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Existence and nonexistence of radial solutions of the Dirichlet problem for a class of general k-Hessian equations*

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Abstract. In this paper, we establish the existence and nonexistence of radial solutions of the Dirichlet problem for a class of general k-Hessian equations in a ball. Under some suitable local growth conditions for nonlinearity, several new results are obtained by using the fixed point theorem.

Keywords: k-Hessian equation, existence and nonexistence, multiplicity, radial solutions.

1 Introduction

In this paper, we consider the existence and nonexistence of radial solutions for the following Dirichlet problem of the general k-Hessian equation:

$$\begin{aligned}&\mathfrak{B}\big(S_k^{1/k}\big(\lambda\big(D^2u\big)\big)\big)S_k^{1/k}\big(\lambda\big(D^2u\big)\big)=f\big(|x|,-u\big)\quad \text{in } \Omega\subset\mathbb{R}^N\ (N>2k),\\&u=0\quad \text{on } \partial\Omega,\end{aligned}\tag{1}$$

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where Ω is a unit ball, $f : C(\Omega \times \mathbb{R} \to [0, +\infty))$, $\mathfrak{B} \in \mathcal{X}$ is a nonlinear operator with the following property:

$$\mathcal{X} = \bigl\{\mathfrak{B} \in C^2\bigl([0, +\infty), [0, +\infty)\bigr)\colon \text{there exists a constant } \sigma > 0 \\ \text{such that for any } 0 < c < 1,\ \mathfrak{B}(cs) \leqslant c^\sigma \mathfrak{B}(s)\bigr\}.$$

$S_k^{1/k}(\lambda(D^2u))$ is defined as the k-Hessian operator by

$$S_k\bigl(\lambda\bigl(D^2u\bigr)\bigr) = \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant N} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, \quad k = 1, 2, \dots, N,$$

where $\lambda_1, \lambda_2, \dots, \lambda_N$ are the eigenvalues of the Hessian matrix D^2u, and $\lambda(D^2u) = (\lambda_1, \lambda_2, \dots, \lambda_N)$ is the vector of eigenvalues of D^2u. Clearly, $S_k(\lambda(D^2u))$ is a second-order fully nonlinear differential operator for $k > 1$, which is the sum of all $k \times k$ principal minors of the Hessian matrix of D^2u. On the other hand, the k-Hessian operator can also be written in the divergence form

$$S_k\bigl(\lambda\bigl(D^2u\bigr)\bigr) = \frac{1}{k}\sum_{i,j=1}^{N} \bigl(S_k^{ij} u_i\bigr)_j,$$

where $S_k^{ij} = \partial S_k(D^2u)/\partial u_{ij}$, and for more details, the reader is referred to [13, 29, 30]. It is easy to see that the k-Hessian operator is a generalization of both the Monge–Ampère operator [1, 2] when $k = N$ and the Laplace operator [15] when $k = 1$. This implies that the k-Hessian operator constructs a discrete collection of partial differential operators including the Monge–Ampère operator and the Laplace operator as special cases.

In many existing work for the k-Hessian equation, mathematical theories are constructed with no background on modeling or exploration of their applications. We thus briefly review here some potential applications in physics and applied mathematics. In [9, 11], Escudero used the k-Hessian equation to model various phenomena of condensed matter and statistical physics. In addition, the k-Hessian equation is also regarded as an important class of fully nonlinear operators related to an object of geometric investigation [31, 32] and study of quasilinear parabolic problems [26].

There are many rich literatures concerning the k-Hessian equation. For example, Caffarelli, Nirenberg, and Spruck [3] first studied the existence and a priori estimate of the smooth solutions for the k-Hessian equation

$$\begin{aligned} &S_k\bigl(\lambda\bigl(D^2u\bigr)\bigr) = f \quad \text{in } \Omega \subset \mathbb{R}^N, \\ &u = \varphi \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

Then the work was extended to more general equations in [21, 28], and for more recent results, we refer the reader to [4–6, 10, 12, 17, 26, 27, 33]. In [20] and [12], the regularity for a more general class of fully nonlinear elliptic equations was obtained under nondivergence form. Recently, Covei [6] considered the existence of positive radial solutions for a Hessian equation with weights

$$S_k^{1/k}\bigl(\lambda\bigl(D^2u\bigr)\bigr) = p\bigl(|x|\bigr)h(u), \quad x \in \mathbb{R}^N\ (N > 2k), \tag{3}$$

and a system of two Hessian equations

$$S_k^{1/k}\big(\lambda\big(D^2u\big)\big) = p\big(|x|\big)f(u,v), \quad x \in \mathbb{R}^N \ (N > 2k),$$
$$S_k^{1/k}\big(\lambda\big(D^2v\big)\big) = q\big(|x|\big)g(u,v), \quad x \in \mathbb{R}^N \ (N > 2k).$$

By using a successive approximation technique, a necessary condition and a sufficient condition for a positive radial solution to be large were established.

Inspired by the above work, in this paper, we establish the existence and nonexistence of radial solutions for the k-Hessian equation (1) based on some fixed point theorems. Noticing that the k-Hessian equation (1) involves a nonlinear operator $\mathfrak{B}$, so it includes many interesting and important cases. In particular, if $\mathfrak{B}(x) = x^{k-1}$, then the k-Hessian equation (1) reduces to the Hessian equation (2), which has been studied by many authors [3, 4, 14, 21, 28] via to different methods such as the variational method, the Perron's method, and so on. Moreover, if $\mathfrak{B}(x) = \mathrm{const} \neq 0$, then the k-Hessian equation becomes Hessian equation (3). Ji and Bao [17], Covei [6] considered the necessary and sufficient conditions for the existence of positive radial solutions. When $\mathfrak{B}(x) = |x|^{p-2}$, $p \geqslant 2$, the k-Hessian equation (1) becomes

$$\varphi_p\big(S_k^{1/k}\big(\lambda\big(D^2u\big)\big)\big) = f\big(|x|, -u\big) \quad \text{in } \Omega \subset \mathbb{R}^N,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

which is a p-Poisson–Hessian equation, and few work were reported. Thus, the k-Hessian equation (1) is a generalization of fully nonlinear elliptic equations involving many important cases. To the best of our knowledge, no results have been reported on the existence and nonexistence of radial solutions for the k-Hessian equation (1), and this is the first paper using the Leggett–Williams' fixed point theorem and the Leray–Schauder nonlinear alternative theorem to study the k-Hessian equation involving a nonlinear operator.

Before we give a detailed description of our main results, we first establish the following property of the inverse operator of the operator $s\mathfrak{B}(s)$.

Proposition 1. *If* $\mathfrak{B} \in \mathcal{X}$*, let* $\mathfrak{L}(s) = s\mathfrak{B}(s)$*, then* $\mathfrak{L}$ *has a nonnegative increasing inverse mapping* $\mathfrak{L}^{-1}(s)$*, and for any* $0 < b < 1$*,*

$$\mathfrak{L}^{-1}(bs) \geqslant b^{1/(1+\sigma)}\mathfrak{L}^{-1}(s).$$

Proof. Firstly, we prove that $\mathfrak{B}$ is an increasing operator if $\mathfrak{B} \in \mathcal{X}$. In fact, for any $\mathfrak{B} \in \mathcal{X}$ and $s, t \in [0, +\infty)$, without loss of the generality, let $0 \leqslant s < t$. If $s = 0$, obviously $\mathfrak{B}(s) \leqslant \mathfrak{B}(t)$ holds. If $s \neq 0$, let $c_0 = s/t$, then $0 < c_0 < 1$. It follows from the property of $\mathfrak{B}$ that

$$\mathfrak{B}(s) = \mathfrak{B}(c_0 t) \leqslant c_0^\sigma \mathfrak{B}(t) \leqslant \mathfrak{B}(t),$$

which implies that $\mathfrak{B}$ is an increasing operator. Thus, we have $\mathfrak{L}'(s) = (s\mathfrak{B}(s))' > 0$ for any $s > 0$, which implies that $\mathfrak{L}$ is a bijection on $(0, \infty)$ and has a nonnegative increasing inverse mapping $\mathfrak{L}^{-1}(s)$.

On the other hand, for any $0 < b < 1$, let $c = b^{1/(1+\sigma)}$, then $0 < c < 1$. Thus, we have

$$\mathfrak{L}(cx) = cx\mathfrak{B}(cx) \leqslant c^{1+\sigma}x\mathfrak{B}(x) = c^{1+\sigma}\mathfrak{L}(x) \quad \text{for } x > 0.$$

Consequently, let $s = \mathfrak{L}(x)$, then

$$c\mathfrak{L}^{-1}(s) = cx \leqslant \mathfrak{L}^{-1}\bigl(c^{1+\sigma}\mathfrak{L}(x)\bigr) = \mathfrak{L}^{-1}(bs),$$

that is

$$b^{1/(1+\sigma)}\mathfrak{L}^{-1}(s) \leqslant \mathfrak{L}^{-1}(bs).$$

The proof is completed. □

Remark 1. Clearly, if $r \geqslant 1$, then we have

$$\mathfrak{L}^{-1}(rs) \leqslant r^{1/(1+\sigma)}\mathfrak{L}^{-1}(s). \tag{4}$$

Remark 2. The operator set $\mathcal{X}$ includes a large class of operators and the standard type of operators is $\mathfrak{B}(s) = \sum_{i=1}^{n} s^{\alpha_i}$, $\alpha_i > 0$. In fact, take $\sigma = \min\{\alpha_1, \dots, \alpha_n\} > 0$, then for any $0 < c < 1$, one has

$$\mathfrak{B}(cs) \leqslant c^{\sigma}\mathfrak{B}(s).$$

2 Preliminary results on radial solutions

In this paper, we only focus on the classical solutions of the k-Hessian equation (1), namely, a function $u(t)$ of class $C^2[0,1]$ satisfies the k-Hessian equation (1). In the rest of this paper, t is used as an independent variable of functions, and r as radiuses of balls in the cone.

For $B_R := \{x \in \mathbb{R}^N\colon |x| < R\}$ and radial function $u(r)$ with $r = \sqrt{\sum_{i=1}^{N} x_i^2}$, we have the following properties.

Lemma 1. (*See* [17].) *Assume $v(r) \in C^2[0,R)$ is radially symmetric, and $v'(0) = 0$. Then the function $u(|x|) = v(r)$ with $r = |x| < R$ is $C^2(B_R)$, and*

$$\lambda\bigl(D^2u\bigr) = \begin{cases} (v''(r), \frac{v'(r)}{r}, \dots, \frac{v'(r)}{r}), & r \in (0,R),\\ (v''(0), v''(0), \dots, v''(0)), & r = 0, \end{cases}$$

$$S_k\bigl(\lambda\bigl(D^2u\bigr)\bigr) = \begin{cases} C_{N-1}^{k-1}v''(r)(\frac{v'(r)}{r})^{k-1} + C_{N-1}^{k}(\frac{v'(r)}{r})^k, & r \in (0,R),\\ C_N^k(v''(0))^k, & r = 0. \end{cases}$$

Notice

$$\begin{aligned} v'(0) &= \lim_{r\to 0}\frac{v(r)-v(0)}{r-0} = \lim_{\xi\to 0} v'(\xi)\\ &= \lim_{\xi\to 0}\left(\frac{k}{\xi^{N-k}}\int_0^{\xi}\frac{s^{N-1}}{C_{N-1}^{k-1}}\bigl[\mathfrak{L}^{-1}\bigl(f\bigl(r, -v(s)\bigr)\bigr)\bigr]^k\,\mathrm{d}s\right)^{1/k} = 0, \end{aligned}$$

then we have

$$\lim_{r\to 0}\frac{v'(r)}{r}=v''(0).$$

Thus, from Lemma 1 and Proposition 1 we get the following lemma.

Lemma 2. *The function $u\in C^2(B_1)$ is a radial solution of equation* (1) *if and only if $v(r)$ is a solution of the ODE*

$$\begin{aligned}&C_{N-1}^{k-1}v''(r)\biggl(\frac{v'(r)}{r}\biggr)^{k-1}+C_{N-1}^{k}\biggl(\frac{v'(r)}{r}\biggr)^{k}\\&\quad=\bigl[\mathfrak{L}^{-1}\bigl(f\bigl(r,-v(r)\bigr)\bigr)\bigr]^k,\quad r\in(0,1),\\&v'(0)=0,\qquad v(1)=0.\end{aligned}\tag{5}$$

Now with a simple transformation $\varphi=-v$, (5) can be rewritten as follows:

$$\begin{aligned}&\biggl[\frac{r^{N-k}}{k}\bigl(-\varphi'(r)\bigr)^k\biggr]'=\frac{r^{N-1}}{C_{N-1}^{k-1}}\bigl[\mathfrak{L}^{-1}\bigl(f\bigl(r,\varphi(r)\bigr)\bigr)\bigr]^k,\quad r\in(0,1),\\&\varphi'(0)=0,\qquad\varphi(1)=0.\end{aligned}$$

Then $u(|x|)=-\varphi(r)$ is a radial solution of equation (1) if and only if $\varphi(r)$ is a solution of the integral equation

$$\varphi(r)=\int_r^1\Biggl(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\bigl[\mathfrak{L}^{-1}\bigl(f\bigl(s,\varphi(s)\bigr)\bigr)\bigr]^k\,\mathrm{d}s\Biggr)^{1/k}\mathrm{d}t,\quad r\in(0,1).$$

Define the Banach space $E=C[0,1]$ with the usual supremum normal $\|\varphi(x)\|=\max_{x\in[0,1]}|\varphi(x)|$, and define a nonlinear operator $\mathfrak{F}$ on E as follows:

$$(\mathfrak{F}\varphi)(r)=\int_r^1\Biggl(\frac{k}{t^{N-k}}\int_0^t\frac{s^{N-1}}{C_{N-1}^{k-1}}\bigl[\mathfrak{L}^{-1}\bigl(f\bigl(s,\varphi(s)\bigr)\bigr)\bigr]^k\,\mathrm{d}s\Biggr)^{1/k}\mathrm{d}t,\quad r\in[0,1].$$

We will establish conditions for the existence, nonexistence, and multiplicity of radial solutions for equation (1) in Sections 3–5, respectively.

3 Existence results

Nonlinear functional analysis method plays an important role for studying nonlinear ordinary differential equations and partial differential equations [7, 16, 18, 19, 23–25, 34–37, 40–49]. Many fixed point theorems have been developed to solve various boundary value problems of differential equations [7, 38, 39]. In this section, our main tool to establish a existence result of solution for k-Hessian equation (1) is the following Leray–Schauder nonlinear alternative theorem [8].

Lemma 3. *Let E be a real Banach space, Ω be a bounded open subset of E, $0 \in \Omega$, $\mathfrak{L} : \overline{\Omega} \to E$ is a completely continuous operator. Then either there exist $\varphi \in \partial\Omega$ and $\mu > 1$ such that $\mathfrak{L}(\varphi) = \mu\varphi$ or there exists a fixed point $\varphi^* \in \overline{\Omega}$.*

Theorem 1. *Assume that there exist a nondecreasing function $\psi : [0,+\infty) \to [0,+\infty)$ and a function $a(t) \in C[0,1]$ such that*

$$\bigl|f(t,u)\bigr| \leqslant a(t)\psi\bigl(|u|\bigr), \quad t \in [0,1]. \tag{6}$$

Then the k-Hessian equation (1) *has at least one solution if there exists a real number $m > 0$ such that*

$$\|a\|\psi(m) \leqslant \mathfrak{L}\biggl(2m\biggl(\frac{NC_{N-1}^{k-1}}{k}\biggr)^{1/k}\biggr). \tag{7}$$

Proof. Firstly, we prove that the operator $\mathfrak{F}$ is completely continuous. Clearly, continuity of the operator $\mathfrak{F}$ follows from the continuity of f.

Let $D \subset E$ be any bounded set. Then there exists a constant $L > 0$ such that $|f(r,\varphi)| \leqslant L$ for any $(r,\varphi) \in [0,1] \times D$, thus, we have

$$\begin{aligned}\bigl|(\mathfrak{F}\varphi)(r)\bigr| &\leqslant \int_r^1 \biggl(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}\bigl|\mathfrak{L}^{-1}\bigl(f\bigl(s,\varphi(s)\bigr)\bigr)\bigr|^k\,\mathrm{d}s\biggr)^{1/k}\mathrm{d}t \\ &\leqslant \mathfrak{L}^{-1}(L)\int_0^1 \biggl(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}\,\mathrm{d}s\biggr)^{1/k}\mathrm{d}t \\ &\leqslant \frac{1}{2}\mathfrak{L}^{-1}(L)\biggl(\frac{k}{NC_{N-1}^{k-1}}\biggr)^{1/k} = \mathrm{const}.\end{aligned}$$

Therefore, $\mathfrak{F}(D)$ is uniformly bounded.

Now we show that $\mathfrak{F}(D)$ is equicontinuous on $[0,1]$. For any $(r,\varphi) \in [0,1] \times D$, we have

$$\begin{aligned}\biggl|\frac{\mathrm{d}}{\mathrm{d}r}(\mathfrak{F}\varphi)(r)\biggr| &= \biggl(\frac{k}{r^{N-k}}\int_0^r \frac{s^{N-1}}{C_{N-1}^{k-1}}\bigl|\mathfrak{L}^{-1}\bigl(f\bigl(s,\varphi(s)\bigr)\bigr)\bigr|^k\,\mathrm{d}s\biggr)^{1/k} \\ &\leqslant \mathfrak{L}^{-1}(L)\biggl(\frac{k}{NC_{N-1}^{k-1}}\biggr)^{1/k} r \leqslant \mathfrak{L}^{-1}(L)\biggl(\frac{k}{NC_{N-1}^{k-1}}\biggr)^{1/k},\end{aligned}$$

and then, for any $\varphi \in D$ and $r_1, r_2 \in [0,1]$, we get

$$\begin{aligned}&\bigl|(\mathfrak{F}\varphi)(r_1) - (\mathfrak{F}\varphi)(r_2)\bigr| \\ &\quad = \biggl|\int_{t_2}^{t_1} \frac{\mathrm{d}}{\mathrm{d}r}(\mathfrak{F}\varphi)(r)\,\mathrm{d}r\biggr| \leqslant \mathfrak{L}^{-1}(L)\biggl(\frac{k}{NC_{N-1}^{k-1}}\biggr)^{1/k}|t_1 - t_2|.\end{aligned} \tag{8}$$

It follows from (8) that $\mathfrak{F}(E)$ is equicontinuous on $[0,1]$. Thus, according to the Ascoli–Arzela theorem, $\mathfrak{F}$ is a completely continuous operator.

Now we consider $B_m = \{\varphi \in C[0,1]\colon \|\varphi\| \leqslant m\}$. It follows from the Leray–Schauder nonlinear alternative theorem that either the operator $\mathfrak{F}$ has a fixed point or there exists $\varphi \in \partial B_m$ such that $\mathfrak{F}\varphi = \mu\varphi$ for some $\mu > 1$. We assert that the latter conclusion does not hold. Otherwise, there exist some $\varphi_0 \in \partial B_m$ and some $\mu > 1$ such that $\mathfrak{F}\varphi_0 = \mu\varphi_0$. Thus, it follows from (6)–(7) that

$$\begin{aligned}
\mu m &= \mu\|\varphi_0\| = \|\mathfrak{F}\varphi_0\| \\
&\leqslant \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left|\mathfrak{L}^{-1}\bigl(f\bigl(s,\varphi_0(s)\bigr)\bigr)\right|^k \mathrm{d}s\right)^{1/k} \mathrm{d}t \\
&\leqslant \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left|\mathfrak{L}^{-1}\bigl(a(s)\psi\bigl(\left|\varphi_0(s)\right|\bigr)\bigr)\right|^k \mathrm{d}s\right)^{1/k} \mathrm{d}t \\
&\leqslant \mathfrak{L}^{-1}\bigl(\|a\|\psi(m)\bigr) \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \mathrm{d}s\right)^{1/k} \mathrm{d}t \\
&\leqslant \frac{1}{2}\left(\frac{k}{NC_{N-1}^{k-1}}\right)^{1/k} \mathfrak{L}^{-1}\bigl(\|a\|\psi(m)\bigr) \leqslant m,
\end{aligned}$$

that is $\mu \leqslant 1$, which leads to a contraction with $\mu > 1$. In consequence, the operator $\mathfrak{F}$ has a fixed point in $C[0,1]$ with $\|\varphi\| \leqslant m$. This further implies that problem (1) has at least one solution on $[0,1]$ if (7) holds. The proof is completed. $\square$

By Theorem 1 we have the following corollary.

Corollary 1. *Assume that there exists a function* $a(t) \in C[0,1]$ *such that*

$$\left|f(t,u)\right| \leqslant a(t), \quad t \in [0,1].$$

Then the k*-Hessian equation* (1) *has at least one solution.*

4 Nonexistence results

In this section, we are interested in the nonexistence result of solutions for the k-Hessian equation (1) with a parameter μ:

$$\begin{aligned}
&\mathfrak{B}\bigl(S_k^{1/k}\bigl(\lambda\bigl(D^2u\bigr)\bigr)\bigr)S_k^{1/k}\bigl(\lambda\bigl(D^2u\bigr)\bigr) = \mu f\bigl(|x|,-u\bigr) \quad \text{in } \Omega \subset \mathbb{R}^N, \\
&u = 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{9}$$

Let

$$f_0 = \lim_{|\varphi|\to 0^+} \max_{r\in[0,1]} \frac{f(r,\varphi)}{\mathfrak{L}(|\varphi|)}, \qquad f_\infty = \lim_{|\varphi|\to +\infty} \min_{r\in[0,1]} \frac{f(r,\varphi)}{\mathfrak{L}(|\varphi|)},$$

then we have the following nonexistence result of solutions.

Theorem 2. *Assume that $f_0 < \infty$ and $f_\infty < \infty$, then there exists $\mu_0 > 0$ such that the k-Hessian equation* (9) *has no solutions for* $0 < \mu < \mu_0$.

Proof. It follows from $f_0 < \infty$ and $f_\infty < \infty$ that

(i) for any $\varepsilon_1 > 0$, there exists a constant $\delta > 0$ such that $f(r,\varphi) < (f_0 + \varepsilon_1)\mathfrak{L}(|\varphi|)$ for $0 < \varphi < \delta$ and $r \in [0,1]$;

(ii) for any $\varepsilon_2 > 0$, there exists a constant $M > 0$ such that $|f(r,\varphi)| < (f_\infty + \varepsilon_2) \times \mathfrak{L}(|\varphi|)$ for $\varphi > M$ and $r \in [0,1]$.

Without loss of generality, take $\delta < M$ and

$$\varepsilon = \max\left\{ f_0 + \varepsilon_1,\ f_\infty + \varepsilon_2,\ \max_{(r,\varphi)\in[0,1]\times[\delta,M]} \frac{f(r,\varphi)}{\mathfrak{L}(|\varphi|)} \right\},$$

then for any $(r,\varphi) \in [0,1] \times [0,\infty)$, we have $f(r,\varphi) \leqslant \varepsilon \mathfrak{L}(|\varphi|)$.

Let $\mu_0 = 1/\varepsilon$, assume that there is a solution φ_0 of the k-Hessian equation (9). We will show that this leads to a contradiction for $0 < \mu < \mu_0$. Since $\|\mathfrak{F}_\mu \varphi_0\| = \|\varphi_0\|$ for $r \in [0,1]$ and $N > 2k$, then for $0 < \mu < \mu_0$, we have

$$\begin{aligned}
\|\varphi_0\| = \|\mathfrak{F}_\mu \varphi_0\| &= \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left| \mathfrak{L}^{-1}\big(\mu f\big(s, \varphi_0(s)\big)\big) \right|^k \mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&\leqslant \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left| \mathfrak{L}^{-1}\big(\mu_0 \varepsilon \mathfrak{L}\big(|\varphi_0|\big)\big) \right|^k \mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&\leqslant \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \mathrm{d}s \right)^{1/k} \mathrm{d}t \|\varphi_0\| \\
&\leqslant \frac{1}{2} \left(\frac{k}{N C_{N-1}^{k-1}} \right)^{1/k} \|\varphi_0\| < \|\varphi_0\|,
\end{aligned}$$

which is a contradiction. The proof is completed. □

5 Results on multiple solutions

In order to obtain the multiplicity of radial solutions of (1), we need the following Leggett–Williams fixed point theorem.

Definition 1. Let P be a cone in a real Banach space E. A mapping α is called a nonnegative continuous concave functional on P if it satisfies

(i) $\alpha : P \to [0, +\infty)$ is continuous;

(ii) $\alpha(\lambda u + (1-\lambda)v) \geqslant \lambda\alpha(u) + (1-\lambda)\alpha(v)$ for all $u, v \in P$ and $\lambda \in [0,1]$.

Suppose that α is a nonnegative continuous concave functional on P, for constants $0 < a < b$ and $c > 0$, define the following convex sets:

$$P_c = \{u \in P\colon \|u\| \leqslant c\},$$
$$P(\alpha, a, b) = \{u \in P\colon a \leqslant \alpha(u),\ \|u\| \leqslant b\}.$$

Lemma 4 [Leggett–Williams fixed point theorem]. (*See* [22].) *Let $\mathfrak{F} : P_c \to P_c$ be a completely continuous operator, α be a nonnegative continuous concave functional on P satisfying $\alpha(u) \leqslant \|u\|$ for all $u \in P_c$. Assume there exist some constants $0 < d < a < b \leqslant c$ such that*

(i) $\{u \in P(\alpha, a, b)\colon \alpha(u) > a\} \neq \phi$ *and* $\alpha(\mathfrak{F}u) > a$ *for* $u \in P(\alpha, a, b)$;
(ii) $\|\mathfrak{F}u\| < d$ *for* $\|u\| \leqslant d$;
(iii) $\alpha(\mathfrak{F}u) > a$ *for* $u \in P(\alpha, a, c)$ *with* $\|\mathfrak{F}u\| > b$.

Then $\mathfrak{F}$ has at least three fixed points u_1, u_2, u_3 satisfying

$$\|u_1\| < d, \qquad a < \alpha(u_2), \qquad \|u_3\| > d \quad \text{with } \alpha(u_3) < a.$$

Now let

$$P = \{\varphi \in E\colon \varphi(x) \geqslant 0 \text{ and } \varphi \text{ is nonincreasing on } [0,1]\},$$

then P is a cone in E. We still consider the nonlinear operator $\mathfrak{F}$ on E:

$$(\mathfrak{F}\varphi)(r) = \int\limits_r^1 \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \big[\mathfrak{L}^{-1}\big(f\big(s, \varphi(s)\big)\big)\big]^k \,\mathrm{d}s \right)^{1/k} \mathrm{d}t, \quad r \in [0,1].$$

For any $\varphi \in P$, clearly, $(\mathfrak{F}\varphi)(r) \geqslant 0$ for all $r \in [0,1]$, $(\mathfrak{F}\varphi)'(0) = (\mathfrak{F}\varphi)(1) = 0$, and

$$(\mathfrak{F}\varphi)'(r) = -\left(\frac{k}{r^{N-k}} \int\limits_0^r \frac{s^{N-1}}{C_{N-1}^{k-1}} \Big[\mathfrak{L}^{-1}\big(f\big(s, \varphi(s)\big)\big)\Big]^k \mathrm{d}s \right)^{1/k} \leqslant 0, \quad r \in [0,1].$$

Thus, we have $\mathfrak{F} : P \to P$. On the other hand, by the standard argument we know that $\mathfrak{F}$ is continuous and compact, also see [43]. So, from the above facts we have the following lemma.

Lemma 5. *$\mathfrak{F} : P \to P$ is continuous and compact.*

Now for some $\mu_0 \in (0, 1/2)$, define a nonnegative continuous concave functional $\alpha : P \to [0, +\infty)$

$$\alpha(\varphi) = \min_{0 \leqslant r \leqslant 1-\mu_0} \{\varphi(r)\}, \quad \varphi \in P,$$

then we have $\alpha(\varphi) \leqslant \|\varphi\|$ for $\varphi \in P$ and $\alpha(\varphi) = \varphi(1-\mu_0)$, $\alpha(\mathfrak{F}\varphi) = (\mathfrak{F}\varphi)(1-\mu_0)$ for all $\varphi \in P$.

In what follows, we define two constants:

$$\lambda_1 = \left(\frac{NC_{N-1}^{k-1}}{k}\right)^{(1+\sigma)/k}, \qquad \lambda_2 = \frac{2}{\mu_0(2-\mu_0)}\left[\frac{NC_{N-1}^{k-1}}{k}\right]^{1/k},$$

where σ is defined by $\mathfrak{B}$, which depends on the operator $\mathfrak{B}$. Clearly, it follows from $N > 2k$ that $\lambda_1 > 1$.

Theorem 3. *Assume that there exist four constants a, b, c, d such that $0 < d < a < \mu_0(1-\mu_0)b < b \leqslant c$, and the following conditions are satisfied*:

(A1) $f(r,\varphi) < \lambda_1 \mathfrak{L}(d)$ *for all* $(r,\varphi) \in [0,1]\times[0,d]$;
(A2) $f(r,\varphi) \geqslant \mathfrak{L}(\lambda_2 a)$ *for all* $(r,\varphi) \in [1-\mu_0,1]\times[a,b]$;
(A3) $f(r,\varphi) \leqslant \lambda_1 \mathfrak{L}(c)$ *for all* $(r,\varphi) \in [0,1]\times[0,c]$;
(A4) $\min_{[0,1-\mu_0]\times[a,c]} f(s,r) \geqslant \mathfrak{L}(a/(\mu_0(1-\mu_0)b)) \max_{[0,1]\times[0,c]} f(s,r)$.

Then the k-Hessian equation (1) *has at least three radial solutions satisfying*

$$\|\varphi_1\| < d, \qquad \min_{r\in[0,1-\mu_0]}(\varphi_2)(r) > a, \qquad \|\varphi_3\| > d \quad \textit{with} \min_{r\in[0,1-\mu_0]}(\varphi_3)(r) < a.$$

Proof. Firstly, from the definition of α we have $\alpha(\varphi) \leqslant \|\varphi\|$ for $\varphi \in P$. Now we prove that $\mathfrak{F} : \overline{P}_c \to \overline{P}_c$, and for any $\varphi \in \overline{P}_d$, there is $\|\mathfrak{F}\varphi\| \leqslant d$.

In fact, for any $\varphi \in \overline{P}_c$, we have $\|\varphi\| \leqslant c$, and it follows from (A3) and (4) that

$$\begin{aligned}
\|\mathfrak{F}\varphi\| &= \max_{r\in[0,1]} \int_r^1 \left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathfrak{L}^{-1}\big(f\big(s,\varphi(s)\big)\big)\right]^k \mathrm{d}s\right)^{1/k} \mathrm{d}t \\
&= \int_0^1 \left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}\left[\mathfrak{L}^{-1}\big(f\big(s,\varphi(s)\big)\big)\right]^k \mathrm{d}s\right)^{1/k} \mathrm{d}t \\
&\leqslant \int_0^1 \left(\frac{k}{t^{N-k}}\int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}}\big(\lambda_1^{1/(1+\sigma)}c\big)^k \mathrm{d}s\right)^{1/k} \mathrm{d}t \\
&= \lambda_1^{1/(1+\sigma)}\left(\frac{k}{NC_{N-1}^{k-1}}\right)^{1/k} c = c,
\end{aligned}$$

which implies that $\mathfrak{F} : \overline{P}_c \to \overline{P}_c$. In the same way, we have $\|\mathfrak{F}\varphi\| < d$ for any $\varphi \in \overline{P}_d$. Thus, condition (ii) of Lemma 4 is satisfied.

Secondly, we show that condition (i) of Lemma 4 also holds. To do this, take $\varphi_0 = \mu_0(1-\mu_0)b$, we have $\varphi_0 = \mu_0(1-\mu_0)b > a$, and then $\alpha(\varphi_0) > a$. In addition, since $\mu_0 \in (0,1/2)$, we have $\|\varphi_0\| = \mu_0(1-\mu_0)b \leqslant b$. Thus,

$$\{\varphi \in P(\alpha,a,b)\colon \alpha(\varphi) > a\} \neq \phi.$$

Now for any $\varphi \in P(\alpha, a, b)$, we have $\alpha(\varphi) \geqslant a$ and $\|\varphi\| \leqslant b$, and then $a \leqslant \varphi(r) \leqslant b$ for any $r \in [0, 1-\mu_0]$. Thus, it results from (A2) that

$$\begin{aligned}
\alpha\bigl(\mathfrak{F}\varphi(r)\bigr) &= \min_{r\in[0,1-\mu_0]} \mathfrak{F}\varphi(r) = (\mathfrak{F}\varphi)(1-\mu_0) \\
&= \int\limits_{1-\mu_0}^{1} \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \bigl[\mathfrak{L}^{-1}\bigl(f\bigl(s,\varphi(s)\bigr)\bigr)\bigr]^k \,\mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&\geqslant \lambda_2 a \left(\frac{k}{C_{N-1}^{k-1}} \right)^{1/k} \int\limits_{1-\mu_0}^{1} \left(\frac{1}{t^{N-k}} \int\limits_0^t s^{N-1} \,\mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&= \lambda_2 a \left(\frac{k}{C_{N-1}^{k-1}} \right)^{1/k} \frac{2\mu_0 - \mu_0^2}{2N^{\frac{1}{k}}} = a.
\end{aligned}$$

So, condition (i) of Lemma 4 holds.

Thirdly, we verify that condition (iii) of Lemma 4 is satisfied. For any $\varphi \in P(\alpha, a, c)$ with $\|\mathfrak{F}\varphi\| > b$, we have

$$\alpha\bigl(\varphi(r)\bigr) = \min_{0\leqslant r\leqslant 1-\mu_0} \bigl\{\varphi(r)\bigr\} \geqslant a, \quad \|\varphi\| \leqslant c.$$

Then we have

$$a \leqslant \varphi(r) \leqslant c \quad \forall r \in [0, 1-\mu_0]$$

and

$$\begin{aligned}
\|\mathfrak{F}\varphi\| &= \max_{r\in[0,1]} \int\limits_r^1 \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \bigl[\mathfrak{L}^{-1}\bigl(f\bigl(s,\varphi(s)\bigr)\bigr)\bigr]^k \,\mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&= \int\limits_0^1 \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \bigl[\mathfrak{L}^{-1}\bigl(f\bigl(s,\varphi(s)\bigr)\bigr)\bigr]^k \,\mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&\leqslant \mathfrak{L}^{-1}\Bigl(\max_{[0,1]\times[0,c]} f(s,r) \Bigr) \left(\frac{k}{N C_{N-1}^{k-1}} \right)^{1/k}.
\end{aligned} \tag{10}$$

On the other hand, for $\varphi \in P(\alpha, a, c)$ with $\|\mathfrak{F}\varphi\| > b$, by (10) and (A4), we have

$$\begin{aligned}
\alpha\bigl(\mathfrak{F}\varphi(r)\bigr) &= \min_{r\in[0,1-\mu_0]} \mathfrak{F}\varphi(r) \\
&= \int\limits_{1-\mu_0}^{1} \left(\frac{k}{t^{N-k}} \int\limits_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \bigl[\mathfrak{L}^{-1}\bigl(f\bigl(s,\varphi(s)\bigr)\bigr)\bigr]^k \,\mathrm{d}s \right)^{1/k} \mathrm{d}t \\
&\geqslant \mathfrak{L}^{-1}\Bigl(\min_{[0,1-\mu_0]\times[a,c]} f(s,r) \Bigr) \left(\frac{k}{N C_{N-1}^{k-1}} \right)^{1/k} \int\limits_{1-\mu_0}^{1} \left(\frac{1}{t^{N-k}} \int\limits_0^t s^{N-1} \,\mathrm{d}s \right)^{1/k} \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \mathfrak{L}^{-1}\Big(\min_{[0,1-\mu_0]\times[a,c]} f(s,r)\Big)\bigg(\frac{k}{NC_{N-1}^{k-1}}\bigg)^{1/k}\mu_0(1-\mu_0)\\
&\geqslant \frac{a}{b}\mathfrak{L}^{-1}\Big(\max_{[0,1]\times[0,c]} f(s,r)\Big)\bigg(\frac{k}{NC_{N-1}^{k-1}}\bigg)^{1/k} \geqslant \frac{a}{b}\|\mathfrak{F}\varphi\| > a.
\end{aligned}
$$

Thus, all hypotheses of the Leggett–Williams theorem are satisfied. So, according to the Leggett–Williams theorem, the k-Hessian equation (1) has at least three radial solutions satisfying

$$\|\varphi_1\| < d, \qquad \min_{r\in[0,1-\mu_0]}(\varphi_2)(r) > a, \qquad \|\varphi_3\| > d \quad \text{with} \min_{r\in[0,1-\mu_0]}(\varphi_3)(r) < a. \quad \square$$

Corollary 2. *Assume that there exist three constants a, b, d such that* $0 < d < a < \mu_0(1-\mu_0)b$ *and* (A1), (A2)*, and the following condition are satisfied*:

(A3′) $f(r,\varphi) \leqslant \lambda_1\mathfrak{L}(b)$ *for all* $(r,\varphi) \in [0,1]\times[0,b]$.

Then the k-Hessian equation (1) *has at least three radial solutions satisfying*

$$\|\varphi_1\| < d, \qquad \min_{r\in[0,1-\mu_0]}(\varphi_2)(r) > a, \qquad \|\varphi_3\| > d \quad \textit{with} \min_{r\in[0,1-\mu_0]}(\varphi_3)(r) < a.$$

Proof. Take $c = b$, then we have $0 < d < a < \mu_0(1-\mu_0)b < b = c$, thus, (A1), (A2), and (A3) of Theorem 3 hold. In addition, if $b = c$, condition (i) of Lemma 4 implies condition (iii). By Theorem 3 the conclusion of Corollary 2 is true. $\square$

Corollary 3. *Suppose that* (A2) *and* (A4) *hold. In addition, assume the following conditions are satisfied*:

(A5′) $$\lim_{s\to0^+}\max_{r\in[0,1]}\frac{f(r,s)}{\mathfrak{L}(s)} < \lambda_1;$$

(A6′) $$\lim_{s\to1^-}\max_{r\in[0,1]}\frac{f(r,s)}{\mathfrak{L}(s)} < \lambda_1.$$

Then the k-Hessian equation (1) *has at least three radial solutions satisfying*

$$\|\varphi_1\| < d, \qquad \min_{r\in[0,1-\mu_0]}(\varphi_2)(r) > a, \qquad \|\varphi_3\| > d \quad \textit{with} \min_{r\in[0,1-\mu_0]}(\varphi_3)(r) < a.$$

Proof. In fact, clearly, (A5) implies (A1). So, we only need to prove that there exists a positive constant c with $c \geqslant b$ such that $\mathfrak{F} : \overline{P}_c \to \overline{P}_c$.

It follows from (A6) that there exists a constant $\delta > 0$ such that

$$f(r,s) \leqslant \lambda_1\mathfrak{L}(s) \quad \forall r\in[0,1],\ s \geqslant \delta.$$

Taking

$$M = \max_{(r,s)\in[0,1]\times[0,\delta]} f(r,s),$$

then we have

$$f(r,s) \leqslant M + \lambda_1 \mathfrak{L}(s) \quad \forall (r,s) \in [0,1] \times [0,1).$$

Let

$$c > \max\left\{b,\ \mathfrak{L}^{-1}\left(\frac{M}{(2^{1+\sigma}-1)\lambda_1}\right)\right\}$$

for any $\varphi \in \overline{P}_c$, one has

$$\begin{aligned}
\|\mathfrak{F}\varphi\| &= \max_{r\in[0,1]} \int_r^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(f\big(s,\varphi(s)\big)\big)\right]^k \mathrm{d}s\right)^{1/k} \mathrm{d}t \\
&\leqslant \int_0^1 \left(\frac{k}{t^{N-k}} \int_0^t \frac{s^{N-1}}{C_{N-1}^{k-1}} \left[\mathfrak{L}^{-1}\big(M + \lambda_1\mathfrak{L}(c)\big)\right]^k \mathrm{d}s\right)^{1/k} \mathrm{d}t \\
&= \frac{1}{2}\left(\frac{k}{NC_{N-1}^{k-1}}\right)^{1/k} \mathfrak{L}^{-1}\big(M + \lambda_1\mathfrak{L}(c)\big) \\
&\leqslant \frac{1}{2}\left(\frac{k}{NC_{N-1}^{k-1}}\right)^{1/k} \mathfrak{L}^{-1}\big(2^{1+\sigma}\lambda_1\mathfrak{L}(c)\big) \\
&\leqslant \frac{1}{2}\left(\frac{k}{NC_{N-1}^{k-1}}\right)^{1/k} \times 2\lambda_1^{1+\sigma} c = c,
\end{aligned}$$

which implies that $\mathfrak{F} : \overline{P}_c \to \overline{P}_c$. □

Corollary 4. *Assume that there exist constants* $0 < d_1 < a_1 < \mu_0(1-\mu_0)b_1 < d_2 < a_2 < \mu_0(1-\mu_0)b_2 < \cdots < d_n < a_n < \mu_0(1-\mu_0)b_n$ *such that the following conditions hold*:

(A1′) $f(r,\varphi) \leqslant \lambda_1\mathfrak{L}(d_i)$ *for all* $(r,\varphi) \in [0,1] \times [0,d_i]$, $1 \leqslant i \leqslant n$;
(A2′) $f(r,\varphi) \geqslant \mathfrak{L}(\lambda_2 a_i)$ *for all* $(r,\varphi) \in [\mu_0, 1-\mu_0] \times [a_i,b_i]$, $1 \leqslant i \leqslant n$;
(A3′) $f(r,\varphi) \leqslant \lambda_1\mathfrak{L}(b_i)$ *for all* $(r,\varphi) \in [0,1] \times [0,b_i]$, $1 \leqslant i \leqslant n$.

Then the k-Hessian equation (1) *has at least* $2n-1$ *radial solutions.*

6 Numerical examples

In this section, we present some examples to illustrate our main results.

Example 1. Consider the existence of radial solutions for the following Dirichlet problem of the 2-Hessian equation:

$$\begin{aligned}
&\big(S_2^{1/2}\big(\lambda\big(D^2u\big)\big)\big)^2 S_2^{1/2}\big(\lambda\big(D^2u\big)\big) = |x|^{1/2}\sin^2(-u) \quad \text{in } \Omega \subset \mathbb{R}^5, \\
&u = 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{11}$$

where Ω is a unit ball. Then the 2-Hessian equation (11) has at least one solution.

Proof. In fact, here $k=2$ and $f(t,u)=t^{1/2}\sin^2 u$. Since $\mathfrak{B}(x)=x^2$, we have $\mathfrak{L}(x)=x^3$ and

$$\bigl|f(t,u)\bigr| = \bigl|t^{1/2}\sin^2 u\bigr| \leqslant t^{1/2}u^2 = a(t)\psi(u), \quad (t,u)\in[0,1]\times[0,+\infty),$$

where $a(t)=t^{1/2}$, $\psi(u)=u^2$. Take $m=2$, then

$$\|a\|\psi(m) = 4 \leqslant \mathfrak{L}\left(2m\left(\frac{NC_{N-1}^{k-1}}{k}\right)^{1/k}\right) = \left(4\left(\frac{5C_4^1}{2}\right)^{1/2}\right)^3 = 640\sqrt{10},$$

and by Theorem 1, the 2-Hessian equation (11) has at least one solution. $\square$

Example 2. Consider the nonexistence of radial solutions for the following Dirichlet problem of the 2-Hessian equation with a parameter

$$\begin{aligned}&\bigl(S_2^{1/2}\bigl(\lambda\bigl(D^2u\bigr)\bigr)\bigr)^2 S_2^{1/2}\bigl(\lambda\bigl(D^2u\bigr)\bigr) = -\mu\bigl(|x|^{1/2}+1\bigr)u^3 \quad \text{in } \Omega\subset\mathbb{R}^5,\\ &u=0 \quad \text{on } \partial\Omega,\end{aligned} \tag{12}$$

where Ω is a unit ball. Then there exists $\mu_0>0$ such that the 2-Hessian equation (12) has no solutions for $0<\mu<\mu_0$.

Proof. In fact, here $k=2, N=5$ and $f(r,\varphi)=(r^{1/2}+1)\varphi^3$, $\mathfrak{B}(x)=x^2$. Thus, $\mathfrak{L}(x)=x^3$ and

$$f_0 = \lim_{|\varphi|\to 0^+}\max_{r\in[0,1]}\frac{f(r,\varphi)}{\mathfrak{L}(|\varphi|)} = 2, \qquad f_\infty = \lim_{|\varphi|\to+\infty}\min_{r\in[0,1]}\frac{f(r,\varphi)}{\mathfrak{L}(|\varphi|)} = 1.$$

By Theorem 2 there exists $\mu_0>0$ such that the 2-Hessian equation (12) has no solutions for $0<\mu<\mu_0$. $\square$

Example 3. Consider the existence of multiple radial solutions for the following Dirichlet problem of the 2-Hessian equation:

$$\begin{aligned}&\bigl(S_2^{1/2}\bigl(\lambda\bigl(D^2u\bigr)\bigr)\bigr)^2 S_2^{1/2}\bigl(\lambda\bigl(D^2u\bigr)\bigr) = f\bigl(|x|,-u\bigr) \quad \text{in } \Omega\subset\mathbb{R}^5,\\ &u=0 \quad \text{on } \partial\Omega,\end{aligned} \tag{13}$$

where Ω is a unit ball and

$$f(r,\varphi) = \begin{cases} \mathrm{e}^{-r}+\frac12\varphi^2, & 0\leqslant\varphi\leqslant 1,\\ \mathrm{e}^{-r}+\frac12+[18\cdot(\frac{36}{5})^3\cdot 10^{3/2}-\frac12](\varphi-1), & 1\leqslant\varphi\leqslant 2,\\ \mathrm{e}^{-r}+(\frac{36}{5})^3\cdot 10^{3/2}(20-\varphi), & 2\leqslant\varphi\leqslant 18,\\ \mathrm{e}^{-r}+2\cdot(\frac{36}{5})^3\times 10^{\frac32}, & \varphi\geqslant 18. \end{cases} \tag{14}$$

Then the 2-Hessian equation (14) has at least three solutions satisfying

$$\|\varphi_1\|<1, \qquad \min_{r\in[0,3/5]}(\varphi_2)(r)>2, \qquad \|\varphi_3\|>18 \quad \text{with} \quad \min_{r\in[0,3/5]}(\varphi_3)(r)<2.$$

Proof. Since $k = 2$, $N = 5$, $\sigma = 2$, $\mathfrak{L}(x) = x^3$. Take $d = 1$, $a = 2$, $b = 18$, $c = 20$, $\mu_0 = 2/5$, then we have

$$\lambda_1 = \left(\frac{NC_{N-1}^{k-1}}{k}\right)^{(1+\sigma)/k} = 10^{3/2}, \qquad \lambda_2 = \frac{2}{\mu_0(2-\mu_0)}\left[\frac{NC_{N-1}^{k-1}}{k}\right]^{1/k} = \frac{25}{8} \cdot 10^{1/2}.$$

Now we check conditions (A1)–(A4) of Theorem 3.

Firstly, for any $(r, \varphi) \in [0, 1] \times [0, 1]$, we have

$$f(r, \varphi) \leqslant \max_{(r,\varphi)\in[0,1]\times[0,1]} f(r, \varphi) = \frac{3}{2} < \lambda_1 \mathfrak{L}(d) = 10^{3/2}.$$

So, condition (A1) holds.

Next, we consider the interval $(r, \varphi) \in [3/5, 1] \times [2, 18]$, one has

$$\begin{aligned} f(r, \varphi) &\geqslant \min_{(r,\varphi)\in[3/5,1]\times[2,18]} f(r, \varphi) = \mathrm{e}^{-1} + 2 \cdot \left(\frac{36}{5}\right)^3 \cdot 10^{3/2} \\ &\geqslant \mathfrak{L}(\lambda_2 a) = \left(\frac{25}{4}\right)^3 \cdot 10^{3/2}. \end{aligned}$$

Thus, condition (A2) is satisfied.

Thirdly, we focus on $(r, \varphi) \in [0, 1] \times [0, 20]$, and we get

$$\begin{aligned} f(r, \varphi) &\leqslant \max_{(r,\varphi)\in[0,1]\times[0,20]} f(r, \varphi) = 1 + 18 \cdot \left(\frac{36}{5}\right)^3 \cdot 10^{3/2} \\ &< \lambda_1 \mathfrak{L}(c) = 8000 \cdot 10^{3/2}. \end{aligned}$$

Therefore, condition (A2) also holds.

In the end, we check (A4). In fact, we have

$$\begin{aligned} &\min_{[0,1-\mu_0]\times[a,c]} f(s, r) = \min_{[0,3/5]\times[2,20]} f(s, r) = \mathrm{e}^{-3/5} + 2 \cdot \left(\frac{36}{5}\right)^3 \cdot 10^{3/2} = 23607, \\ &\mathfrak{L}\left(\frac{a}{\mu_0(1-\mu_0)b}\right) \max_{[0,1]\times[0,c]} f(s, r) \\ &\quad = \left(\frac{2}{\frac{2}{5} \cdot \frac{3}{5} \cdot 18}\right)^3 \max_{[0,1]\times[0,20]} f(s, r) = \left(\frac{25}{54}\right)^3 \left(1 + 18 \cdot \left(\frac{36}{5}\right)^3 \cdot 10^{3/2}\right) \\ &\quad = 21082, \end{aligned}$$

which implies (A4) is satisfied.

Thus, all of the conditions of Theorem 3 are satisfied. By Theorem 3 the 2-Hessian equation (13) has at least three solutions satisfying

$$\|\varphi_1\| < 1, \qquad \min_{r\in[0,3/5]} (\varphi_2)(r) > 2, \qquad \|\varphi_3\| > 18 \quad \text{with} \quad \min_{r\in[0,3/5]} (\varphi_3)(r) < 2. \qquad \square$$

7 Conclusion

The k-Hessian equation is a class of very important fully nonlinear and nonuniformly elliptic partial differential equations, which fill up the gap between the Monge–Ampère and Poisson equations. In this paper, we introduce a nonlinear operator $\mathfrak{B}$ such that k-Hessian equation we studied include many important and interesting cases. To establish the existence, nonexistence, and multiplicity of radial solutions to Dirichlet problems of k-Hessian equations with a nonlinear operator in a ball, we adopt the Leray–Schauder alternative theorem and the Leggett–Williams fixed point theorem as well as some suitable growth conditions for nonlinearity. Our work improves and generalizes some recent work such as [3, 4, 6, 17, 21, 28].

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