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Positive solutions for a system of fractional differential equations with p-Laplacian operator and multi-point boundary conditions

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Abstract. We investigate the existence and nonexistence of positive solutions for a system of nonlinear Riemann–Liouville fractional differential equations with parameters and p-Laplacian operator subject to multi-point boundary conditions, which contain fractional derivatives. The proof of our main existence results is based on the Guo–Krasnosel'skii fixed-point theorem.

Keywords: Riemann–Liouville fractional differential equations, p-Laplacian operator, multi-point boundary conditions, positive solutions, existence, nonexistence.

1 Introduction

We consider the system of nonlinear ordinary fractional differential equations with r_1-Laplacian and r_2-Laplacian operators

$$\begin{aligned} &D_{0+}^{\alpha_1}\big(\varphi_{r_1}\big(D_{0+}^{\beta_1}u(t)\big)\big) + \lambda f\big(t,u(t),v(t)\big) = 0, \quad t\in(0,1),\\ &D_{0+}^{\alpha_2}\big(\varphi_{r_2}\big(D_{0+}^{\beta_2}v(t)\big)\big) + \mu g\big(t,u(t),v(t)\big) = 0, \quad t\in(0,1), \end{aligned} \tag{S}$$

with the multi-point boundary conditions

$$\begin{aligned} &u^{(j)}(0)=0,\ \ j=0,\dots,n-2; \quad D_{0+}^{\beta_1}u(0)=0,\ \ D_{0+}^{p_1}u(1)=\sum_{i=1}^{N} a_i D_{0+}^{q_1}u(\xi_i),\\ &v^{(j)}(0)=0,\ \ j=0,\dots,m-2; \quad D_{0+}^{\beta_2}v(0)=0,\ \ D_{0+}^{p_2}v(1)=\sum_{i=1}^{M} b_i D_{0+}^{q_2}v(\eta_i), \end{aligned} \tag{BC}$$

where $\alpha_1,\ \alpha_2 \in (0,1]$, $\beta_1 \in (n-1,n]$, $\beta_2 \in (m-1,m]$, $n,\ m \in \mathbb{N}$, $n,m \geqslant 3$, $p_1,p_2,q_1,q_2 \in \mathbb{R}$, $p_1 \in [1,n-2]$, $p_2 \in [1,m-2]$, $q_1 \in [0,p_1]$, $q_2 \in [0,p_2]$, $\xi_i,\ a_i \in \mathbb{R}$ for all $i=1,\dots,N$ ($N\in\mathbb{N}$), $0<\xi_1<\cdots<\xi_N \leqslant 1$, $\eta_i,\ b_i \in \mathbb{R}$ for all $i=1,\dots,M$

($M \in \mathbb{N}$), $0 < \eta_1 < \cdots < \eta_M \leqslant 1$, $r_1, r_2 > 1$, $\varphi_{r_i}(s) = |s|^{r_i-2}s$, $\varphi_{r_i}^{-1} = \varphi_{\varrho_i}$, $1/r_i + 1/\varrho_i = 1$, $i = 1, 2$, $\lambda, \mu > 0$, $f, g \in C([0,1] \times [0,\infty) \times [0,\infty), [0,\infty))$, and D_{0+}^k denotes the Riemann–Liouville derivative of order k (for $k = \alpha_1, \beta_1, \alpha_2, \beta_2, p_1, q_1, p_2, q_2$).

Under some assumptions on the functions f and g, we give intervals for the parameters λ and μ such that positive solutions of (S)–(BC) exist. A positive solution of problem (S)–(BC) is a pair of functions $(u, v) \in (C([0,1], [0,\infty)))^2$ satisfying (S) and (BC) with $u(t) > 0$ for all $t \in (0,1]$, or $v(t) > 0$ for all $t \in (0,1]$. The nonexistence of positive solutions for the above problem is also studied.

Systems with fractional differential equations without p-Laplacian operator subject to various multi-point or Riemann–Stieltjes integral boundary conditions were studied in the last years in [6–13, 15, 16, 20, 21, 23, 24]. Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, biology, economics, control theory, signal and image processing, aerodynamics, viscoelasticity, electromagnetics, and so on (see [1–3, 5, 14, 17–19, 22]).

The paper is organized as follows. In Section 2, we investigate two nonlocal boundary value problems for fractional differential equations with p-Laplacian, and we present some properties of the associated Green functions. Section 3 contains the main existence theorems for the positive solutions with respect to a cone for our problem (S)–(BC) based on the Guo–Krasnosel'skii fixed-point theorem (see [4]). In Section 4, we study the nonexistence of positive solutions of (S)–(BC), and in Section 5, an example is given to support our results. In Appendix we prove a relation between the supremum limits of two functions, which is used in the proof of the second existence result.

2 Auxiliary results

First, we consider the nonlinear fractional differential equation

$$D_{0+}^{\alpha_1}\left(\varphi_{r_1}\left(D_{0+}^{\beta_1}u(t)\right)\right) + h(t) = 0, \quad t \in (0,1), \tag{1}$$

with the boundary conditions

$$u^{(j)}(0) = 0,\ \ j = 0, \dots, n-2; \quad D_{0+}^{\beta_1}u(0) = 0,\ \ D_{0+}^{p_1}u(1) = \sum_{i=1}^{N} a_i D_{0+}^{q_1}u(\xi_i), \tag{2}$$

where $\alpha_1 \in (0,1]$, $\beta_1 \in (n-1, n]$, $n \in \mathbb{N}$, $n \geqslant 3$, $p_1, q_1 \in \mathbb{R}$, $p_1 \in [1, n-2]$, $q_1 \in [0, p_1]$, $\xi_i, a_i \in \mathbb{R}$ for all $i = 1, \dots, N$ ($N \in \mathbb{N}$), $0 < \xi_1 < \cdots < \xi_N \leqslant 1$, and $h \in C[0,1]$.

If we denote by $\varphi_{r_1}(D_{0+}^{\beta_1}u(t)) = x(t)$, then problem (1)–(2) is equivalent to the following two boundary value problems:

$$D_{0+}^{\alpha_1}x(t) + h(t) = 0, \quad 0 < t < 1, \tag{3}$$

with the boundary condition

$$x(0) = 0, \tag{4}$$

and

$$D_{0^+}^{\beta_1} u(t) = \varphi_{\varrho_1} x(t), \quad 0 < t < 1, \tag{5}$$

with the boundary conditions

$$u^{(j)}(0) = 0, \ \ j = 0, \ldots, n-2; \quad D_{0^+}^{p_1} u(1) = \sum_{i=1}^{N} a_i D_{0^+}^{q_1} u(\xi_i). \tag{6}$$

For the first problem (3)–(4), the function

$$x(t) = -I_{0^+}^{\alpha_1} h(t) = -\frac{1}{\Gamma(\alpha_1)} \int\limits_0^t (t-s)^{\alpha_1 - 1} h(s)\,\mathrm{d}s, \quad t \in [0,1] \tag{7}$$

is solution of (3)–(4).

For the second problem (5)–(6), if $\varDelta_1 = \Gamma(\beta_1)/\Gamma(\beta_1 - p_1) - \Gamma(\beta_1)/\Gamma(\beta_1 - q_1) \times \sum_{i=1}^{N} a_i \xi_i^{\beta_1 - q_1 - 1} \neq 0$, then by [7, Lemma 2.2] we deduce that the function

$$u(t) = -\int\limits_0^1 G_1(t,s) \varphi_{\varrho_1} x(s)\,\mathrm{d}s, \quad t \in [0,1], \tag{8}$$

is solution of (5)–(6). Here the Green functions G_1, g_1, g_2 are given by

$$G_1(t,s) = g_1(t,s) + \frac{t^{\beta_1 - 1}}{\varDelta_1} \sum_{i=1}^{N} a_i g_2(\xi_i, s), \tag{9}$$

$$g_1(t,s) = \frac{1}{\Gamma(\beta_1)} \begin{cases} t^{\beta_1 - 1}(1-s)^{\beta_1 - p_1 - 1} - (t-s)^{\beta_1 - 1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\beta_1 - 1}(1-s)^{\beta_1 - p_1 - 1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{10}$$

$$g_2(t,s) = \frac{1}{\Gamma(\beta_1 - q_1)} \begin{cases} t^{\beta_1 - q_1 - 1}(1-s)^{\beta_1 - p_1 - 1} - (t-s)^{\beta_1 - q_1 - 1}, \\ \quad 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\beta_1 - q_1 - 1}(1-s)^{\beta_1 - p_1 - 1}, \quad 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{11}$$

Therefore, by (7) and (8) we obtain the following lemma.

Lemma 1. *If $\varDelta_1 \neq 0$, then the function $u \in C[0,1]$ given by*

$$u(t) = \int\limits_0^1 G_1(t,s) \varphi_{\varrho_1}\bigl(I_{0^+}^{\alpha_1} h(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1], \tag{12}$$

is solution of problem (1)–(2).

Next, we consider the nonlinear fractional differential equation

$$D_{0+}^{\alpha_2}\big(\varphi_{r_2}\big(D_{0+}^{\beta_2}v(t)\big)\big)+k(t)=0,\quad t\in(0,1), \tag{13}$$

with the boundary conditions

$$v^{(j)}(0)=0,\ \ j=0,\dots,m-2;\quad D_{0+}^{\beta_2}v(0)=0,\quad D_{0+}^{p_2}v(1)=\sum_{i=1}^{M}b_iD_{0+}^{q_2}v(\eta_i), \tag{14}$$

where $\alpha_2\in(0,1]$, $\beta_2\in(m-1,m]$, $m\in\mathbb{N}$, $m\geqslant 3$, $p_2,q_2\in\mathbb{R}$, $p_2\in[1,m-2]$, $q_2\in[0,p_2]$, $\eta_i,b_i\in\mathbb{R}$ for all $i=1,\dots,M$ ($M\in\mathbb{N}$), $0<\eta_1<\dots<\eta_M\leqslant 1$, and $k\in C[0,1]$.

We denote by $\Delta_2=\Gamma(\beta_2)/\Gamma(\beta_2-p_2)-\Gamma(\beta_2)/\Gamma(\beta_2-q_2)\sum_{i=1}^{M}b_i\eta_i^{\beta_2-q_2-1}$ and by G_2, g_3, g_4 the following Green functions:

$$G_2(t,s)=g_3(t,s)+\frac{t^{\beta_2-1}}{\Delta_2}\sum_{i=1}^{M}b_ig_4(\eta_i,s), \tag{15}$$

$$g_3(t,s)=\frac{1}{\Gamma(\beta_2)}\begin{cases}t^{\beta_2-1}(1-s)^{\beta_2-p_2-1}-(t-s)^{\beta_2-1}, & 0\leqslant s\leqslant t\leqslant 1,\\ t^{\beta_2-1}(1-s)^{\beta_2-p_2-1}, & 0\leqslant t\leqslant s\leqslant 1,\end{cases} \tag{16}$$

$$g_4(t,s)=\frac{1}{\Gamma(\beta_2-q_2)}\begin{cases}t^{\beta_2-q_2-1}(1-s)^{\beta_2-p_2-1}-(t-s)^{\beta_2-q_2-1},\\ \quad 0\leqslant s\leqslant t\leqslant 1,\\ t^{\beta_2-q_2-1}(1-s)^{\beta_2-p_2-1},\\ \quad 0\leqslant t\leqslant s\leqslant 1.\end{cases} \tag{17}$$

In a similar manner as above, we obtain the following result.

Lemma 2. *If $\Delta_2\neq 0$, then the function $v\in C[0,1]$ given by*

$$v(t)=\int_0^1 G_2(t,s)\varphi_{\varrho_2}\big(I_{0+}^{\alpha_2}k(s)\big)\,\mathrm{d}s,\quad t\in[0,1], \tag{18}$$

is solution of problem (13)–(14).

For some properties of the functions g_i, $i=1,\dots,4$, given by (10), (11), (16), and (17), we refer the reader to [7, Lemma 2.3]. We present now some properties of the Green functions G_1 and G_2 that will be used in the next sections.

Lemma 3. (*See* [7].) *Assume that $a_i,b_j\geqslant 0$ for all $i=1,\dots,N$ and $j=1,\dots,M$ and $\Delta_1,\Delta_2>0$. Then for the functions G_1, G_2 given by* (9) *and* (15)*, respectively, the following hold*:

(i) $G_1,G_2:[0,1]\times[0,1]\to[0,\infty)$ *are continuous functions*;
(ii) $G_1(t,s)\leqslant J_1(s)$ *for all* $t,s\in[0,1]$, *where* $J_1(s)=h_1(s)+\sum_{i=1}^{N}a_ig_2(\xi_i,s)/\Delta_1$, $s\in[0,1]$, *and* $h_1(s)=(1-s)^{\beta_1-p_1-1}(1-(1-s)^{p_1})/\Gamma(\beta_1)$, $s\in[0,1]$;

(iii) $G_1(t,s) \geqslant t^{\beta_1-1}J_1(s)$ *for all* $t,s \in [0,1]$;
(iv) $G_2(t,s) \leqslant J_2(s)$ *for all* $t,s \in [0,1]$, *where* $J_2(s) = h_3(s) + \sum_{i=1}^{M} b_i g_4(\eta_i, s)/\Delta_2$, $s \in [0,1]$, *and* $h_3(s) = (1-s)^{\beta_2-p_2-1}(1-(1-s)^{p_2})/\Gamma(\beta_2)$, $s \in [0,1]$;
(v) $G_2(t,s) \geqslant t^{\beta_2-1}J_2(s)$ *for all* $t,s \in [0,1]$.

3 Existence of positive solutions

In this section, we present sufficient conditions on the functions f, g and intervals for the parameters λ, μ such that positive solutions with respect to a cone for our problem (S)–(BC) exist.

We present now the assumptions that we will use in the sequel.

(H1) $\alpha_1, \alpha_2 \in (0,1]$, $\beta_1 \in (n-1,n]$, $\beta_2 \in (m-1,m]$, $n,m \in \mathbb{N}$, $n,m \geqslant 3$, $p_1,p_2,q_1,q_2 \in \mathbb{R}$, $p_1 \in [1,n-2]$, $p_2 \in [1,m-2]$, $q_1 \in [0,p_1]$, $q_2 \in [0,p_2]$, $\xi_i \in \mathbb{R}$, $a_i \geqslant 0$ for all $i = 1,\dots,N$ ($N \in \mathbb{N}$), $0 < \xi_1 < \dots < \xi_N \leqslant 1$, $\eta_i \in \mathbb{R}$, $b_i \geqslant 0$ for all $i = 1,\dots,M$ ($M \in \mathbb{N}$), $0 < \eta_1 < \dots < \eta_M \leqslant 1$, $\lambda,\mu > 0$, $\Delta_1 = \Gamma(\beta_1)/\Gamma(\beta_1-p_1) - (\Gamma(\beta_1)/\Gamma(\beta_1-q_1))\sum_{i=1}^{N} a_i \xi_i^{\beta_1-q_1-1} > 0$, $\Delta_2 = \Gamma(\beta_2)/\Gamma(\beta_2-p_2) - (\Gamma(\beta_2)/\Gamma(\beta_2-q_2))\sum_{i=1}^{M} b_i \eta_i^{\beta_2-q_2-1} > 0$, $r_i > 1$, $\varphi_{r_i}(s) = |s|^{r_i-2}s$, $\varphi_{r_i}^{-1} = \varphi_{\varrho_i}$, $\varrho_i = r_i/(r_i-1)$, $i = 1,2$.
(H2) The functions $f,g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous.

For $[c_1,c_2] \subset [0,1]$ with $0 < c_1 < c_2 \leqslant 1$, we introduce the following extreme limits:

$$f_0^s = \limsup_{\substack{u+v\to 0^+\\ u,v\geqslant 0}} \max_{t\in[0,1]} \frac{f(t,u,v)}{(u+v)^{r_1-1}}, \qquad g_0^s = \limsup_{\substack{u+v\to 0^+\\ u,v\geqslant 0}} \max_{t\in[0,1]} \frac{g(t,u,v)}{(u+v)^{r_2-1}},$$

$$f_0^i = \liminf_{\substack{u+v\to 0^+\\ u,v\geqslant 0}} \min_{t\in[c_1,c_2]} \frac{f(t,u,v)}{(u+v)^{r_1-1}}, \qquad g_0^i = \liminf_{\substack{u+v\to 0^+\\ u,v\geqslant 0}} \min_{t\in[c_1,c_2]} \frac{g(t,u,v)}{(u+v)^{r_2-1}},$$

$$f_\infty^s = \limsup_{\substack{u+v\to \infty\\ u,v\geqslant 0}} \max_{t\in[0,1]} \frac{f(t,u,v)}{(u+v)^{r_1-1}}, \qquad g_\infty^s = \limsup_{\substack{u+v\to \infty\\ u,v\geqslant 0}} \max_{t\in[0,1]} \frac{g(t,u,v)}{(u+v)^{r_2-1}},$$

$$f_\infty^i = \liminf_{\substack{u+v\to \infty\\ u,v\geqslant 0}} \min_{t\in[c_1,c_2]} \frac{f(t,u,v)}{(u+v)^{r_1-1}}, \qquad g_\infty^i = \liminf_{\substack{u+v\to \infty\\ u,v\geqslant 0}} \min_{t\in[c_1,c_2]} \frac{g(t,u,v)}{(u+v)^{r_2-1}}.$$

By using Lemmas 1 and 2 (relations (12) and (18)) a solution of the nonlinear system of integral equations

$$u(t) = \lambda^{\varrho_1-1} \int_0^1 G_1(t,s)\varphi_{\varrho_1}\bigl(I_{0+}^{\alpha_1} f\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s, \quad t \in [0,1],$$
$$v(t) = \mu^{\varrho_2-1} \int_0^1 G_2(t,s)\varphi_{\varrho_2}\bigl(I_{0+}^{\alpha_2} g\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s, \quad t \in [0,1],$$

is solution of problem (S)–(BC).

We consider the Banach space $X = C[0,1]$ with the supremum norm $\|\cdot\|$ and the Banach space $Y = X \times X$ with the norm $\|(u,v)\|_Y = \|u\| + \|v\|$. We define the cones

$$P_1 = \{u \in X\colon\ u(t) \geqslant t^{\beta_1-1}\|u\|\ \forall t \in [0,1]\} \subset X,$$
$$P_2 = \{v \in X\colon\ v(t) \geqslant t^{\beta_2-1}\|v\|\ \forall t \in [0,1]\} \subset X,$$

and $P = P_1 \times P_2 \subset Y$.

We define now the operators $Q_1, Q_2 : Y \to X$ and $Q : Y \to Y$ by

$$Q_1(u,v)(t) = \lambda^{\varrho_1-1} \int_0^1 G_1(t,s)\varphi_{\varrho_1}\bigl(I_{0+}^{\alpha_1} f\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s, \quad t \in [0,1],$$
$$Q_2(u,v)(t) = \mu^{\varrho_2-1} \int_0^1 G_2(t,s)\varphi_{\varrho_2}\bigl(I_{0+}^{\alpha_2} g\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s, \quad t \in [0,1],$$

and $Q(u,v) = (Q_1(u,v), Q_2(u,v))$, $(u,v) \in Y$. If (u,v) is a fixed point of operator Q, then (u,v) is a solution of problem (S)–(BC).

Lemma 4. *If* (H1)–(H2) *hold, then $Q : P \to P$ is a completely continuous operator.*

Proof. Let $(u,v) \in P$ be an arbitrary element. Because $Q_1(u,v)$ and $Q_2(u,v)$ satisfy problem (1)–(2) for $h(t) = \lambda f(t,u(t),v(t))$, $t \in [0,1]$ and problem (13)–(14) for $k(t) = \mu g(t,u(t),v(t))$, $t \in [0,1]$, respectively, then we obtain

$$Q_1(u,v)(t) \leqslant \lambda^{\varrho_1-1} \int_0^1 J_1(s)\varphi_{\varrho_1}\bigl(I_{0+}^{\alpha_1} f\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s, \quad t \in [0,1],$$
$$Q_2(u,v)(t) \leqslant \mu^{\varrho_2-1} \int_0^1 J_2(s)\varphi_{\varrho_2}\bigl(I_{0+}^{\alpha_2} g\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s, \quad t \in [0,1],$$

and so

$$\bigl\|Q_1(u,v)\bigr\| \leqslant \lambda^{\varrho_1-1} \int_0^1 J_1(s)\varphi_{\varrho_1}\bigl(I_{0+}^{\alpha_1} f\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s,$$
$$\bigl\|Q_2(u,v)\bigr\| \leqslant \mu^{\varrho_2-1} \int_0^1 J_2(s)\varphi_{\varrho_2}\bigl(I_{0+}^{\alpha_2} g\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s.$$

Therefore, for all $t \in [0,1]$, we conclude that

$$Q_1(u,v)(t) \geqslant \lambda^{\varrho_1-1} \int_0^1 t^{\beta_1-1} J_1(s)\varphi_{\varrho_1}\bigl(I_{0+}^{\alpha_1} f\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s \geqslant t^{\beta_1-1}\bigl\|Q_1(u,v)\bigr\|,$$

$$Q_2(u,v)(t) \geqslant \mu^{\varrho_2-1}\int_0^1 t^{\beta_2-1}J_2(s)\varphi_{\varrho_2}\bigl(I_{0+}^{\alpha_2}g\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s \geqslant t^{\beta_2-1}\bigl\|Q_2(u,v)\bigr\|.$$

Hence, $Q(u,v) = (Q_1(u,v), Q_2(u,v)) \in P$, and then $Q(P) \subset P$. By the continuity of the functions f, g, G_1, G_2 and the Ascoli–Arzela theorem we can show that Q_1 and Q_2 are completely continuous operators (compact operators, that is they map bounded sets into relatively compact sets, and continuous), and then Q is a completely continuous operator. □

For $[c_1, c_2] \subset [0,1]$ with $0 < c_1 < c_2 \leqslant 1$, we denote by

$$A = \frac{\int_{c_1}^{c_2}(s-c_1)^{\alpha_1(\varrho_1-1)}J_1(s)\,\mathrm{d}s}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}, \qquad B = \frac{\int_0^1 s^{\alpha_1(\varrho_1-1)}J_1(s)\,\mathrm{d}s}{(\Gamma(\alpha_1+1))^{\varrho_1-1}},$$
$$C = \frac{\int_{c_1}^{c_2}(s-c_1)^{\alpha_2(\varrho_2-1)}J_2(s)\,\mathrm{d}s}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}, \qquad D = \frac{\int_0^1 s^{\alpha_2(\varrho_2-1)}J_2(s)\,\mathrm{d}s}{(\Gamma(\alpha_2+1))^{\varrho_2-1}},$$

where J_1 and J_2 are defined in Lemma 3.

First, for $f_0^s, g_0^s, f_\infty^i, g_\infty^i \in (0,\infty)$ and numbers $\alpha_1', \alpha_2' \geqslant 0$, $\widetilde{\alpha}_1, \widetilde{\alpha}_2 > 0$ such that $\alpha_1' + \alpha_2' = 1$ and $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$, we define the numbers L_1, L_2, L_3, L_4, L_2', L_4' by

$$L_1 = \frac{1}{f_\infty^i}\left(\frac{\alpha_1'}{\gamma\gamma_1 A}\right)^{r_1-1}, \quad L_2 = \frac{1}{f_0^s}\left(\frac{\widetilde{\alpha}_1}{B}\right)^{r_1-1}, \quad L_3 = \frac{1}{g_\infty^i}\left(\frac{\alpha_2'}{\gamma\gamma_2 C}\right)^{r_2-1},$$
$$L_4 = \frac{1}{g_0^s}\left(\frac{\widetilde{\alpha}_2}{D}\right)^{r_2-1}, \qquad L_2' = \frac{1}{f_0^s B^{r_1-1}}, \qquad L_4' = \frac{1}{g_0^s D^{r_2-1}},$$

where $\gamma_1 = c_1^{\beta_1-1}$, $\gamma_2 = c_1^{\beta_2-1}$, $\gamma = \min\{\gamma_1, \gamma_2\}$.

Theorem 1. *Assume that* (H1) *and* (H2) *hold,* $[c_1, c_2] \subset [0,1]$ *with* $0 < c_1 < c_2 \leqslant 1$, $\alpha_1', \alpha_2' \geqslant 0$, $\widetilde{\alpha}_1, \widetilde{\alpha}_2 > 0$ *such that* $\alpha_1' + \alpha_2' = 1$, $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$.

- (i) *If* $f_0^s, g_0^s, f_\infty^i, g_\infty^i \in (0,\infty)$, $L_1 < L_2$, *and* $L_3 < L_4$, *then for each* $\lambda \in (L_1, L_2)$ *and* $\mu \in (L_3, L_4)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).
- (ii) *If* $f_0^s = 0$, $g_0^s, f_\infty^i, g_\infty^i \in (0,\infty)$, *and* $L_3 < L_4'$, *then for each* $\lambda \in (L_1, \infty)$ *and* $\mu \in (L_3, L_4')$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).
- (iii) *If* $g_0^s = 0$, $f_0^s, f_\infty^i, g_\infty^i \in (0,\infty)$, *and* $L_1 < L_2'$, *then for each* $\lambda \in (L_1, L_2')$ *and* $\mu \in (L_3, \infty)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).
- (iv) *If* $f_0^s = g_0^s = 0$, $f_\infty^i, g_\infty^i \in (0,\infty)$, *then for each* $\lambda \in (L_1, \infty)$ *and* $\mu \in (L_3, \infty)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).
- (v) *If* $f_0^s, g_0^s \in (0,\infty)$ *and at least one of* f_∞^i, g_∞^i *is* ∞, *then for each* $\lambda \in (0, L_2)$ *and* $\mu \in (0, L_4)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).
- (vi) *If* $f_0^s = 0$, $g_0^s \in (0,\infty)$, *and at least one of* f_∞^i, g_∞^i *is* ∞, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0, L_4')$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).

(vii) *If* $f_0^s \in (0,\infty)$, $g_0^s = 0$, *and at least one of* f_∞^i, g_∞^i *is* ∞, *then for each* $\lambda \in (0, L_2')$ *and* $\mu \in (0,\infty)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).

(viii) *If* $f_0^s = g_0^s = 0$ *and at least one of* f_∞^i, g_∞^i *is* ∞, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0,\infty)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).

Proof. We consider the above cone $P \subset Y$ and the operators Q_1, Q_2, and Q. Because the proofs of the above cases are similar, in what follows, we will prove some representative cases.

(i) We have $f_0^s, g_0^s, f_\infty^i, g_\infty^i \in (0,\infty)$, $L_1 < L_2$, and $L_3 < L_4$. Let $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$. We consider $\varepsilon > 0$ such that $\varepsilon < f_\infty^i$, $\varepsilon < g_\infty^i$, and

$$\frac{1}{f_\infty^i - \varepsilon}\left(\frac{\alpha_1'}{\gamma\gamma_1 A}\right)^{r_1-1} \leqslant \lambda \leqslant \frac{1}{f_0^s + \varepsilon}\left(\frac{\widetilde{\alpha}_1}{B}\right)^{r_1-1},$$
$$\frac{1}{g_\infty^i - \varepsilon}\left(\frac{\alpha_2'}{\gamma\gamma_2 C}\right)^{r_2-1} \leqslant \mu \leqslant \frac{1}{g_0^s + \varepsilon}\left(\frac{\widetilde{\alpha}_2}{D}\right)^{r_2-1}.$$

By using (H2) and the definitions of f_0^s and g_0^s, we deduce that there exists $R_1 > 0$ such that

$$f(t,u,v) \leqslant \big(f_0^s + \varepsilon\big)(u+v)^{r_1-1}, \qquad g(t,u,v) \leqslant \big(g_0^s + \varepsilon\big)(u+v)^{r_2-1}$$

for all $t \in [0,1]$ and $u, v \geqslant 0$, $u + v \leqslant R_1$.

We define the set $\Omega_1 = \{(u,v) \in Y\colon \|(u,v)\|_Y < R_1\}$. Now let $(u,v) \in P \cap \partial\Omega_1$, that is $(u,v) \in P$ with $\|(u,v)\|_Y = R_1$, or, equivalently, $\|u\| + \|v\| = R_1$. Then $u(t) + v(t) \leqslant R_1$ for all $t \in [0,1]$, and by Lemma 3 we obtain

$$\begin{aligned}
&Q_1(u,v)(t)\\
&\quad\leqslant \lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{\int_0^s (s-\tau)^{\alpha_1-1} f(\tau, u(\tau), v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad\leqslant \lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{\int_0^s (s-\tau)^{\alpha_1-1}(f_0^s+\varepsilon)(u(\tau)+v(\tau))^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad\leqslant \lambda^{\varrho_1-1}\big(f_0^s+\varepsilon\big)^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{\int_0^s (s-\tau)^{\alpha_1-1}(\|u\|+\|v\|)^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad= \lambda^{\varrho_1-1}\big(f_0^s+\varepsilon\big)^{\varrho_1-1}\big(\|u\|+\|v\|\big)\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{\int_0^s (s-\tau)^{\alpha_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad= \lambda^{\varrho_1-1}\big(f_0^s+\varepsilon\big)^{\varrho_1-1}\big\|(u,v)\big\|_Y\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{s^{\alpha_1}}{\alpha_1\Gamma(\alpha_1)}\right)\mathrm{d}s
\end{aligned}$$

$$= \lambda^{\varrho_1-1}\big(f_0^s+\varepsilon\big)^{\varrho_1-1}\big\|(u,v)\big\|_Y \int\limits_0^1 J_1(s)\frac{s^{\alpha_1(\varrho_1-1)}}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}\,\mathrm{d}s$$
$$= \lambda^{\varrho_1-1}\big(f_0^s+\varepsilon\big)^{\varrho_1-1}B\big\|(u,v)\big\|_Y \leqslant \widetilde{\alpha}_1\big\|(u,v)\big\|_Y \quad \forall t\in[0,1].$$

Therefore, we have $\|Q_1(u,v)\| \leqslant \widetilde{\alpha}_1\|(u,v)\|_Y$.

In a similar manner, we conclude

$$\begin{aligned}
&Q_2(u,v)(t)\\
&\quad\leqslant \mu^{\varrho_2-1}\int\limits_0^1 J_2(s)\varphi_{\varrho_2}\bigg(\frac{\int_0^s(s-\tau)^{\alpha_2-1}g(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\bigg)\,\mathrm{d}s\\
&\quad\leqslant \mu^{\varrho_2-1}\int\limits_0^1 J_2(s)\varphi_{\varrho_2}\bigg(\frac{\int_0^s(s-\tau)^{\alpha_2-1}(g_0^s+\varepsilon)(u(\tau)+v(\tau))^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\bigg)\,\mathrm{d}s\\
&\quad\leqslant \mu^{\varrho_2-1}(g_0^s+\varepsilon)^{\varrho_2-1}\int\limits_0^1 J_2(s)\varphi_{\varrho_2}\bigg(\frac{\int_0^s(s-\tau)^{\alpha_2-1}(\|u\|+\|v\|)^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\bigg)\,\mathrm{d}s\\
&\quad= \mu^{\varrho_2-1}\big(g_0^s+\varepsilon\big)^{\varrho_2-1}\big(\|u\|+\|v\|\big)\int\limits_0^1 J_2(s)\varphi_{\varrho_2}\bigg(\frac{\int_0^s(s-\tau)^{\alpha_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\bigg)\,\mathrm{d}s\\
&\quad= \mu^{\varrho_2-1}\big(g_0^s+\varepsilon\big)^{\varrho_2-1}\big\|(u,v)\big\|_Y\int\limits_0^1 J_2(s)\varphi_{\varrho_2}\bigg(\frac{s^{\alpha_2}}{\alpha_2\Gamma(\alpha_2)}\bigg)\,\mathrm{d}s\\
&\quad= \mu^{\varrho_2-1}\big(g_0^s+\varepsilon\big)^{\varrho_2-1}\big\|(u,v)\big\|_Y\int\limits_0^1 J_2(s)\frac{s^{\alpha_2(\varrho_2-1)}}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}\,\mathrm{d}s\\
&\quad= \mu^{\varrho_2-1}\big(g_0^s+\varepsilon\big)^{\varrho_2-1}D\big\|(u,v)\big\|_Y \leqslant \widetilde{\alpha}_2\big\|(u,v)\big\|_Y \quad \forall t\in[0,1].
\end{aligned}$$

Hence, we get $\|Q_2(u,v)\| \leqslant \widetilde{\alpha}_2\|(u,v)\|_Y$.

Therefore, for $(u,v)\in P\cap\partial\Omega_1$, we deduce

$$\begin{aligned}
\big\|Q(u,v)\big\|_Y &= \big\|Q_1(u,v)\big\| + \big\|Q_2(u,v)\big\| \leqslant \widetilde{\alpha}_1\big\|(u,v)\big\|_Y + \widetilde{\alpha}_2\big\|(u,v)\big\|_Y\\
&= \big\|(u,v)\big\|_Y.
\end{aligned}\tag{19}$$

Next, by the definitions of f_∞^i and g_∞^i there exists $\overline{R}_2>0$ such that

$$f(t,u,v) \geqslant \big(f_\infty^i-\varepsilon\big)(u+v)^{r_1-1}, \qquad g(t,u,v) \geqslant \big(g_\infty^i-\varepsilon\big)(u+v)^{r_2-1}$$

for all $t\in[c_1,c_2]$ and $u,v\geqslant 0$, $u+v\geqslant \overline{R}_2$.

We consider $R_2 = \max\{2R_1, \overline{R}_2/\gamma\}$ and define $\Omega_2 = \{(u,v) \in Y : \|(u,v)\|_Y < R_2\}$. Then for $(u,v) \in P \cap \partial\Omega_2$, we obtain

$$u(t) + v(t) \geqslant \min_{t\in[c_1,c_2]} t^{\beta_1-1}\|u\| + \min_{t\in[c_1,c_2]} t^{\beta_2-1}\|v\| = c_1^{\beta_1-1}\|u\| + c_1^{\beta_2-1}\|v\| \\ = \gamma_1\|u\| + \gamma_2\|v\| \geqslant \gamma\big\|(u,v)\big\|_Y = \gamma R_2 \geqslant \overline{R}_2 \quad \forall t \in [c_1,c_2].$$

Then, by Lemma 3 we conclude

$$\begin{aligned}
&Q_1(u,v)(c_1)\\
&\quad\geqslant \lambda^{\varrho_1-1}\int_0^1 c_1^{\beta_1-1}J_1(s)\varphi_{\varrho_1}\big(I_{0^+}^{\alpha_1}f\big(s,u(s),v(s)\big)\big)\,\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1}f(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1}(f_\infty^i-\varepsilon)(u(\tau)+v(\tau))^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1}(f_\infty^i-\varepsilon)(\gamma\|(u,v)\|_Y)^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad= \lambda^{\varrho_1-1}c_1^{\beta_1-1}\big(f_\infty^i-\varepsilon\big)^{\varrho_1-1}\gamma\big\|(u,v)\big\|_Y\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s(s-\tau)^{\alpha_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad= \lambda^{\varrho_1-1}c_1^{\beta_1-1}\big(f_\infty^i-\varepsilon\big)^{\varrho_1-1}\gamma\big\|(u,v)\big\|_Y\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{(s-c_1)^{\alpha_1}}{\alpha_1\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad= \gamma\gamma_1\lambda^{\varrho_1-1}\big(f_\infty^i-\varepsilon\big)^{\varrho_1-1}\big\|(u,v)\big\|_Y\int_{c_1}^{c_2} J_1(s)\frac{(s-c_1)^{\alpha_1(\varrho_1-1)}}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}\,\mathrm{d}s\\
&\quad= \gamma\gamma_1\lambda^{\varrho_1-1}\big(f_\infty^i-\varepsilon\big)^{\varrho_1-1}A\big\|(u,v)\big\|_Y \geqslant \alpha_1'\big\|(u,v)\big\|_Y.
\end{aligned}$$

Therefore, we obtain $\|Q_1(u,v)\| \geqslant Q_1(u,v)(c_1) \geqslant \alpha_1'\|(u,v)\|_Y$.

In a similar manner, we deduce

$$\begin{aligned}
&Q_2(u,v)(c_1)\\
&\quad\geqslant \mu^{\varrho_2-1}\int_0^1 c_1^{\beta_2-1}J_2(s)\varphi_{\varrho_2}\big(I_{0^+}^{\alpha_2}g\big(s,u(s),v(s)\big)\big)\,\mathrm{d}s\\
&\quad\geqslant \mu^{\varrho_2-1}c_1^{\beta_2-1}\int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1}g(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s
\end{aligned}$$

$$\geqslant \mu^{\varrho_2-1}c_1^{\beta_2-1}\int\limits_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\left(\frac{\int_{c_1}^{s}(s-\tau)^{\alpha_2-1}(g_\infty^i-\varepsilon)(u(\tau)+v(\tau))^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s$$

$$\geqslant \mu^{\varrho_2-1}c_1^{\beta_2-1}\int\limits_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\left(\frac{\int_{c_1}^{s}(s-\tau)^{\alpha_2-1}(g_\infty^i-\varepsilon)(\gamma\|(u,v)\|_Y)^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s$$

$$= \mu^{\varrho_2-1}c_1^{\beta_2-1}\big(g_\infty^i-\varepsilon\big)^{\varrho_2-1}\gamma\big\|(u,v)\big\|_Y\int\limits_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\left(\frac{\int_{c_1}^{s}(s-\tau)^{\alpha_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s$$

$$= \mu^{\varrho_2-1}c_1^{\beta_2-1}\big(g_\infty^i-\varepsilon\big)^{\varrho_2-1}\gamma\big\|(u,v)\big\|_Y\int\limits_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\left(\frac{(s-c_1)^{\alpha_2}}{\alpha_2\Gamma(\alpha_2)}\right)\mathrm{d}s$$

$$= \gamma\gamma_2\mu^{\varrho_2-1}\big(g_\infty^i-\varepsilon\big)^{\varrho_2-1}\big\|(u,v)\big\|_Y\int\limits_{c_1}^{c_2} J_2(s)\frac{(s-c_1)^{\alpha_2(\varrho_2-1)}}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}\,\mathrm{d}s$$

$$= \gamma\gamma_2\mu^{\varrho_2-1}\big(g_\infty^i-\varepsilon\big)^{\varrho_2-1}C\big\|(u,v)\big\|_Y \geqslant \alpha_2'\big\|(u,v)\big\|_Y.$$

Hence, we get $\|Q_2(u,v)\| \geqslant Q_2(u,v)(c_1) \geqslant \alpha_2'\|(u,v)\|_Y$.
Then for $(u,v) \in P\cap\partial\Omega_2$, we obtain

$$\big\|Q(u,v)\big\|_Y = \big\|Q_1(u,v)\big\| + \big\|Q_2(u,v)\big\| \geqslant (\alpha_1'+\alpha_2')\big\|(u,v)\big\|_Y = \big\|(u,v)\big\|_Y. \quad (20)$$

By using Lemma 4, (19), (20), and the Guo–Krasnosel'skii fixed-point theorem we conclude that the operator Q has a fixed point $(u,v) \in P\cap(\overline{\Omega}_2\setminus\Omega_1)$, so $u(t) \geqslant t^{\beta_1-1}\|u\|$, $v(t) \geqslant t^{\beta_2-1}\|v\|$ for all $t \in [0,1]$ and $R_1 \leqslant \|u\|+\|v\| \leqslant R_2$. If $\|u\| > 0$, then $u(t) > 0$ for all $t \in (0,1]$, and if $\|v\| > 0$, then $v(t) > 0$ for all $t \in (0,1]$.

(iii) We have $g_0^s = 0$, $f_0^s, f_\infty^i, g_\infty^i \in (0,\infty)$, and $L_1 < L_2'$. Let $\lambda \in (L_1, L_2')$ and $\mu \in (L_3,\infty)$. Instead of the numbers $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ used in the first case, we choose $\widetilde{\alpha}_1'$ such that $\widetilde{\alpha}_1' \in (B(\lambda f_0^s)^{\varrho_1-1}, 1)$ and $\widetilde{\alpha}_2' = 1-\widetilde{\alpha}_1'$. The choice of the $\widetilde{\alpha}_1'$ is possible because $\lambda < 1/(f_0^s B^{r_1-1})$. Then let $\varepsilon > 0$ with $\varepsilon < f_\infty^i$, $\varepsilon < g_\infty^i$, and

$$\frac{1}{f_\infty^i-\varepsilon}\left(\frac{\alpha_1'}{\gamma\gamma_1 A}\right)^{r_1-1} \leqslant \lambda \leqslant \frac{1}{f_0^s+\varepsilon}\left(\frac{\widetilde{\alpha}_1'}{B}\right)^{r_1-1},$$
$$\frac{1}{g_\infty^i-\varepsilon}\left(\frac{\alpha_2'}{\gamma\gamma_2 C}\right)^{r_2-1} \leqslant \mu \leqslant \frac{1}{\varepsilon}\left(\frac{\widetilde{\alpha}_2'}{D}\right)^{r_2-1}.$$

By using (H2) and the definitions of f_0^s and g_0^s we deduce that there exists $R_1 > 0$ such that

$$f(t,u,v) \leqslant \big(f_0^s+\varepsilon\big)(u+v)^{r_1-1}, \qquad g(t,u,v) \leqslant \varepsilon(u+v)^{r_2-1}$$

for all $t \in [0,1]$ and $u,v \geqslant 0$, $u+v \leqslant R_1$.

We define the set $\Omega_1 = \{(u,v) \in Y\colon \|(u,v)\|_Y < R_1\}$. In a similar manner as in the proof of case (i), for any $(u,v) \in P \cap \partial\Omega_1$, we obtain

$$Q_1(u,v)(t) \leqslant \lambda^{\varrho_1-1}\big(f_0^s+\varepsilon\big)^{\varrho_1-1}B\big\|(u,v)\big\|_Y \leqslant \widetilde{\alpha}_1'\big\|(u,v)\big\|_Y \quad \forall t \in [0,1],$$
$$Q_2(u,v)(t) \leqslant \mu^{\varrho_2-1}\varepsilon^{\varrho_2-1}D\big\|(u,v)\big\|_Y \leqslant \widetilde{\alpha}_2'\big\|(u,v)\big\|_Y \quad \forall t \in [0,1],$$

and so $\|Q(u,v)\|_Y \leqslant (\widetilde{\alpha}_1' + \widetilde{\alpha}_2')\|(u,v)\|_Y = \|(u,v)\|_Y$.

The second part of the proof is the same as the corresponding one from case (i). For Ω_2 defined in case (i) and for any $(u,v) \in P \cap \partial\Omega_2$, we conclude

$$Q_1(u,v)(c_1) \geqslant \gamma\gamma_1\lambda^{\varrho_1-1}\big(f_\infty^i-\varepsilon\big)^{\varrho_1-1}A\big\|(u,v)\big\|_Y \geqslant \alpha_1'\big\|(u,v)\big\|_Y,$$
$$Q_2(u,v)(c_1) \geqslant \gamma\gamma_2\mu^{\varrho_2-1}\big(g_\infty^i-\varepsilon\big)^{\varrho_2-1}C\big\|(u,v)\big\|_Y \geqslant \alpha_2'\big\|(u,v)\big\|_Y,$$

and then $\|Q(u,v)\|_Y \geqslant (\alpha_1' + \alpha_2')\|(u,v)\|_Y = \|(u,v)\|_Y$.

Therefore, we deduce the conclusion of the theorem.

(vi) We consider here $f_0^s = 0$, $g_0^s \in (0,\infty)$, and $f_\infty^i = \infty$. Let $\lambda \in (0,\infty)$ and $\mu \in (0, L_4')$. Instead of the numbers $\widetilde{\alpha}_1$ and $\widetilde{\alpha}_2$ used in the first case, we choose $\widetilde{\alpha}_2'$ such that $\widetilde{\alpha}_2' \in (D(\mu g_0^s)^{\varrho_2-1}, 1)$ and $\widetilde{\alpha}_1' = 1 - \widetilde{\alpha}_2'$. The choice of the $\widetilde{\alpha}_2'$ is possible because $\mu < 1/(g_0^s D^{r_2-1})$. Then let $\varepsilon > 0$ such that

$$\varepsilon\left(\frac{1}{\gamma\gamma_1 A}\right)^{r_1-1} \leqslant \lambda \leqslant \frac{1}{\varepsilon}\left(\frac{\widetilde{\alpha}_1'}{B}\right)^{r_1-1}, \qquad \mu \leqslant \frac{1}{g_0^s+\varepsilon}\left(\frac{\widetilde{\alpha}_2'}{D}\right)^{r_2-1}.$$

By using (H2) and the definitions of f_0^s and g_0^s we deduce that there exists $R_1 > 0$ such that

$$f(t,u,v) \leqslant \varepsilon(u+v)^{r_1-1}, \qquad g(t,u,v) \leqslant \big(g_0^s+\varepsilon\big)(u+v)^{r_2-1}$$

for all $t \in [0,1]$ and $u, v \geqslant 0$, $u + v \leqslant R_1$.

We define the set $\Omega_1 = \{(u,v) \in Y\colon \|(u,v)\|_Y < R_1\}$. In a similar manner as in the proof of case (i), for any $(u,v) \in P \cap \partial\Omega_1$, we obtain

$$Q_1(u,v)(t) \leqslant \lambda^{\varrho_1-1}\varepsilon^{\varrho_1-1}B\big\|(u,v)\big\|_Y \leqslant \widetilde{\alpha}_1'\big\|(u,v)\big\|_Y \quad \forall t \in [0,1],$$
$$Q_2(u,v)(t) \leqslant \mu^{\varrho_2-1}\big(g_0^s+\varepsilon\big)^{\varrho_2-1}D\big\|(u,v)\big\|_Y \leqslant \widetilde{\alpha}_2'\big\|(u,v)\big\|_Y \quad \forall t \in [0,1],$$

and so $\|Q(u,v)\|_Y \leqslant (\widetilde{\alpha}_1' + \widetilde{\alpha}_2')\|(u,v)\|_Y = \|(u,v)\|_Y$.

For the second part of the proof, by the definition of f_∞^i there exists $\overline{R}_2 > 0$ such that

$$f(t,u,v) \geqslant \frac{1}{\varepsilon}(u+v)^{r_1-1} \quad \forall t \in [c_1,c_2],\ u,v \geqslant 0,\ u+v \geqslant \overline{R}_2.$$

We consider $R_2 = \max\{2R_1, \overline{R}_2/\gamma\}$ and define $\Omega_2 = \{(u,v) \in Y\colon \|(u,v)\|_Y < R_2\}$. Then for $(u,v) \in P \cap \partial\Omega_2$, we deduce as in case (i) that $u(t) + v(t) \geqslant \gamma R_2 \geqslant \overline{R}_2$ for all $t \in [c_1, c_2]$.

Then by Lemma 3 we have

$$\begin{aligned}
&Q_1(u,v)(c_1)\\
&\quad\geqslant \lambda^{\varrho_1-1}\int_0^1 c_1^{\beta_1-1}J_1(s)\varphi_{\varrho_1}\bigl(I_{0+}^{\alpha_1}f\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\biggl(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1}f(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\biggr)\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\biggl(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1}\frac{1}{\varepsilon}(u(\tau)+v(\tau))^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\biggr)\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\biggl(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1}\frac{1}{\varepsilon}(\gamma\|(u,v)\|_Y)^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\biggr)\mathrm{d}s\\
&\quad= \lambda^{\varrho_1-1}c_1^{\beta_1-1}\biggl(\frac{1}{\varepsilon}\biggr)^{\varrho_1-1}\gamma\|(u,v)\|_Y\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\biggl(\frac{(s-c_1)^{\alpha_1}}{\Gamma(\alpha_1+1)}\biggr)\mathrm{d}s\\
&\quad= \lambda^{\varrho_1-1}c_1^{\beta_1-1}\biggl(\frac{1}{\varepsilon}\biggr)^{\varrho_1-1}\gamma\|(u,v)\|_Y\int_{c_1}^{c_2} J_1(s)\frac{(s-c_1)^{\alpha_1(\varrho_1-1)}}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}\mathrm{d}s\\
&\quad= \gamma\gamma_1\lambda^{\varrho_1-1}\biggl(\frac{1}{\varepsilon}\biggr)^{\varrho_1-1}A\bigl\|(u,v)\bigr\|_Y\\
&\quad\geqslant \bigl\|(u,v)\bigr\|_Y.
\end{aligned}$$

So we conclude that $\|Q_1(u,v)\| \geqslant Q_1(u,v)(c_1) \geqslant \|(u,v)\|_Y$ and $\|Q(u,v)\|_Y \geqslant \|Q_1(u,v)\| \geqslant \|(u,v)\|_Y$.

Therefore, we deduce the conclusion of the theorem.

(viii) We consider $f_0^s = g_0^s = 0$ and $g_\infty^i = \infty$. Let $\lambda \in (0,\infty)$ and $\mu \in (0,\infty)$, and let $\varepsilon > 0$ such that

$$\lambda \leqslant \frac{1}{\varepsilon(2B)^{r_1-1}}, \qquad \varepsilon\biggl(\frac{1}{\gamma\gamma_2 C}\biggr)^{r_2-1} \leqslant \mu \leqslant \frac{1}{\varepsilon(2D)^{r_2-1}}.$$

By using (H2) and the definition of f_0^s and g_0^s we deduce that there exists $R_1 > 0$ such that

$$f(t,u,v) \leqslant \varepsilon(u+v)^{r_1-1} \qquad g(t,u,v) \leqslant \varepsilon(u+v)^{r_2-1}$$

for all $t \in [0,1]$, $u, v \geqslant 0$, $u + v \leqslant R_1$.

We define the set $\Omega_1 = \{(u,v) \in Y\colon \|(u,v)\|_Y < R_1\}$. In a similar manner as in the proof of case (i), for any $(u,v) \in P \cap \partial\Omega_1$, we obtain

$$Q_1(u,v)(t) \leqslant \lambda^{\varrho_1-1}\varepsilon^{\varrho_1-1}B\big\|(u,v)\big\|_Y \leqslant \frac{1}{2}\big\|(u,v)\big\|_Y \quad \forall t \in [0,1],$$
$$Q_2(u,v)(t) \leqslant \mu^{\varrho_2-1}\varepsilon^{\varrho_2-1}D\big\|(u,v)\big\|_Y \leqslant \frac{1}{2}\big\|(u,v)\big\|_Y, \quad \forall t \in [0,1],$$

and so $\|Q(u,v)\|_Y \leqslant (1/2+1/2)\|(u,v)\|_Y = \|(u,v)\|_Y$.

For the second part of the proof, by the definition of g_∞^i there exists $\overline{R}_2 > 0$ such that

$$g(t,u,v) \geqslant \frac{1}{\varepsilon}(u+v)^{r_2-1} \quad \forall t \in [c_1,c_2],\ u,v \geqslant 0,\ u+v \geqslant \overline{R}_2.$$

We consider $R_2 = \max\{2R_1, \overline{R}_2/\gamma\}$ and define $\Omega_2 = \{(u,v) \in Y\colon \|(u,v)\|_Y < R_2\}$. Then for $(u,v) \in P \cap \partial\Omega_2$, we deduce as in case (i) that $u(t)+v(t) \geqslant \gamma R_2 \geqslant \overline{R}_2$ for all $t \in [c_1,c_2]$.

Then by Lemma 3 we have

$$\begin{aligned}
&Q_2(u,v)(c_1)\\
&\quad\geqslant \mu^{\varrho_2-1}\int_0^1 c_1^{\beta_2-1}J_2(s)\varphi_{\varrho_2}\big(I_{0+}^{\alpha_2}g\big(s,u(s),v(s)\big)\big)\,\mathrm{d}s\\
&\quad\geqslant \mu^{\varrho_2-1}c_1^{\beta_2-1}\int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1}g(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s\\
&\quad\geqslant \mu^{\varrho_2-1}c_1^{\beta_2-1}\int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1}\frac{1}{\varepsilon}(u(\tau)+v(\tau))^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s\\
&\quad\geqslant \mu^{\varrho_2-1}c_1^{\beta_2-1}\int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1}\frac{1}{\varepsilon}(\gamma\|(u,v)\|_Y)^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s\\
&\quad= \mu^{\varrho_2-1}c_1^{\beta_2-1}\left(\frac{1}{\varepsilon}\right)^{\varrho_2-1}\gamma\big\|(u,v)\big\|_Y\int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\left(\frac{(s-c_1)^{\alpha_2}}{\Gamma(\alpha_2+1)}\right)\mathrm{d}s\\
&\quad= \mu^{\varrho_2-1}c_1^{\beta_2-1}\left(\frac{1}{\varepsilon}\right)^{\varrho_2-1}\gamma\big\|(u,v)\big\|_Y\int_{c_1}^{c_2} J_2(s)\frac{(s-c_1)^{\alpha_2(\varrho_2-1)}}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}\mathrm{d}s\\
&\quad= \gamma\gamma_2\mu^{\varrho_2-1}\left(\frac{1}{\varepsilon}\right)^{\varrho_2-1}C\big\|(u,v)\big\|_Y \geqslant \big\|(u,v)\big\|_Y.
\end{aligned}$$

Then we conclude that $\|Q_2(u,v)\| \geqslant Q_2(u,v)(c_1) \geqslant \|(u,v)\|_Y$ and $\|Q(u,v)\|_Y \geqslant \|Q_2(u,v)\| \geqslant \|(u,v)\|_Y$.

Therefore, we deduce the conclusion of the theorem. □

In what follows, for $f_0^i, g_0^i, f_\infty^s, g_\infty^s \in (0,\infty)$ and numbers $\alpha_1', \alpha_2' \geqslant 0$, $\widetilde{\alpha}_1, \widetilde{\alpha}_2 > 0$ such that $\alpha_1' + \alpha_2' = 1$ and $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$, we define the numbers $\widetilde{L}_1$, $\widetilde{L}_2$, $\widetilde{L}_3$, $\widetilde{L}_4$, $\widetilde{L}_2'$, and $\widetilde{L}_4'$ by

$$\widetilde{L}_1 = \frac{1}{f_0^i}\left(\frac{\alpha_1'}{\gamma\gamma_1 A}\right)^{r_1-1}, \quad \widetilde{L}_2 = \frac{1}{f_\infty^s}\left(\frac{\widetilde{\alpha}_1}{B}\right)^{r_1-1}, \quad \widetilde{L}_3 = \frac{1}{g_0^i}\left(\frac{\alpha_2'}{\gamma\gamma_2 C}\right)^{r_2-1},$$
$$\widetilde{L}_4 = \frac{1}{g_\infty^s}\left(\frac{\widetilde{\alpha}_2}{D}\right)^{r_2-1}, \qquad \widetilde{L}_2' = \frac{1}{f_\infty^s B^{r_1-1}}, \qquad \widetilde{L}_4' = \frac{1}{g_\infty^s D^{r_2-1}}.$$

Theorem 2. *Assume that* (H1) *and* (H2) *hold,* $[c_1, c_2] \subset [0,1]$ *with* $0 < c_1 < c_2 \leqslant 1$, $\alpha_1', \alpha_2' \geqslant 0$, $\widetilde{\alpha}_1, \widetilde{\alpha}_2 > 0$ *such that* $\alpha_1' + \alpha_2' = 1$, $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 = 1$.

(i) *If* $f_0^i, g_0^i, f_\infty^s, g_\infty^s \in (0,\infty)$, $\widetilde{L}_1 < \widetilde{L}_2$, *and* $\widetilde{L}_3 < \widetilde{L}_4$, *then for each* $\lambda \in (\widetilde{L}_1, \widetilde{L}_2)$ *and* $\mu \in (\widetilde{L}_3, \widetilde{L}_4)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).

(ii) *If* $f_0^i, g_0^i, f_\infty^s \in (0,\infty)$, $g_\infty^s = 0$, *and* $\widetilde{L}_1 < \widetilde{L}_2'$, *then for each* $\lambda \in (\widetilde{L}_1, \widetilde{L}_2')$ *and* $\mu \in (\widetilde{L}_3, \infty)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).

(iii) *If* $f_0^i, g_0^i, g_\infty^s \in (0,\infty)$, $f_\infty^s = 0$, *and* $\widetilde{L}_3 < \widetilde{L}_4'$, *then for each* $\lambda \in (\widetilde{L}_1, \infty)$ *and* $\mu \in (\widetilde{L}_3, \widetilde{L}_4')$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).

(iv) *If* $f_0^i, g_0^i \in (0,\infty)$, $f_\infty^s = g_\infty^s = 0$, *then for each* $\lambda \in (\widetilde{L}_1, \infty)$ *and* $\mu \in (\widetilde{L}_3, \infty)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).

(v) *If* $f_\infty^s, g_\infty^s \in (0,\infty)$ *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0, \widetilde{L}_2)$ *and* $\mu \in (0, \widetilde{L}_4)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).

(vi) *If* $f_\infty^s \in (0,\infty)$, $g_\infty^s = 0$, *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0, \widetilde{L}_2')$ *and* $\mu \in (0,\infty)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).

(vii) *If* $f_\infty^s = 0$, $g_\infty^s \in (0,\infty)$, *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0, \widetilde{L}_4')$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).

(viii) *If* $f_\infty^s = g_\infty^s = 0$ *and at least one of* f_0^i, g_0^i *is* ∞, *then for each* $\lambda \in (0,\infty)$ *and* $\mu \in (0,\infty)$, *there exists a positive solution* $(u(t), v(t))$, $t \in [0,1]$, *for* (S)–(BC).

Proof. We consider the cone $P \subset Y$ and the operators Q_1, Q_2, and Q defined at the beginning of this section. Because the proofs of the above cases are similar, in what follows we will prove some representative cases.

(i) We have $f_0^i, g_0^i, f_\infty^s, g_\infty^s \in (0,\infty)$, $\widetilde{L}_1 < \widetilde{L}_2$, and $\widetilde{L}_3 < \widetilde{L}_4$. Let $\lambda \in (\widetilde{L}_1, \widetilde{L}_2)$ and $\mu \in (\widetilde{L}_3, \widetilde{L}_4)$. We consider $\varepsilon > 0$ such that $\varepsilon < f_0^i$, $\varepsilon < g_0^i$, and

$$\frac{1}{f_0^i - \varepsilon}\left(\frac{\alpha_1'}{\gamma\gamma_1 A}\right)^{r_1-1} \leqslant \lambda \leqslant \frac{1}{f_\infty^s + \varepsilon}\left(\frac{\widetilde{\alpha}_1}{B}\right)^{r_1-1},$$
$$\frac{1}{g_0^i - \varepsilon}\left(\frac{\alpha_2'}{\gamma\gamma_2 C}\right)^{r_2-1} \leqslant \mu \leqslant \frac{1}{g_\infty^s + \varepsilon}\left(\frac{\widetilde{\alpha}_2}{D}\right)^{r_2-1}.$$

By using (H2) and the definitions of f_0^i and g_0^i we deduce that there exists $R_3 > 0$ such that

$$f(t,u,v) \geqslant \left(f_0^i - \varepsilon\right)(u+v)^{r_1-1}, \qquad g(t,u,v) \geqslant \left(g_0^i - \varepsilon\right)(u+v)^{r_2-1}$$

for all $t \in [c_1, c_2]$, $u, v \geqslant 0$, $u + v \leqslant R_3$.

We denote $\Omega_3 = \{(u,v) \in Y\colon\ \|(u,v)\|_Y < R_3\}$. Let $(u,v) \in P$ with $\|(u,v)\|_Y = R_3$, that is $\|u\| + \|v\| = R_3$. Because $u(t) + v(t) \leqslant \|u\| + \|v\| = R_3$ for all $t \in [0,1]$, then by Lemma 3 we obtain

$$\begin{aligned}
&Q_1(u,v)(c_1)\\
&\quad\geqslant \lambda^{\varrho_1-1}\int_0^1 c_1^{\beta_1-1} J_1(s)\varphi_{\varrho_1}\left(I_{0+}^{\alpha_1} f\left(s,u(s),v(s)\right)\right)\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1} f(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1} (f_0^i-\varepsilon)(u(\tau)+v(\tau))^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1} (f_0^i-\varepsilon)(\gamma\|(u,v)\|_Y)^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad= \lambda^{\varrho_1-1}c_1^{\beta_1-1}\left(f_0^i-\varepsilon\right)^{\varrho_1-1}\gamma\left\|(u,v)\right\|_Y\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad= \gamma\gamma_1\lambda^{\varrho_1-1}\left(f_0^i-\varepsilon\right)^{\varrho_1-1}A\left\|(u,v)\right\|_Y \geqslant \alpha_1'\left\|(u,v)\right\|_Y.
\end{aligned}$$

Therefore, we conclude $\|Q_1(u,v)\| \geqslant Q_1(u,v)(c_1) \geqslant \alpha_1'\|(u,v)\|_Y$.

In a similar manner, we deduce

$$\begin{aligned}
&Q_2(u,v)(c_1)\\
&\quad\geqslant \mu^{\varrho_2-1}\int_0^1 c_1^{\beta_2-1} J_2(s)\varphi_{\varrho_2}\left(I_{0+}^{\alpha_2} g\left(s,u(s),v(s)\right)\right)\mathrm{d}s\\
&\quad\geqslant \mu^{\varrho_2-1}c_1^{\beta_2-1}\int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1} g(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s\\
&\quad\geqslant \mu^{\varrho_2-1}c_1^{\beta_2-1}\int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1} (g_0^i-\varepsilon)(u(\tau)+v(\tau))^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s
\end{aligned}$$

$$\geqslant \mu^{\varrho_2-1} c_1^{\beta_2-1} \int\limits_{c_1}^{c_2} J_2(s) \varphi_{\varrho_2}\left(\frac{\int_{c_1}^{s} (s-\tau)^{\alpha_2-1} (g_0^i-\varepsilon)(\gamma \|(u,v)\|_Y)^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)} \right) \mathrm{d}s$$

$$= \mu^{\varrho_2-1} c_1^{\beta_2-1} \bigl(g_0^i-\varepsilon\bigr)^{\varrho_2-1} \gamma \bigl\|(u,v)\bigr\|_Y \int\limits_{c_1}^{c_2} J_2(s) \varphi_{\varrho_2}\left(\frac{\int_{c_1}^{s} (s-\tau)^{\alpha_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)} \right) \mathrm{d}s$$

$$= \gamma\gamma_2 \mu^{\varrho_2-1} \bigl(g_0^i-\varepsilon\bigr)^{\varrho_2-1} C \bigl\|(u,v)\bigr\|_Y \geqslant \alpha_2' \bigl\|(u,v)\bigr\|_Y.$$

Hence, we get $\|Q_2(u,v)\| \geqslant Q_2(u,v)(c_1) \geqslant \alpha_2'\|(u,v)\|_Y$.

Then for $(u,v) \in P \cap \partial\Omega_2$, we obtain

$$\bigl\|Q(u,v)\bigr\|_Y = \bigl\|Q_1(u,v)\bigr\| + \bigl\|Q_2(u,v)\bigr\| \geqslant (\alpha_1' + \alpha_2')\bigl\|(u,v)\bigr\|_Y = \bigl\|(u,v)\bigr\|_Y. \quad (21)$$

Now we define the functions $f^*, g^* : [0,1] \times [0,\infty) \to [0,\infty)$ by

$$f^*(t,x) = \max_{0 \leqslant u+v \leqslant x} f(t,u,v) \quad \forall t \in [0,1],\ x \in [0,\infty),$$
$$g^*(t,x) = \max_{0 \leqslant u+v \leqslant x} g(t,u,v) \quad \forall t \in [0,1],\ x \in [0,\infty).$$

Then

$$f(t,u,v) \leqslant f^*(t,x), \quad g(t,u,v) \leqslant g^*(t,x) \quad \forall t \in [0,1],\ u,v \geqslant 0,\ u+v \leqslant x.$$

The functions $f^*(t,\cdot)$, $g^*(t,\cdot)$ are nondecreasing for every $t \in [0,1]$, and they satisfy the conditions (see Appendix)

$$\limsup_{x \to \infty} \max_{t \in [0,1]} \frac{f^*(t,x)}{x^{r_1-1}} = f_\infty^s, \qquad \limsup_{x \to \infty} \max_{t \in [0,1]} \frac{g^*(t,x)}{x^{r_2-1}} = g_\infty^s.$$

Therefore, for $\varepsilon > 0$, there exists $\overline{R}_4 > 0$ such that for all $x \geqslant \overline{R}_4$ and $t \in [0,1]$, we have

$$\frac{f^*(t,x)}{x^{r_1-1}} \leqslant \limsup_{x \to \infty} \max_{t \in [0,1]} \frac{f^*(t,x)}{x^{r_1-1}} + \varepsilon = f_\infty^s + \varepsilon,$$
$$\frac{g^*(t,x)}{x^{r_2-1}} \leqslant \limsup_{x \to \infty} \max_{t \in [0,1]} \frac{g^*(t,x)}{x^{r_2-1}} + \varepsilon = g_\infty^s + \varepsilon,$$

and so $f^*(t,x) \leqslant (f_\infty^s + \varepsilon)x^{r_1-1}$ and $g^*(t,x) \leqslant (g_\infty^s + \varepsilon)x^{r_2-1}$.

We consider $R_4 = \max\{2R_3, \overline{R}_4\}$ and denote $\Omega_4 = \{(u,v) \in Y\colon \|(u,v)\|_Y < R_4\}$. Let $(u,v) \in P \cap \partial\Omega_4$. By the definitions of f^* and g^* we conclude

$$f\bigl(t,u(t),v(t)\bigr) \leqslant f^*\bigl(t, \bigl\|(u,v)\bigr\|_Y\bigr) \quad \forall t \in [0,1],$$
$$g\bigl(t,u(t),v(t)\bigr) \leqslant g^*\bigl(t, \bigl\|(u,v)\bigr\|_Y\bigr) \quad \forall t \in [0,1]. \quad (22)$$

Then for all $t \in [0,1]$, we obtain

$$\begin{aligned} Q_1(u,v)(t) &\leqslant \lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\biggl(\frac{\int_0^s (s-\tau)^{\alpha_1-1} f(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\biggr)\mathrm{d}s \\ &\leqslant \lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\biggl(\frac{\int_0^s (s-\tau)^{\alpha_1-1} f^*(\tau,\|(u,v)\|_Y)\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\biggr)\mathrm{d}s \\ &\leqslant \lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\biggl(\frac{(f_\infty^s+\varepsilon)\|(u,v)\|_Y^{r_1-1}\int_0^s (s-\tau)^{\alpha_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\biggr)\mathrm{d}s \\ &= \lambda^{\varrho_1-1}\bigl(f_\infty^s+\varepsilon\bigr)^{\varrho_1-1}B\bigl\|(u,v)\bigr\|_Y \leqslant \widetilde{\alpha}_1\bigl\|(u,v)\bigr\|_Y \quad \forall t\in[0,1], \end{aligned}$$

and so $\|Q_1(u,v)\| \leqslant \widetilde{\alpha}_1\|(u,v)\|_Y$.

In a similar manner, we deduce

$$\begin{aligned} Q_2(u,v)(t) &\leqslant \mu^{\varrho_2-1}\int_0^1 J_2(s)\varphi_{\varrho_2}\biggl(\frac{\int_0^s (s-\tau)^{\alpha_2-1} g(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\biggr)\mathrm{d}s \\ &\leqslant \mu^{\varrho_2-1}\int_0^1 J_2(s)\varphi_{\varrho_2}\biggl(\frac{\int_0^s (s-\tau)^{\alpha_2-1} g^*(\tau,\|(u,v)\|_Y)\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\biggr)\mathrm{d}s \\ &\leqslant \mu^{\varrho_2-1}\int_0^1 J_2(s)\varphi_{\varrho_2}\biggl(\frac{(g_\infty^s+\varepsilon)\|(u,v)\|_Y^{r_2-1}\int_0^s (s-\tau)^{\alpha_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\biggr)\mathrm{d}s \\ &= \mu^{\varrho_2-1}\bigl(g_\infty^s+\varepsilon\bigr)^{\varrho_2-1}D\bigl\|(u,v)\bigr\|_Y \leqslant \widetilde{\alpha}_2\bigl\|(u,v)\bigr\|_Y \quad \forall t\in[0,1], \end{aligned}$$

and then $\|Q_2(u,v)\| \leqslant \widetilde{\alpha}_2\|(u,v)\|_Y$.

Therefore, for $(u,v) \in P\cap\partial\Omega_2$ it follows that

$$\bigl\|Q(u,v)\bigr\|_Y = \bigl\|Q_1(u,v)\bigr\| + \bigl\|Q_2(u,v)\bigr\| \leqslant (\widetilde{\alpha}_1+\widetilde{\alpha}_2)\bigl\|(u,v)\bigr\|_Y = \bigl\|(u,v)\bigr\|_Y. \quad (23)$$

By using Lemma 4, (21), (23), and the Guo–Krasnosel'skii fixed-point theorem we conclude that Q has a fixed point $(u,v) \in P\cap(\overline{\Omega}_4\setminus\Omega_3)$.

(iii) We have $f_\infty^s = 0$, $f_0^i, g_0^i, g_\infty^s \in (0,\infty)$, and $\widetilde{L}_3 < \widetilde{L}_4'$. Let $\lambda \in (\widetilde{L}_1,\infty)$ and $\mu \in (\widetilde{L}_3, \widetilde{L}_4')$. We choose $\widetilde{\alpha}_2' \in (D(\mu g_\infty^s)^{\varrho_2-1}, 1)$ and $\widetilde{\alpha}_1' = 1-\widetilde{\alpha}_2'$. Let $\varepsilon > 0$ with $\varepsilon < f_0^i$, $\varepsilon < g_0^i$, and

$$\begin{gathered} \frac{1}{f_0^i-\varepsilon}\biggl(\frac{\alpha_1'}{\gamma\gamma_1 A}\biggr)^{r_1-1} \leqslant \lambda \leqslant \frac{1}{\varepsilon}\biggl(\frac{\widetilde{\alpha}_1'}{B}\biggr)^{r_1-1}, \\ \frac{1}{g_0^i-\varepsilon}\biggl(\frac{\alpha_2'}{\gamma\gamma_2 C}\biggr)^{r_2-1} \leqslant \mu \leqslant \frac{1}{g_\infty^s+\varepsilon}\biggl(\frac{\widetilde{\alpha}_2'}{D}\biggr)^{r_2-1}. \end{gathered}$$

The first part of the proof is the same as the corresponding one from case (i). For Ω_3 defined in case (i), for $(u,v) \in P \cap \partial\Omega_3$, we obtain

$$Q_1(u,v)(c_1) \geqslant \gamma\gamma_1\lambda^{\varrho_1-1}\big(f_0^i-\varepsilon\big)^{\varrho_1-1}A\big\|(u,v)\big\|_Y \geqslant \alpha_1'\big\|(u,v)\big\|_Y,$$
$$Q_2(u,v)(c_1) \geqslant \gamma\gamma_2\mu^{\varrho_2-1}\big(g_0^i-\varepsilon\big)^{\varrho_2-1}C\big\|(u,v)\big\|_Y \geqslant \alpha_2'\big\|(u,v)\big\|_Y,$$

and so $\|Q(u,v)\|_Y \geqslant (\alpha_1'+\alpha_2')\|(u,v)\|_Y = \|(u,v)\|_Y$.

For the second part, we use the same functions f^* and g^* from case (i), which satisfy in this case the conditions

$$\lim_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x^{r_1-1}} = 0, \qquad \limsup_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x^{r_2-1}} = g_\infty^s.$$

Therefore, for $\varepsilon > 0$ there exists $\overline{R}_4 > 0$ such that for all $x \geqslant \overline{R}_4$ and $t \in [0,1]$, we have

$$\frac{f^*(t,x)}{x^{r_1-1}} \leqslant \lim_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x^{r_1-1}} + \varepsilon = \varepsilon,$$
$$\frac{g^*(t,x)}{x^{r_2-1}} \leqslant \limsup_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x^{r_2-1}} + \varepsilon = g_\infty^s + \varepsilon,$$

and so $f^*(t,x) \leqslant \varepsilon x^{r_1-1}$ and $g^*(t,x) \leqslant (g_\infty^s+\varepsilon)x^{r_2-1}$.

We consider $R_4 = \max\{2R_3, \overline{R}_4\}$ and denote $\Omega_4 = \{(u,v) \in Y\colon \|(u,v)\|_Y < R_4\}$. Let $(u,v) \in P \cap \partial\Omega_4$. By the definitions of f^* and g^* we obtain relations (22). In addition, in a similar manner as in the proof of case (i), we conclude

$$Q_1(u,v)(t) \leqslant \lambda^{\varrho_1-1}\varepsilon^{\varrho_1-1}\big\|(u,v\big\|_Y B \leqslant \widetilde{\alpha}_1'\big\|(u,v)\big\|_Y \quad \forall t \in [0,1],$$
$$Q_2(u,v)(t) \leqslant \mu^{\varrho_2-1}\big(g_\infty^s+\varepsilon\big)^{\varrho_2-1}\big\|(u,v)\big\|_Y D \leqslant \widetilde{\alpha}_2'\big\|(u,v)\big\|_Y \quad \forall t \in [0,1],$$

and so $\|Q(u,v)\|_Y \leqslant (\widetilde{\alpha}_1'+\widetilde{\alpha}_2')\|(u,v\|_Y = \|(u,v)\|_Y$.

Therefore, we deduce the conclusion of the theorem.

(vi) We consider here $g_\infty^s = 0$, $f_\infty^s \in (0,\infty)$, and $g_0^i = \infty$. Let $\lambda \in (0, \widetilde{L}_2')$ and $\mu \in (0,\infty)$. We choose $\widetilde{\alpha}_1' \in (B(\lambda f_\infty^s)^{\varrho_1-1}, 1)$ and $\widetilde{\alpha}_2' = 1-\widetilde{\alpha}_1$, and let $\varepsilon > 0$ such that

$$\lambda \leqslant \frac{1}{f_\infty^s+\varepsilon}\left(\frac{\widetilde{\alpha}_1'}{B}\right)^{r_1-1}, \quad \varepsilon\left(\frac{1}{\gamma\gamma_2 C}\right)^{r_2-1} \leqslant \mu \leqslant \frac{1}{\varepsilon}\left(\frac{\widetilde{\alpha}_2'}{D}\right)^{r_2-1}.$$

By (H2) and the definition of g_0^i we deduce that there exists $R_3 > 0$ such that

$$g(t,u,v) \geqslant \frac{1}{\varepsilon}(u+v)^{r_2-1} \quad \forall t \in [c_1,c_2],\ u,v \geqslant 0,\ u+v \leqslant R_3.$$

We define $\Omega_3 = \{(u,v) \in Y\colon \|(u,v)\|_Y < R_3\}$. Let $(u,v) \in P$ with $\|(u,v)\|_Y = R_3$, that is $\|u\|+\|v\| = R_3$. Because $u(t)+v(t) \leqslant \|u\|+\|v\| = R_3$ for all $t \in [0,1]$, then by Lemma 3 we obtain

$$\begin{aligned}
&Q_2(u,v)(c_1)\\
&\quad\geqslant \mu^{\varrho_2-1}\int_0^1 c_1^{\beta_2-1}J_2(s)\varphi_{\varrho_2}\bigl(I_{0+}^{\alpha_2}g\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s\\
&\quad\geqslant \mu^{\varrho_2-1}c_1^{\beta_2-1}\int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\biggl(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1}g(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\biggr)\,\mathrm{d}s\\
&\quad\geqslant \mu^{\varrho_2-1}c_1^{\beta_2-1}\int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\biggl(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1}\frac{1}{\varepsilon}(u(\tau)+v(\tau))^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\biggr)\,\mathrm{d}s\\
&\quad\geqslant \mu^{\varrho_2-1}c_1^{\beta_2-1}\int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\biggl(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1}\frac{1}{\varepsilon}(\gamma\|(u,v)\|_Y)^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\biggr)\,\mathrm{d}s\\
&\quad= \mu^{\varrho_2-1}c_1^{\beta_2-1}\biggl(\frac{1}{\varepsilon}\biggr)^{\varrho_2-1}\gamma\bigl\|(u,v)\bigr\|_Y\int_{c_1}^{c_2} J_2(s)\frac{(s-c_1)^{\alpha_2(\varrho_2-1)}}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}\,\mathrm{d}s\\
&\quad= \gamma\gamma_2\mu^{\varrho_2-1}\biggl(\frac{1}{\varepsilon}\biggr)^{\varrho_2-1}C\bigl\|(u,v)\bigr\|_Y\geqslant \bigl\|(u,v)\bigr\|_Y.
\end{aligned}$$

Hence, $\|Q_2(u,v)\| \geqslant Q_2(u,v)(c_1) \geqslant \|(u,v)\|_Y$ and $\|Q(u,v)\|_Y \geqslant \|Q_2(u,v)\| \geqslant \|(u,v)\|_Y$.

For the second part of the proof, we consider the functions f^* and g^* from case (i), which satisfy in this case the conditions

$$\limsup_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x^{r_1-1}} = f_\infty^s,\qquad \lim_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x^{r_2-1}} = 0.$$

Then for $\varepsilon > 0$, there exists $\overline{R}_4 > 0$ such that for all $x \geqslant \overline{R}_4$ and $t \in [0,1]$, we have

$$\begin{aligned}
\frac{f^*(t,x)}{x^{r_1-1}} &\leqslant \limsup_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x^{r_1-1}}+\varepsilon = f_\infty^s+\varepsilon,\\
\frac{g^*(t,x)}{x^{r_2-1}} &\leqslant \lim_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x^{r_2-1}}+\varepsilon = \varepsilon,
\end{aligned}$$

and so $f^*(t,x) \leqslant (f_\infty^s+\varepsilon)x^{r_1-1}$ and $g^*(t,x) \leqslant \varepsilon x^{r_2-1}$.

We consider $R_4 = \max\{2R_3, \overline{R}_4\}$ and denote by $\Omega_4 = \{(u,v) \in Y{:}\ \|(u,v)\|_Y < R_4\}$. Let $(u,v) \in P \cap \partial\Omega_4$. By the definitions of f^* and g^* we deduce relations (22). In addition, in a similar manner as in the proof of case (i), we conclude

$$\begin{aligned}
Q_1(u,v)(t) &\leqslant \lambda^{\varrho_1-1}\bigl(f_\infty^s+\varepsilon\bigr)^{\varrho_1-1}\bigl\|(u,v)\bigr\|_Y B \leqslant \widetilde{\alpha}_1'\bigl\|(u,v)\bigr\|_Y \quad \forall t\in[0,1],\\
Q_2(u,v)(t) &\leqslant \mu^{\varrho_2-1}\varepsilon^{\varrho_2-1}\bigl\|(u,v)\bigr\|_Y D \leqslant \widetilde{\alpha}_2'\bigl\|(u,v)\bigr\|_Y \quad \forall t\in[0,1],
\end{aligned}$$

and so $\|Q(u,v)\|_Y \leqslant (\widetilde{\alpha}_1'+\widetilde{\alpha}_2')\|(u,v\|_Y = \|(u,v)\|_Y$.

Therefore, we obtain the conclusion of the theorem.

(viii) We consider $f_\infty^s = g_\infty^s = 0$ and $f_0^i = \infty$. Let $\lambda \in (0, \infty)$ and $\mu \in (0, \infty)$. We choose $\varepsilon > 0$ such that

$$\varepsilon\left(\frac{1}{\gamma\gamma_1 A}\right)^{r_1-1} \leqslant \lambda \leqslant \frac{1}{\varepsilon(2B)^{r_1-1}}, \qquad \mu \leqslant \frac{1}{\varepsilon(2D)^{r_2-1}}.$$

By using (H2) and the definition of f_0^i we deduce that there exists $R_3 > 0$ such that

$$f(t,u,v) \geqslant \frac{1}{\varepsilon}(u+v)^{r_1-1} \quad \forall t \in [c_1, c_2],\ u, v \geqslant 0,\ u+v \leqslant R_3.$$

We denote $\Omega_3 = \{(u,v) \in Y\colon \|(u,v)\|_Y < R_3\}$. Let $(u,v) \in P$ with $\|(u,v)\|_Y = R_3$, that is $\|u\| + \|v\| = R_3$. Because $u(t) + v(t) \leqslant \|u\| + \|v\| = R_3$ for all $t \in [0,1]$, then by Lemma 3 we obtain

$$\begin{aligned}
&Q_1(u,v)(c_1)\\
&\quad\geqslant \lambda^{\varrho_1-1}\int_0^1 c_1^{\beta_1-1} J_1(s)\varphi_{\varrho_1}\bigl(I_{0+}^{\alpha_1} f\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1} f(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1} \frac{1}{\varepsilon}(u(\tau)+v(\tau))^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad\geqslant \lambda^{\varrho_1-1}c_1^{\beta_1-1}\int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1} \frac{1}{\varepsilon}(\gamma\|(u,v)\|_Y)^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\quad= \lambda^{\varrho_1-1}c_1^{\beta_1-1}\left(\frac{1}{\varepsilon}\right)^{\varrho_1-1}\gamma\bigl\|(u,v)\bigr\|_Y\int_{c_1}^{c_2} J_1(s)\frac{(s-c_1)^{\alpha_1(\varrho_1-1)}}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}\,\mathrm{d}s\\
&\quad= \gamma\gamma_1\lambda^{\varrho_1-1}\left(\frac{1}{\varepsilon}\right)^{\varrho_1-1} A\bigl\|(u,v)\bigr\|_Y \geqslant \bigl\|(u,v)\bigr\|_Y.
\end{aligned}$$

Hence, $\|Q_1(u,v)\| \geqslant Q_1(u,v)(c_1) \geqslant \|(u,v)\|_Y$ and $\|Q(u,v)\|_Y \geqslant \|Q_1(u,v)\| \geqslant \|(u,v)\|_Y$.

For the second part of the proof, we consider the functions f^* and g^* from case (i), which satisfy in this case the conditions

$$\lim_{x\to\infty}\max_{t\in[0,1]}\frac{f^*(t,x)}{x^{r_1-1}} = 0, \qquad \lim_{x\to\infty}\max_{t\in[0,1]}\frac{g^*(t,x)}{x^{r_2-1}} = 0.$$

Then for $\varepsilon > 0$, there exists $\overline{R}_4 > 0$ such that for all $x \geqslant \overline{R}_4$ and $t \in [0,1]$, we have

$$\frac{f^*(t,x)}{x^{r_1-1}} \leqslant \lim_{x\to\infty} \max_{t\in[0,1]} \frac{f^*(t,x)}{x^{r_1-1}} + \varepsilon = \varepsilon, \qquad \frac{g^*(t,x)}{x^{r_2-1}} \leqslant \lim_{x\to\infty} \max_{t\in[0,1]} \frac{g^*(t,x)}{x^{r_2-1}} + \varepsilon = \varepsilon,$$

and so $f^*(t,x) \leqslant \varepsilon x^{r_1-1}$ and $g^*(t,x) \leqslant \varepsilon x^{r_2-1}$.

We consider $R_4 = \max\{2R_3, \overline{R}_4\}$ and denote $\Omega_4 = \{(u,v) \in Y\colon \|(u,v)\|_Y < R_4\}$. Let $(u,v) \in P \cap \partial\Omega_4$. By the definitions of f^* and g^* we obtain relations (22). In addition, in a similar manner as in the proof of case (i), we deduce

$$Q_1(u,v)(t) \leqslant \lambda^{\varrho_1-1}\varepsilon^{\varrho_1-1}\big\|(u,v)\big\|_Y B \leqslant \frac{1}{2}\big\|(u,v)\big\|_Y \quad \forall t \in [0,1],$$
$$Q_2(u,v)(t) \leqslant \mu^{\varrho_2-1}\varepsilon^{\varrho_2-1}\big\|(u,v)\big\|_Y D \leqslant \frac{1}{2}\big\|(u,v)\big\|_Y \quad \forall t \in [0,1],$$

and so $\|Q(u,v)\|_Y \leqslant (1/2+1/2)\|(u,v\|_Y = \|(u,v)\|_Y$.

Therefore, we obtain the conclusion of the theorem. □

4 Nonexistence of positive solutions

In this section, we present intervals for λ and μ for which there exist no positive solutions of problem (S)–(BC) viewed as fixed points of operator Q.

Theorem 3. *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers M_1, M_2 such that*

$$f(t,u,v) \leqslant M_1(u+v)^{r_1-1}, \quad g(t,u,v) \leqslant M_2(u+v)^{r_2-1} \quad \forall t \in [0,1],\ u,v \geqslant 0, \tag{24}$$

then there exist positive constants λ_0 and μ_0 such that for every $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, the boundary value problem (S)–(BC) *has no positive solution.*

Proof. We define $\lambda_0 = 1/(M_1(2B)^{r_1-1})$ and $\mu_0 = 1/(M_2(2D)^{r_2-1})$, where

$$B = \frac{\int_0^1 s^{\alpha_1(\varrho_1-1)} J_1(s)\,\mathrm{d}s}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}, \qquad D = \frac{\int_0^1 s^{\alpha_2(\varrho_2-1)} J_2(s)\,\mathrm{d}s}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}.$$

We will prove that for every $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$, problem (S)–(BC) has no positive solution.

Let $\lambda \in (0,\lambda_0)$ and $\mu \in (0,\mu_0)$. We suppose that (S)–(BC) has a positive solution $(u(t),v(t))$, $t \in [0,1]$. Then we obtain

$$u(t) = Q_1(u,v)(t) = \lambda^{\varrho_1-1}\int_0^1 G_1(t,s)\varphi_{\varrho_1}\big(I_{0+}^{\alpha_1} f\big(s,u(s),v(s)\big)\big)\,\mathrm{d}s$$
$$\leqslant \lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{\int_0^s (s-\tau)^{\alpha_1-1} f(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s$$

$$
\begin{aligned}
&\leqslant \lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}\left(\frac{\int_0^s (s-\tau)^{\alpha_1-1}M_1(u(\tau)+v(\tau))^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&\leqslant \lambda^{\varrho_1-1}\int_0^1 J_1(s)\varphi_{\varrho_1}(M_1)\varphi_{\varrho_1}\left(\frac{\int_0^s (s-\tau)^{\alpha_1-1}(\|u\|+\|v\|)^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s\\
&= \lambda^{\varrho_1-1}M_1^{\varrho_1-1}B\big\|(u,v)\big\|_Y \quad \forall t\in[0,1]
\end{aligned}
$$

and

$$
\begin{aligned}
v(t) = Q_2(u,v)(t) &= \mu^{\varrho_2-1}\int_0^1 G_2(t,s)\varphi_{\varrho_2}\big(I_{0+}^{\alpha_2}g\big(s,u(s),v(s)\big)\big)\,\mathrm{d}s\\
&\leqslant \mu^{\varrho_2-1}\int_0^1 J_2(s)\varphi_{\varrho_2}\left(\frac{\int_0^s (s-\tau)^{\alpha_2-1}g(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s\\
&\leqslant \mu^{\varrho_2-1}\int_0^1 J_2(s)\varphi_{\varrho_2}\left(\frac{\int_0^s (s-\tau)^{\alpha_2-1}M_2(u(\tau)+v(\tau))^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s\\
&\leqslant \mu^{\varrho_2-1}\int_0^1 J_2(s)\varphi_{\varrho_2}(M_2)\varphi_{\varrho_2}\left(\frac{\int_0^s (s-\tau)^{\alpha_2-1}(\|u\|+\|v\|)^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\right)\mathrm{d}s\\
&= \mu^{\varrho_2-1}M_2^{\varrho_2-1}D\big\|(u,v)\big\|_Y \quad \forall t\in[0,1].
\end{aligned}
$$

Then we deduce

$$
\begin{aligned}
\|u\| &\leqslant \lambda^{\varrho_1-1}M_1^{\varrho_1-1}B\big\|(u,v)\big\|_Y < \lambda_0^{\varrho_1-1}M_1^{\varrho_1-1}B\big\|(u,v)\big\|_Y = \frac{1}{2}\big\|(u,v)\big\|_Y,\\
\|v\| &\leqslant \mu^{\varrho_2-1}M_2^{\varrho_2-1}D\big\|(u,v)\big\|_Y < \mu_0^{\varrho_2-1}M_2^{\varrho_2-1}D\big\|(u,v)\big\|_Y = \frac{1}{2}\big\|(u,v)\big\|_Y,
\end{aligned}
$$

and so $\|(u,v)\|_Y = \|u\| + \|v\| < \|(u,v)\|_Y$, which is a contradiction.

Therefore, the boundary value problem (S)–(BC) has no positive solution. □

Remark 1. In the proof of Theorem 3, we can also define $\lambda_0 = (\alpha_1/B)^{r_1-1}/M_1$ and $\mu_0 = (\alpha_2/D)^{r_2-1}/M_2$ with $\alpha_1, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 = 1$.

Remark 2. If $f_0^s, g_0^s, f_\infty^s, g_\infty^s < \infty$, then there exist positive constants M_1, M_2 such that relation (24) holds, and then we obtain the conclusion of Theorem 3.

Theorem 4. *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers* c_1, c_2 *with* $0 < c_1 < c_2 \leqslant 1$ *and* $m_1 > 0$ *such that*

$$
f(t,u,v) \geqslant m_1(u+v)^{r_1-1} \quad \forall t\in[c_1,c_2],\ u,\ v\geqslant 0, \tag{25}
$$

then there exists a positive constant $\widetilde{\lambda}_0$ *such that for every* $\lambda > \widetilde{\lambda}_0$ *and* $\mu > 0$*, the boundary value problem* (S)–(BC) *has no positive solution.*

Proof. We define $\widetilde{\lambda}_0 = 1/(m_1(\gamma\gamma_1 A)^{r_1-1})$, where

$$A = \frac{1}{(\Gamma(\alpha_1+1))^{\varrho_1-1}} \int_{c_1}^{c_2} (s-c_1)^{\alpha_1(\varrho_1-1)} J_1(s)\,\mathrm{d}s.$$

We will show that for every $\lambda > \widetilde{\lambda}_0$ and $\mu > 0$, problem (S)–(BC) has no positive solution.

Let $\lambda > \widetilde{\lambda}_0$ and $\mu > 0$. We suppose that (S)–(BC) has a positive solution $(u(t), v(t))$, $t \in [0,1]$. Then we obtain

$$\begin{aligned} u(c_1) &= Q_1(u,v)(c_1) = \lambda^{\varrho_1-1} \int_0^1 G_1(c_1,s)\varphi_{\varrho_1}\bigl(I_{0+}^{\alpha_1} f\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s \\ &\geqslant \lambda^{\varrho_1-1} \int_0^1 c_1^{\beta_1-1} J_1(s)\varphi_{\varrho_1}\bigl(I_{0+}^{\alpha_1} f\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s \\ &\geqslant \lambda^{\varrho_1-1} c_1^{\beta_1-1} \int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1} f(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s \\ &\geqslant \lambda^{\varrho_1-1} c_1^{\beta_1-1} \int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1} m_1(u(\tau)+v(\tau))^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s \\ &\geqslant \lambda^{\varrho_1-1} c_1^{\beta_1-1} \int_{c_1}^{c_2} J_1(s)\varphi_{\varrho_1}\left(\frac{\int_{c_1}^s (s-\tau)^{\alpha_1-1} m_1\gamma^{r_1-1}\|(u,v)\|_Y^{r_1-1}\,\mathrm{d}\tau}{\Gamma(\alpha_1)}\right)\mathrm{d}s \\ &= \lambda^{\varrho_1-1}\gamma\gamma_1 m_1^{\varrho_1-1}\bigl\|(u,v)\bigr\|_Y \int_{c_1}^{c_2} J_1(s)\frac{(s-c_1)^{\alpha_1(\varrho_1-1)}}{(\Gamma(\alpha_1+1))^{\varrho_1-1}}\,\mathrm{d}s \\ &= \gamma\gamma_1\lambda^{\varrho_1-1} m_1^{\varrho_1-1} A\bigl\|(u,v)\bigr\|_Y. \end{aligned}$$

Then we conclude

$$\begin{aligned} \|u\| &\geqslant u(c_1) \geqslant \gamma\gamma_1 A(\lambda m_1)^{\varrho_1-1}\bigl\|(u,v)\bigr\|_Y > \gamma\gamma_1 A(\widetilde{\lambda}_0 m_1)^{\varrho_1-1}\bigl\|(u,v)\bigr\|_Y \\ &= \bigl\|(u,v)\bigr\|_Y, \end{aligned}$$

and so $\|(u,v)\|_Y = \|u\| + \|v\| \geqslant \|u\| > \|(u,v)\|_Y$, which is a contradiction.

Therefore, the boundary value problem (S)–(BC) has no positive solution. □

Theorem 5. *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers* c_1, c_2 *with* $0 < c_1 < c_2 \leqslant 1$ *and* $m_2 > 0$ *such that*

$$g(t,u,v) \geqslant m_2(u+v)^{r_2-1} \quad \forall t \in [c_1,c_2],\ u,v \geqslant 0, \tag{26}$$

then there exists a positive constant $\widetilde{\mu}_0$ *such that for every* $\mu > \widetilde{\mu}_0$ *and* $\lambda > 0$*, the boundary value problem* (S)–(BC) *has no positive solution.*

Proof. We define $\widetilde{\mu}_0 = 1/(m_2(\gamma\gamma_2 C)^{r_2-1})$, where

$$C = \frac{1}{(\Gamma(\alpha_2+1))^{\varrho_2-1}} \int_{c_1}^{c_2} (s-c_1)^{\alpha_2(\varrho_2-1)} J_2(s)\,\mathrm{d}s.$$

We will show that for every $\mu > \widetilde{\mu}_0$ and $\lambda > 0$, problem (S)–(BC) has no positive solution.

Let $\mu > \widetilde{\mu}_0$ and $\lambda > 0$. We suppose that (S)–(BC) has a positive solution $(u(t), v(t))$, $t \in [0,1]$. Then we obtain

$$\begin{aligned}
v(c_1) &= Q_2(u,v)(c_1) = \mu^{\varrho_2-1} \int_0^1 G_2(c_1,s)\varphi_{\varrho_2}\bigl(I_{0+}^{\alpha_1} g\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s \\
&\geqslant \mu^{\varrho_2-1} \int_0^1 c_1^{\beta_2-1} J_2(s)\varphi_{\varrho_2}\bigl(I_{0+}^{\alpha_2} g\bigl(s,u(s),v(s)\bigr)\bigr)\,\mathrm{d}s \\
&\geqslant \mu^{\varrho_2-1} c_1^{\beta_2-1} \int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\biggl(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1} g(\tau,u(\tau),v(\tau))\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\biggr)\mathrm{d}s \\
&\geqslant \mu^{\varrho_2-1} c_1^{\beta_2-1} \int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\biggl(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1} m_2(u(\tau)+v(\tau))^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\biggr)\mathrm{d}s \\
&\geqslant \mu^{\varrho_2-1} c_1^{\beta_2-1} \int_{c_1}^{c_2} J_2(s)\varphi_{\varrho_2}\biggl(\frac{\int_{c_1}^s (s-\tau)^{\alpha_2-1} m_2\gamma^{r_2-1}\|(u,v)\|_Y^{r_2-1}\,\mathrm{d}\tau}{\Gamma(\alpha_2)}\biggr)\mathrm{d}s \\
&= \mu^{\varrho_2-1}\gamma\gamma_2 m_2^{\varrho_2-1}\bigl\|(u,v)\bigr\|_Y \int_{c_1}^{c_2} J_2(s)\frac{(s-c_1)^{\alpha_2(\varrho_2-1)}}{(\Gamma(\alpha_2+1))^{\varrho_2-1}}\,\mathrm{d}s \\
&= \gamma\gamma_2\mu^{\varrho_2-1} m_2^{\varrho_2-1} C\bigl\|(u,v)\bigr\|_Y.
\end{aligned}$$

Then we deduce

$$\begin{aligned}
\|v\| &\geqslant v(c_1) \geqslant \gamma\gamma_2 C(\mu m_2)^{\varrho_2-1}\bigl\|(u,v)\bigr\|_Y > \gamma\gamma_2 C(\widetilde{\mu}_0 m_2)^{\varrho_2-1}\bigl\|(u,v)\bigr\|_Y \\
&= \bigl\|(u,v)\bigr\|_Y,
\end{aligned}$$

and so $\|(u,v)\|_Y = \|u\| + \|v\| \geqslant \|v\| > \|(u,v)\|_Y$, which is a contradiction.

Therefore, the boundary value problem (S)–(BC) has no positive solution. □

Theorem 6. *Assume that* (H1) *and* (H2) *hold. If there exist positive numbers* c_1, c_2 *with* $0 < c_1 < c_2 \leqslant 1$ *and* $m_1, m_2 > 0$ *such that*

$$\begin{aligned}
f(t,u,v) &\geqslant m_1(u+v)^{r_1-1} \quad \forall t \in [c_1,c_2],\ u,v \geqslant 0, \\
g(t,u,v) &\geqslant m_2(u+v)^{r_2-1} \quad \forall t \in [c_1,c_2],\ u,v \geqslant 0,
\end{aligned} \tag{27}$$

then there exist positive constants $\hat{\lambda}_0$ *and* $\hat{\mu}_0$ *such that for every* $\lambda > \hat{\lambda}_0$ *and* $\mu > \hat{\mu}_0$*, the boundary value problem* (S)–(BC) *has no positive solution.*

Proof. We define $\hat{\lambda}_0 = 1/(m_1(2\gamma\gamma_1 A)^{r_1-1})$ and $\hat{\mu}_0 = 1/(m_2(2\gamma\gamma_2 C)^{r_2-1})$. Then for every $\lambda > \hat{\lambda}_0$ and $\mu > \hat{\mu}_0$, problem (S)–(BC) has no positive solution. Indeed, let $\lambda > \hat{\lambda}_0$ and $\mu > \hat{\mu}_0$. We suppose that (S)–(BC) has a positive solution $(u(t), v(t))$, $t \in [0,1]$. In a similar manner as that used in the proofs of Theorems 4 and 5, we obtain

$$\begin{aligned}\|u\| &\geqslant u(c_1) \geqslant \gamma\gamma_1 A(\lambda m_1)^{\varrho_1-1}\big\|(u,v)\big\|_Y,\\ \|v\| &\geqslant v(c_1) \geqslant \gamma\gamma_2 C(\mu m_2)^{\varrho_2-1}\big\|(u,v)\big\|_Y,\end{aligned}$$

and so

$$\begin{aligned}\big\|(u,v)\big\|_Y &= \|u\| + \|v\|\\ &\geqslant \gamma\gamma_1 A(\lambda m_1)^{\varrho_1-1}\big\|(u,v)\big\|_Y + \gamma\gamma_2 C(\mu m_2)^{\varrho_2-1}\big\|(u,v)\big\|_Y\\ &> \gamma\gamma_1 A(\hat{\lambda}_0 m_1)^{\varrho_1-1}\big\|(u,v)\big\|_Y + \gamma\gamma_2 C(\hat{\mu}_0 m_2)^{\varrho_2-1}\big\|(u,v)\big\|_Y\\ &= \frac{1}{2}\big\|(u,v)\big\|_Y + \frac{1}{2}\big\|(u,v)\big\|_Y = \big\|(u,v)\big\|_Y,\end{aligned}$$

which is a contradiction. Therefore, the boundary value problem (S)–(BC) has no positive solution. □

Remark 3.

(i) If for c_1, c_2 with $0 < c_1 < c_2 \leqslant 1$, we have $f_0^i, f_\infty^i > 0$ and $f(t,u,v) > 0$ for all $t \in [c_1, c_2]$ and $u, v \geqslant 0$ with $u + v > 0$, then relation (25) holds, and we obtain the conclusion of Theorem 4.

(ii) If for c_1, c_2 with $0 < c_1 < c_2 \leqslant 1$, we have $g_0^i, g_\infty^i > 0$ and $g(t,u,v) > 0$ for all $t \in [c_1, c_2]$ and $u, v \geqslant 0$ with $u + v > 0$, then relation (26) holds, and we obtain the conclusion of Theorem 5.

(iii) If for c_1, c_2 with $0 < c_1 < c_2 \leqslant 1$, we have $f_0^i, f_\infty^i, g_0^i, g_\infty^i > 0$ and $f(t,u,v) > 0$, $g(t,u,v) > 0$ for all $t \in [c_1, c_2]$ and $u, v \geqslant 0$ with $u + v > 0$, then relation (27) holds, and we obtain the conclusion of Theorem 6.

5 An example

Let $\alpha_1 = 1/2$, $\alpha_2 = 1/3$, $n = 3$, $\beta_1 = 7/3$, $m = 4$, $\beta_2 = 15/4$, $p_1 = 1$, $q_1 = 1/3$, $p_2 = 3/2$, $q_2 = 6/5$, $N = 2$, $M = 1$, $\xi_1 = 1/4$, $\xi_2 = 3/4$, $a_1 = 3$, $a_2 = 1/4$, $\eta_1 = 1/3$, $b_1 = 2$, $r_1 = 4$, $\varrho_1 = 4/3$, $\varphi_{r_1}(s) = s^3$, $\varphi_{\varrho_1}(s) = s^{1/3}$, $r_2 = 3$, $\varrho_2 = 3/2$, $\varphi_{r_2}(s) = |s|s$, $\varphi_{\varrho_2}(s) = |s|^{-1/2}s$.

We consider the system of fractional differential equations

$$\begin{aligned}&D_{0+}^{1/2}\big(\varphi_4\big(D_{0+}^{7/3}u(t)\big)\big) + \lambda(t+1)^a\big(\mathrm{e}^{(u(t)+v(t))^3} - 1\big) = 0, \quad t \in (0,1),\\ &D_{0+}^{1/3}\big(\varphi_3\big(D_{0+}^{15/4}v(t)\big)\big) + \mu(2-t)^b\big(u^3(t) + v^3(t)\big) = 0, \quad t \in (0,1),\end{aligned} \tag{S$_0$}$$

with the multi-point boundary conditions

$$
\begin{aligned}
&u(0)=u'(0)=0, \quad D_{0+}^{7/3}u(0)=0, \quad u'(1)=3D_{0+}^{1/3}u\left(\frac{1}{4}\right)+\frac{1}{4}D_{0+}^{1/3}u\left(\frac{3}{4}\right),\\
&v(0)=v'(0)=v''(0)=0, \quad D_{0+}^{15/4}v(0)=0, \quad D_{0+}^{3/2}v(1)=2D_{0+}^{6/5}v\left(\frac{1}{3}\right),
\end{aligned}
\tag{BC$_0$}
$$

where $a, b > 0$.

Here we have $f(t,u,v)=(t+1)^a(\mathrm{e}^{(u+v)^3}-1)$, $g(t,u,v)=(2-t)^b(u^3+v^3)$ for all $t\in[0,1]$, $u,v\geqslant 0$. Then we obtain $\Delta_1\approx 0.21710894>0$, $\Delta_2\approx 2.73417069>0$, and so assumptions (H1) and (H2) are satisfied. In addition, we deduce

$$
g_1(t,s)=\frac{1}{\Gamma(7/3)}\begin{cases} t^{4/3}(1-s)^{1/3}-(t-s)^{4/3}, & 0\leqslant s\leqslant t\leqslant 1,\\ t^{4/3}(1-s)^{1/3}, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}
$$

$$
g_2(t,s)=\begin{cases} t(1-s)^{1/3}-t+s, & 0\leqslant s\leqslant t\leqslant 1,\\ t(1-s)^{1/3}, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}
$$

$$
g_3(t,s)=\frac{1}{\Gamma(15/4)}\begin{cases} t^{11/4}(1-s)^{5/4}-(t-s)^{11/4}, & 0\leqslant s\leqslant t\leqslant 1,\\ t^{11/4}(1-s)^{5/4}, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}
$$

$$
g_4(t,s)=\frac{1}{\Gamma(51/20)}\begin{cases} t^{31/20}(1-s)^{5/4}-(t-s)^{31/20}, & 0\leqslant s\leqslant t\leqslant 1,\\ t^{31/20}(1-s)^{5/4}, & 0\leqslant t\leqslant s\leqslant 1,\end{cases}
$$

$$
\begin{aligned}
G_1(t,s)&=g_1(t,s)+\frac{t^{4/3}}{\Delta_1}\left(3g_2\left(\frac{1}{4},s\right)+\frac{1}{4}g_2\left(\frac{3}{4},s\right)\right),\\
G_2(t,s)&=g_3(t,s)+\frac{2t^{11/4}}{\Delta_2}g_4\left(\frac{1}{3},s\right),\\
h_1(s)&=\frac{s(1-s)^{1/3}}{\Gamma(7/3)}, \qquad h_3(s)=\frac{(1-s)^{5/4}(1-(1-s)^{3/2})}{\Gamma(15/4)}.
\end{aligned}
$$

For the functions J_1 and J_2, we obtain

$$
J_1(s)=\begin{cases} \frac{1}{\Gamma(7/3)}s(1-s)^{1/3}+\frac{1}{\Delta_1}[\frac{15}{16}(1-s)^{1/3}+\frac{13s}{4}-\frac{15}{16}], & 0\leqslant s<\frac{1}{4},\\ \frac{1}{\Gamma(7/3)}s(1-s)^{1/3}+\frac{1}{\Delta_1}[\frac{15}{16}(1-s)^{1/3}+\frac{s}{4}-\frac{3}{16}], & \frac{1}{4}\leqslant s<\frac{3}{4},\\ \frac{1}{\Gamma(7/3)}s(1-s)^{1/3}+\frac{15}{16\Delta_1}(1-s)^{1/3}, & \frac{3}{4}\leqslant s\leqslant 1,\end{cases}
$$

$$
J_2(s)=\begin{cases} \frac{1}{\Gamma(15/4)}(1-s)^{5/4}(1-(1-s)^{3/2})+\frac{2}{3^{31/20}\Delta_2\Gamma(51/20)}[(1-s)^{5/4}\\ \quad -(1-3s)^{31/20}], \quad 0\leqslant s<\frac{1}{3},\\ \frac{1}{\Gamma(15/4)}(1-s)^{5/4}(1-(1-s)^{3/2})+\frac{2}{3^{31/20}\Delta_2\Gamma(51/20)}(1-s)^{5/4},\\ \quad \frac{1}{3}\leqslant s\leqslant 1.\end{cases}
$$

Now we choose $c_1 = 1/4$ and $c_2 = 3/4$, and then we deduce $\gamma_1 = (1/4)^{4/3}$, $\gamma_2 = (1/4)^{11/4}$, $\gamma = \gamma_2$. In addition, we have $f_0^s = (7/4)^a$, $f_\infty^i = \infty$, $g_0^s = 0$, $g_\infty^i = \infty$, $A \approx 1.35668478$, $B \approx 2.51926854$.

By Theorem 1(vii), for any $\lambda \in (0, L_2')$ and $\mu \in (0,\infty)$ with $L_2' = 1/(f_0^s B^3)$, problem (S_0)–(BC_0) has a positive solution $(u(t), v(t))$, $t \in [0,1]$. For example, if $a = 2$, we obtain $L_2' \approx 0.0204221$.

We can also use Theorem 4 because $f(t,u,v) \geqslant (5/4)^a(u+v)^3$ for all $t \in [1/4, 3/4]$ and $u, v \geqslant 0$, that is $m_1 = (5/4)^a$. If $a = 2$, we deduce $\widetilde{\lambda}_0 = 1/(m_1(\gamma\gamma_1 A)^3) \approx 6.0810421 \times 10^6$, and then we conclude that for every $\lambda > \widetilde{\lambda}_0$ and $\mu > 0$, the boundary value problem (S_0)–(BC_0) has no positive solution.

Appendix

In this appendix, we will prove that if

$$\limsup_{\substack{u+v\to\infty\\ u,v\geqslant 0}} \max_{t\in[0,1]} \frac{f(t,u,v)}{(u+v)^{r_1-1}} = f_\infty^s,$$

then

$$\limsup_{x\to\infty} \max_{t\in[0,1]} \frac{f^*(t,x)}{x^{r_1-1}} = f_\infty^s,$$

where $f : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function, $f^*(t,x) = \max_{u,v\geqslant 0,\, u+v\leqslant x} f(t,u,v)$ for $t \in [0,1]$, $x \geqslant 0$, and $r_1 > 1$.

(I) In the case $f_\infty^s \in (0,\infty)$, from the characterization theorem of supremum limit we have:

(a) For all $\varepsilon > 0$, there exists $M(\varepsilon) > 0$ such that for all $u, v \geqslant 0$, $u + v > M(\varepsilon)$, we have
$$\max_{t\in[0,1]} \frac{f(t,u,v)}{(u+v)^{r_1-1}} < f_\infty^s + \varepsilon;$$

(b) For all $\varepsilon, M' > 0$, there exists (u_0, v_0), $u_0, v_0 \geqslant 0$, $u_0 + v_0 > M'$, such that
$$\max_{t\in[0,1]} \frac{f(t,u_0,v_0)}{(u_0+v_0)^{r_1-1}} > f_\infty^s - \varepsilon.$$

Relation (b) is verified for an arbitrary (u,v) with $u+v > M'$ if $\varepsilon > f_\infty^s$ because f has nonnegative values.

From (a), for $\varepsilon > 0$ arbitrary but fixed for the moment, there exists $M_1 = M(\varepsilon/2) > 0$ such that for all $u, v \geqslant 0$, $u + v > M_1$, we have

$$\max_{t\in[0,1]} \frac{f(t,u,v)}{(u+v)^{r_1-1}} < f_\infty^s + \frac{\varepsilon}{2},$$

and then $f(t,u,v) < (f_\infty^s + \varepsilon/2)(u+v)^{r_1-1}$ for all $t \in [0,1]$.

Then for $\varepsilon > 0$, there exists $M_1 > 0$ such that for all $x > M_1$ and $t \in [0,1]$, we obtain

$$\begin{aligned} f^*(t,x) &= \max_{0\leqslant u+v\leqslant x} f(t,u,v) \leqslant \max_{0\leqslant u+v\leqslant M_1} f(t,u,v) + \sup_{M_1<u+v\leqslant x} f(t,u,v) \\ &= f^*(t,M_1) + \sup_{M_1<u+v\leqslant x} f(t,u,v) \\ &\leqslant \max_{t\in[0,1]} f^*(t,M_1) + \sup_{M_1<u+v\leqslant x} \left(f_\infty^s + \frac{\varepsilon}{2}\right)(u+v)^{r_1-1} \\ &\leqslant K_{M_1} + \left(f_\infty^s + \frac{\varepsilon}{2}\right)x^{r_1-1}, \end{aligned}$$

where $K_{M_1} = \max_{t\in[0,1]} f^*(t,M_1)$.

Therefore, for $\varepsilon > 0$, there exist $M_1 > 0$ and $K_{M_1} > 0$ such that

$$\frac{f^*(t,x)}{x^{r_1-1}} \leqslant \frac{K_{M_1}}{x^{r_1-1}} + f_\infty^s + \frac{\varepsilon}{2} \quad \forall x > M_1\ t \in [0,1],$$

and so

$$\max_{t\in[0,1]} \frac{f^*(t,x)}{x^{r_1-1}} \leqslant \frac{K_{M_1}}{x^{r_1-1}} + f_\infty^s + \frac{\varepsilon}{2} \quad \forall x > M_1.$$

Because $\lim_{x\to\infty} 1/x^{r_1-1} = 0$, then for $\varepsilon > 0$, there exists $M_2 \geqslant M_1$ such that $1/x^{r_1-1} < \varepsilon/(2K_{M_1})$ for all $x > M_2$.

So we conclude that for all $\varepsilon > 0$, there exists $M_2 > 0$ such that

$$\max_{t\in[0,1]} \frac{f^*(t,x)}{x^{r_1-1}} < \frac{\varepsilon}{2} + f_\infty^s + \frac{\varepsilon}{2} = f_\infty^s + \varepsilon \quad \forall x > M_2. \tag{28}$$

From relation (b) we deduce that for any $\varepsilon > 0$ and any $M' > 0$, there exists $x_0 = u_0 + v_0 > 0$ such that

$$\max_{t\in[0,1]} \frac{f^*(t,x_0)}{x_0^{r_1-1}} \geqslant \max_{t\in[0,1]} \frac{f(t,u_0,v_0)}{x_0^{r_1-1}} = \max_{t\in[0,1]} \frac{f(t,u_0,v_0)}{(u_0+v_0)^{r_1-1}} > f_\infty^s - \varepsilon.$$

Then we obtain that for all $\varepsilon, M' > 0$, there exists $x_0 > M'$ such that

$$\max_{t\in[0,1]} \frac{f^*(t,x_0)}{x_0^{r_1-1}} > f_\infty^s - \varepsilon. \tag{29}$$

By relations (28), (29) and the characterization theorem for supremum limit we conclude that $\limsup_{x\to\infty} \max_{t\in[0,1]} f^*(t,x)/x^{r_1-1} = f_\infty^s$.

(II) If $f_\infty^s = 0$, then $\limsup_{u+v\to\infty,\, u,v\geqslant 0} \max_{t\in[0,1]} f(t,u,v)/(u+v)^{r_1-1} = 0$ is equivalent to $\lim_{u+v\to\infty,\, u,v\geqslant 0} \max_{t\in[0,1]} f(t,u,v)/(u+v)^{r_1-1} = 0$ because f has nonnegative values. Also $\limsup_{x\to\infty} \max_{t\in[0,1]} f^*(t,x)/x^{r_1-1} = 0$ is equivalent to $\lim_{x\to\infty} \max_{t\in[0,1]} f^*(t,x)/x^{r_1-1} = 0$.

In the same manner as used in case (I) (for the implication (a) ⇒ (28)), we can show that relation

$$\forall\varepsilon > 0,\ \exists M > 0\colon\quad 0 \leqslant \max_{t\in[0,1]} \frac{f(t,u,v)}{(u+v)^{r_1-1}} < \varepsilon \quad \forall u,v \geqslant 0,\ u+v > M,$$

implies the relation

$$\forall\varepsilon > 0,\ \exists\widetilde{M} > 0\colon\quad 0 \leqslant \max_{t\in[0,1]} \frac{f^*(t,x)}{x^{r_1-1}} < \varepsilon \quad \forall x > \widetilde{M},$$

that is $\lim_{x\to\infty} \max_{t\in[0,1]} f^*(t,x)/x^{r_1-1} = 0$.

(III) If $f_\infty^s = \infty$, then by the characterization theorem we have

$$\forall M_1, M_2 > 0,\ \exists(u,v),\ u,v \geqslant 0,\ u+v > M_1\colon\quad \max_{t\in[0,1]} \frac{f(t,u,v)}{(u+v)^{r_1-1}} > M_2.$$

Then we deduce that for any $M_1, M_2 > 0$, there exists $x = u+v > M_1$ such that

$$\max_{t\in[0,1]} \frac{f^*(t,x)}{x^{r_1-1}} \geqslant \max_{t\in[0,1]} \frac{f(t,u,v)}{x^{r_1-1}} = \max_{t\in[0,1]} \frac{f(t,u,v)}{(u+v)^{r_1-1}} > M_2.$$

So we obtain that $\limsup_{x\to\infty} \max_{t\in[0,1]} f^*(t,x)/x^{r_1-1} = \infty$.

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