = \frac{\mathrm{i}}{2\omega^*\tau^*} \left(g_{20} g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{1}{2} g_{21},$$

$$\mu_2 = -\frac{\operatorname{Re}(c_1(0))}{\operatorname{Re}(\lambda'(\tau^*))}, \qquad \beta_2 = 2\operatorname{Re}(c_1(0)),$$

$$T_2 = -\frac{1}{\omega^*\tau^*} \big(\operatorname{Im}(c_1(0)) + \mu_2 \operatorname{Im}\big(\lambda'(\tau^*)\big)\big).$$

According to the general theory in Hassard [12], we know that μ_2 determines the direction of the Hopf bifurcation: the direction of the Hopf bifurcation is forward(backward) when $\mu_2 > 0$ (< 0), that is, the bifurcating periodic solutions exist for $\tau > \tau^*$ $(\tau < \tau^*)$; β_2 determines the stability of the bifurcating periodic solutions: the bifurcating periodic solutions are orbitally asymptotically stable(unstable) when $\beta_2 < 0$ (> 0); and T_2 determines the period of the bifurcating periodic solutions: the period of the bifurcating periodic solutions increases(decreases) when $T_2 > 0$ (< 0).

4 Numerical simulations

In this section, we present some numerical simulations to illustrate the above theoretical results.

Example 1. We choose

(D1) $a = 0.85,\ p = 1.5,\ d_1 = 2,\ d_2 = 1.5,\ g = 1.1,\ l = 1.$

Clearly, (H1) holds for the data (D1). Then we have that $E^*(1.0847, 0.8500)$ is the unique positive equilibrium of system (2), and E^* is asymptotically stable when $\tau = 0$. For this set of parameter values, we also observe that conditions (i) in Proposition 1 are satisfied, so we have that equation (6) has a pair of simply imaginary roots only for $n = 0$.

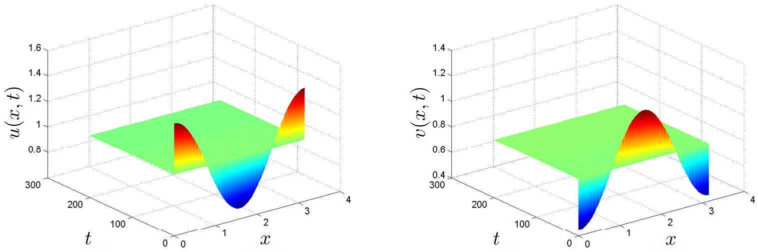


Figure 1. Numerical simulations of system (2) under the data (D1), where $\tau = 0.5 < \tau^* \approx 0.7834$. The positive equilibrium $E^* = (1.0847, 0.8500)$ of system (2) is asymptotically stable.

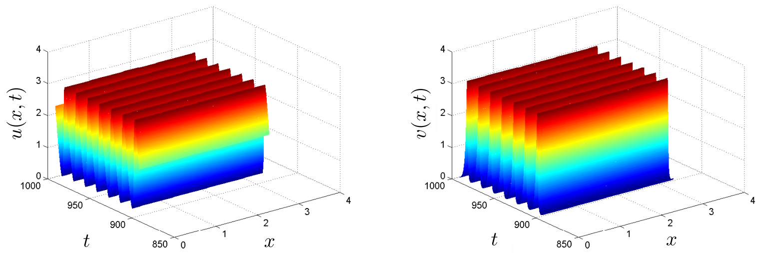


Figure 2. Numerical simulations of system (2) under the data (D1), where $\tau = 1 > \tau^* \approx 0.7834$. The positive equilibrium $E^*(1.0847, 0.8500)$ of system (2) becomes unstable, and the bifurcating periodic solutions from E^* is stable.

From (16) we have

$$\tau_{0,j}^{+} \approx 0.7834 + 10.0194j \quad \text{for } j \in \mathbb{N}_0.$$

So, $\tau^* = \tau_{0,0}^{+} \approx 0.7834$. According to Theorem 1, we have that the equilibrium E^* is locally asymptotically stable for $\tau \in [0, \tau^*)$ (see Fig. 1) and unstable for $\tau > \tau^*$. Then we can say that system (2) undergoes Hopf bifurcation at the positive equilibrium E^* when $\tau = \tau_{0,j}^{+}$, $j \in \mathbb{N}_0$. Using the formula derived in the previous section, we have $c_1(0) \approx -0.4675 - 5.6956\mathrm{i}$. Note that $\operatorname{Re} c_1(0) < 0$, and applying Lemma 4, we know that there exist orbitally stable periodic solutions when $\tau > \tau^* \approx 0.7834$ (see Fig. 2). In the simulations for the Figs. 1–2, the initial values are $u(x,t) = a^{1-p} + 0.4\cos 2x$, $v(x,t) = a - 0.4\cos 2x$, $(x,t) \in [0,\pi] \times [-\tau, 0]$.

Example 2. Let

(D2) $a = 0.91,\ p = 2,\ d_1 = 0.01,\ d_2 = 0.03,\ g = 0.1,\ l = 1.$

For this set of parameter values, the equilibrium $E^*(1.0989, 0.9100)$ is unstable when $\tau = 0$, but the assumption (H4) is satisfied. When $\tau = 0$, we can verify that the characteristic equation (6) has three pairs of roots with positive real parts, and all the other roots of (6) have negative real parts. By calculation, we find that (6) has a pair of

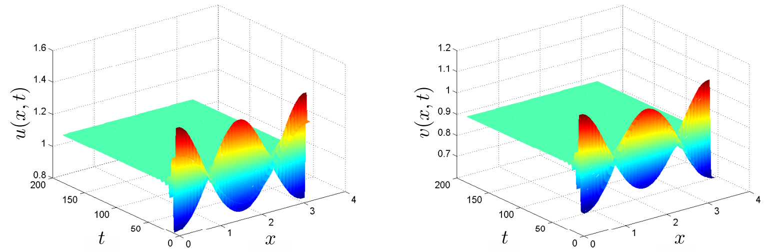


Figure 3. Numerical simulations of system (2) under the data (D2). The positive equilibrium $E^*(1.0989, 0.9100)$ becomes stable when $\tau \in (\tau_{0,0}^-, \tau_{0,0}^+)$. Here $\tau = 4.5$.

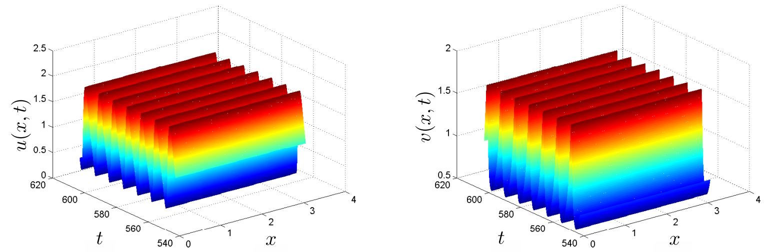


Figure 4. Numerical simulations of spatially homogeneous periodic solutions to system (2) under the data (D2), where $\tau = 3.6 < \tau_{0,0}^- \approx 3.7467$.

simply imaginary roots only for $n = 0, 1, 2$, and

$$\begin{aligned} \tau_{0,j}^+ &\approx 5.4670 + 6.8279j, & \tau_{0,j}^- &\approx 3.7467 + 8.0170j, & j &\in \mathbb{N}_0, \\ \tau_{1,j}^+ &\approx 5.6482 + 6.7064j, & \tau_{1,j}^- &\approx 3.3695 + 7.9554j, & j &\in \mathbb{N}_0, \\ \tau_{2,j}^+ &\approx 6.4845 + 6.6118j, & \tau_{2,j}^- &\approx 2.0496 + 7.4693j, & j &\in \mathbb{N}_0. \end{aligned}$$

Combining Lemma 4 with Corollary 2.4 in Ruan and Wei [22], we know that (6) has a pair of roots with positive real parts crosses the imaginary axis into the left half-plane at each value $\tau = \tau_{j,0}^- (j = 0, 1, 2)$. Consequently, the total multiplicity of roots of (6) in the right half-plane is zero when $\tau \in (\tau_{0,0}^-, \tau_{0,0}^+)$, that is, the equilibrium $E^*(1.0989, 0.9100)$ is unstable when $\tau \in [0, \tau_{0,0}^-)$, and becomes stable when $\tau \in (\tau_{0,0}^-, \tau_{0,0}^+)$ (see Fig. 3).

This means that the stability switches occur. This also implies that delayed feedback control plays a critical role in stabilizing the unstable equilibrium $E^*(1.0989, 0.9100)$. Observing the values of $\tau_{i,j}^{\pm}$ ($j \in \mathbb{N}_0, i = 0, 1, 2$), we find that the total multiplicity of roots of (6) in the right half-plane increases from zero to three as delay crosses the critical value $\tau_{0,0}^+$, and then it begins to decrease and becomes zero when $\tau \in (\tau_{0,1}^-, \tau_{0,1}^+)$. Moreover, (6) has at least a pair of roots with positive real part when $\tau > \tau_{0,1}^+$. By calculation, we have $\operatorname{Re} c_1(0) \approx -0.0606 < 0$ when $\tau_{0,0}^- \approx 3.7467$, and $\operatorname{Re} c_1(0) \approx -0.0134 < 0$ when $\tau_{0,0}^+ \approx 5.4670$. So, we have that when $\tau < \tau_{0,0}^- \approx 3.7467$, there exist stable spatial homogeneous periodic solutions (see Fig. 4).

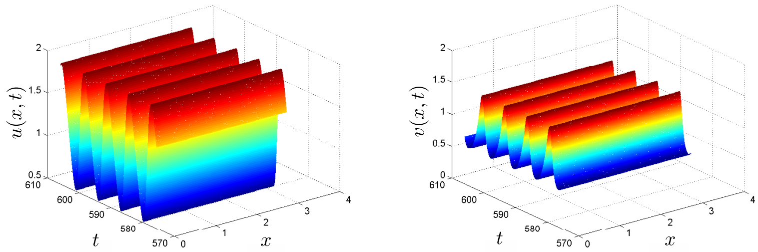


Figure 5. Numerical simulations of spatially homogeneous periodic solutions to system (2) under the data (D2), where $\tau = 5.6 > \tau_{0,0}^{+} \approx 5.4670$.

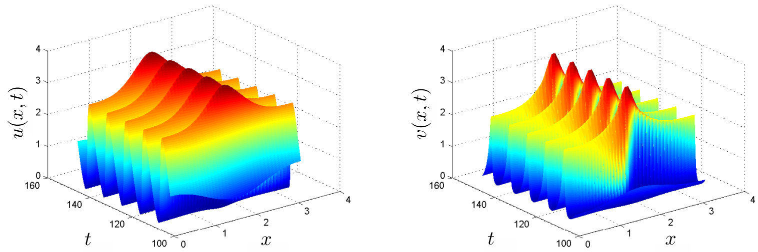


Figure 6. Numerical simulations of spatially inhomogeneous periodic solutions to system (2) under the data (D2), where $\tau = 1.8 < \tau_{2,0}^{-} \approx 2.0496$.

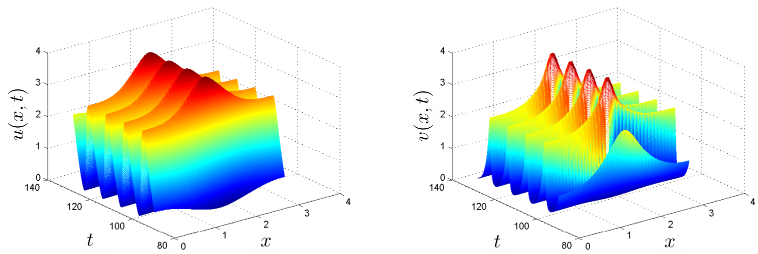
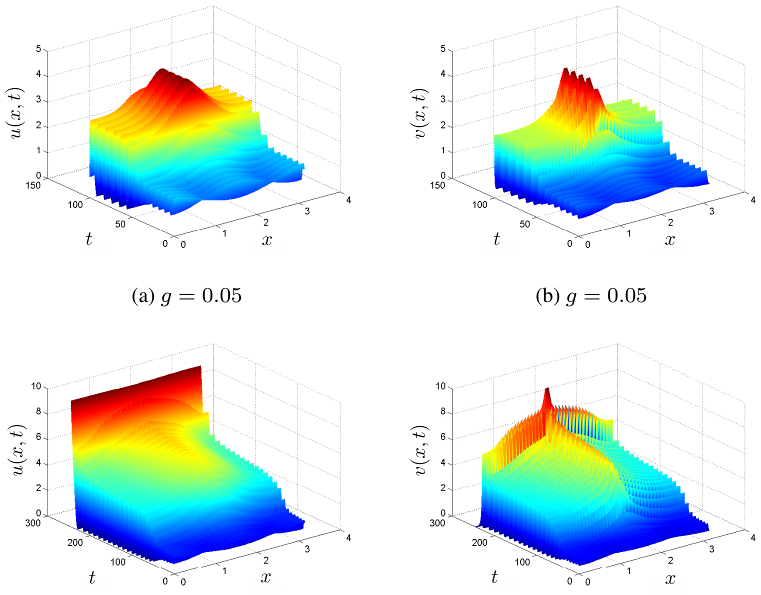


Figure 7. Numerical simulations of spatially inhomogeneous periodic solutions to system (2) under the data (D2), where $\tau = 8 > \tau_{2,0}^{+} \approx 6.4845$.

Similarly, there exist orbitally stable periodic solutions when $\tau > \tau_{0,0}^{+} \approx 5.4670$ (see Fig. 5). Particularly, there also exist spatial inhomogeneous periodic solutions (see Figs. 6–7). In addition, we find that the feedback intensity g would affect the stability of spatial inhomogeneous periodic solutions, that is, when g increases and passes through some critical value (i.e. $g > g* \approx 0.134$) the spatial inhomogeneous periodic solutions become unstable (see Fig. 8). In the simulations for Figs. 3–8, the initial values are $u(x,t) = a^{1-p} + 0.2\cos 2x$, $v(x,t) = a - 0.2\cos 2x$, $(x,t) \in [0,\pi] \times [-\tau, 0]$.



(a) $g = 0.05$ (b) $g = 0.05$

(c) $g = 0.135$ (d) $g = 0.135$

Figure 8. Numerical simulations of spatially inhomogeneous periodic solutions to system (2) for $a = 0.91$, $p = 2$, $d_1 = 0.01$, $d_2 = 0.03$, $\tau = 0.5$.

5 Conclusion

We have studied the effect of delayed feedback on the dynamics of an arbitrary-order autocatalysis model. By selecting the appropriate ratio of diffusion coefficients, the delayed feedback can be used to control the stability of the positive equilibrium. When time delay increases and crosses through some critical values, we found that the positive equilibrium E^* becomes unstable and induces the occurrence of spatial homogeneous periodic solutions. By computing the normal form on the center manifold, we analyzed the direction and stability of the periodic solutions. In addition, feedback intensity affected the stability of the spatial inhomogeneous periodic solutions, that is, as the feedback intensity reaches and exceeds some critical values, the spatial inhomogeneous periodic solutions become unstable. Our study also shows that delayed feedback control can stabilize the unstable equilibrium E^* under certain conditions.

Acknowledgment. The authors greatly appreciate the careful reading and valuable comments from the referees and editor. Their critical comments and helpful suggestions greatly improve the presentation of the manuscript.

References

1. X. Chang, J. Wei, Hopf bifurcation and optimal control in a diffusive predator-prey system with time delay and prey harvesting, *Nonlinear Anal. Model. Control*, **17**(4):379–409, 2012.
2. S. Chen, J. Shi, J. Wei, The effect of delay on a diffusive predator-prey system with Holling type-II predator functional response, *Commun. Pure Appl. Math.*, **12**(1):481–501, 2013.
3. K.L. Cooke, Z. Grossman, Discrete delay, distributed delay and stability switches, *J. Math. Anal. Appl.*, **86**(2):592–627, 1982.
4. F.A. Davidson, B.P. Rynne, A priori bounds and global existence of solutions of the steady-state Sel'kov model, *Proc. R. Soc. Edinb., Sect. A, Math.*, **130**(3):507–516, 2000.
5. T. Faria, Normal forms and Hopf bifurcation for partial differential equations with delays, *Trans. Am. Math. Soc.*, **352**(5):2217–2238, 2000.
6. A.B. Finlayson, J.H. Merkin, Creation of spatial structure by an electric field applied to an ionic cubic autocatalator system, *J. Eng. Math.*, **38**(3):279–296, 2000.
7. E.A. Gaffney, F. Yi, S.S. Lee, The bifurcation analysis of Turing pattern formation induced by delay and diffusion in the Schnakenberg system, *Discrete Contin. Dyn. Syst., Ser. B*, **22**(2):647–668, 2016.
8. M. Ghergu, V. Rădulescu, Turing patterns in general reaction-diffusion systems of Brusselator type, *Commun. Contemp. Math.*, **12**(4):661–679, 2010.
9. P. Ghosh, Control of the Hopf-Turing transition by time-delayed global feedback in a reaction-diffusion system, *Phys. Rev. E*, **84**(1):016222, 2011.
10. G. Guo, B. Li, X. Lin, Hopf bifurcation in spatially homogeneous and inhomogeneous autocatalysis models, *Comput. Math. Appl*, **67**(1):151–163, 2014.
11. G. Guo, B. Li, M. Wei, J. Wu, Hopf bifurcation and steady-state bifurcation for an autocatalysis reaction-diffusion model, *J. Math. Anal. Appl*, **391**(1):265–277, 2012.
12. B.D. Hassard, N.D. Kazarinoff, Y.-H. Wan, *Theory and Applications of Hopf Bifurcation*, Lond. Math. Soc. Lect. Note Ser., Vol. 41, Cambridge Univ. Press, Cambridge, 1981.
13. H. Hu, X. Li, Z. Fang, X. Fu, L. Ji, Q. Li, Inducing and modulating spiral waves by delayed feedback in a uniform oscillatory reaction-diffusion system, *Chem. Phys.*, **371**(1–3):60–65, 2010.
14. J. Jang, W. Ni, M. Tang, Global bifurcation and structure of Turing patterns in the 1-D Lengyel–Epstein model, *J. Dyn. Differ. Equations*, **16**(2):297–320, 2004.
15. J. Jin, J. Shi, J. Wei, F. Yi et al., Bifurcations of patterned solutions in the diffusive Lengyel–Epstein system of CIMA chemical reactions, *Rocky Mt. J. Math.*, **43**(5):1637–1674, 2013.
16. Q. Li, H. Hu, Pattern transitions induced by delay feedback, *J. Math. Chem.*, **127**(15):154510, 2007.
17. P. Liu, J. Shi, Y. Wang, X. Feng, Bifurcation analysis of reaction-diffusion Schnakenberg model, *J. Math. Chem.*, **51**(8):2001–2019, 2013.
18. J.D. Murray, *Mathematical Biology: I. An Introduction*, Interdiscip. Appl. Math., Vol. 17, Springer, New York, 2002.
19. E. Ott, C. Grebogi, J.A. Yorke, Controlling chaos, *Phys. Rev. Lett.*, **64**(11):1196–1199, 1990.

20. R. Peng, Qualitative analysis of steady states to the Sel'kov model, *J. Differ. Equations*, **241**(2):386–398, 2007.
21. R. Peng, M. Wang, Pattern formation in the Brusselator system, *J. Math. Anal. Appl.*, **309**(1):151–166, 2005.
22. S. Ruan, J. Wei, On the zeros of transcendental functions with applications to stability of delay differential equations with two delays, *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.*, **10**:863–874, 2003.
23. M. Wang, Non-constant positive steady states of the Sel'kov model, *J. Differ. Equations*, **190**(2):600–620, 2003.
24. X. Wei, J. Wei, Stability and bifurcation analysis in the photosensitive CDIMA system with delayed feedback control, *Int. J. Bifurcation Chaos Appl. Sci. Eng.*, **27**(11):1750177, 2017.
25. J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Appl. Math. Sci., Vol. 119, Springer, New York, 2012.
26. C. Xu, J. Wei, Hopf bifurcation analysis in a one-dimensional Schnakenberg reaction-diffusion model, *Nonlinear Anal., Real World Appl.*, **13**(4):1961–1977, 2012.
27. F. Yi, J. Wei, J. Shi, Diffusion-driven instability and bifurcation in the Lengyel–Epstein system, *Nonlinear Anal., Real World Appl.*, **9**(3):1038–1051, 2008.
28. J. Zhao, J. Wei, Persistence, Turing instability and Hopf bifurcation in a diffusive plankton system with delay and quadratic closure, *Int. J. Bifurcation Chaos Appl. Sci. Eng.*, **26**(03):1650047, 2016.
29. J. Zhou, C. Mu, Pattern formation of a coupled two-cell Brusselator model, *J. Math. Anal. Appl.*, **2**(366):679–693, 2010.
30. W. Zuo, J. Wei, Stability and bifurcation in a ratio-dependent Holling-III system with diffusion and delay, *Nonlinear Anal. Model. Control*, **19**(1):132–153, 2014.