

Finite-time synchronization of Markovian neural networks with proportional delays and discontinuous activations*

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Abstract. In this paper, finite-time synchronization of neural networks (NNs) with discontinuous activation functions (DAFs), Markovian switching, and proportional delays is studied in the framework of Filippov solution. Since proportional delay is unbounded and different from infinite-time distributed delay and classical finite-time analytical techniques are not applicable anymore, new 1-norm analytical techniques are developed. Controllers with and without the sign function are designed to overcome the effects of the uncertainties induced by Filippov solutions and further synchronize the considered NNs in a finite time. By designing new Lyapunov functionals and using M -matrix method, sufficient conditions are derived to guarantee that the considered NNs realize synchronization in a settling time without introducing any free parameters. It is shown that, though the proportional delay can be unbounded, complete synchronization can still be realized, and the settling time can be explicitly estimated. Moreover, it is discovered that controllers with sign function can reduce the control gains, while controllers without the sign function can overcome chattering phenomenon. Finally, numerical simulations are given to show the effectiveness of theoretical results.

Keywords: neural networks, discontinuous activation, finite-time synchronization, Markovian switching, proportional delays.

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1 Introduction

The past decades witnessed the successful application of continuous NNs in different fields, such as signal processing, pattern recognition, secure communication, and so on [6, 16, 28, 29, 55]. However, when dealing with NNs possessing high-slope nonlinear activations, it is often advantageous to model them with a system of differential equations with discontinuous neuron activations, rather than studying the case where the slope is high but of finite value [35]. Another important example deserves to be mentioned. In [17], in order to solve linear and nonlinear programming problems, Kennedy and Chua introduced a class of NNs, which exploit constraint neurons with a diode-like input-output activations. Moreover, to guarantee satisfaction of the constraints, the diodes are required to possess a very high slope in the conducting region, that is to say, they should approximate the discontinuous characteristic of an ideal diode [8]. Therefore, when dealing with dynamical systems possessing high-slope nonlinear elements, differential equations with discontinuous right-hand side is of great importance to the construction of model. Driven by interests, in recent years, a growing number of scholars have devoted themselves to the study of NNs with discontinuous or non-Lipschitz activation functions [41, 45, 46].

Since the finite switching speed of the neuron amplifiers and the finite speed of signal propagation, delays are actually unavoidable in the electronic implementation. Moreover, delays are sometimes intentionally introduced to accomplish special tasks, such as motion detection via cellular NNs [36]. On the other hand, sometimes the emergence of delays may result in instability since it may originate the onset of nonvanishing oscillations [2, 9, 30]. Therefore, it is significant to study the dynamical behaviors of NNs in the presence of delays. Till to now, abundant theory achievement in this aspect have actually appeared (see, e.g., [19–21, 37, 47, 48] and references therein). Another important delay, proportional delay, cannot be ignored due to its wide existence in real world, such as in web quality of service (QoS) routing decision [7, 40]. Different from infinite-time distributed delays, proportional delay is unbound and time-varying. Therefore, dynamical behaviors of NNs with proportional delays have aroused many scholars' interests. For example, in [56], delay-dependent exponential synchronization of recurrent NNs with multiple proportional delays has been obtained. In [15], finite-time stability of a class of fuzzy cellular NNs with multi-proportional delays was investigated. Finite-time stability of a class of cellular NNs with heterogeneous proportional delays and oscillating leakage coefficients was investigated in [23]. Exponential cluster synchronization of NNs with proportional delays was investigated in [12]. However, most of existing results concerning synchronization of NNs with proportional delays are asymptotic, which means that the synchronization is achieved only in the case that time tends to infinity.

Actually, although different types of synchronization have been considered in the literature, they can be classified into two kinds according to the time of achieving synchronization: (i) infinite-time synchronization (including asymptotic and exponential synchronization) [5, 12, 15, 23, 32, 56]; (ii) finite-time synchronization [1, 3, 22, 34, 38, 42–44, 51, 54]. Finite-time synchronization means that synchronization can be realized in a finite settling time, which is more practical than infinite-time synchronization since the life spans of human beings and machines are limit. On the other hand, secure communication is one

of important applications of synchronization. It is well known that the range of time during which the chaotic oscillators are not synchronized corresponds to the range of time during which the encoded message can unfortunately not be recovered [51]. Therefore, finite-time synchronization technique enables us to recover the transmitted signals in a desired time, while the transmitted signals can only be obtained as time goes to infinity if asymptotic or exponential synchronization technique is utilized. Obviously, compared with infinite-time synchronization, finite-time synchronization improves the efficiency and confidentiality greatly when it is applied to secure communication. Therefore, many researchers have devoted much effort to finite-time synchronization. For instance, finite-time synchronization of uncertain coupled switched NNs under asynchronous switching was investigated in [3]. Authors in [42] studied finite-time synchronization of coupled NNs with DAFs and mixed delays. Author in [51] studied finite-time synchronization for a class of fuzzy cellular NNs with time-varying coefficients and proportional delays. It should be noted that most of existing literatures, including [12, 15, 23, 56] and [38], consider only NNs with continuous activation functions and proportional delays. When the activation functions are discontinuous, the control techniques used in [12, 15, 23, 56] and [38] are not applicable anymore since the discontinuous activation functions induce uncertain Filippov solutions. To the authors' knowledge, few published papers consider finite-time problem for NNs with DAFs and proportional delays, let alone finite-time synchronization and control of them. The main difficulty lies in how to design a simple controller to overcome the effects of both the uncertainties of Filippov solutions and unbounded proportional delays.

The effects of uncertain environmental factors to systems usually exist, which make systems switch from one mode to another at different times, such as stochastic forces and noisy measurements caused by environmental uncertainties. Such switching can be governed by Markov chains with finite state space [11]. In fact, the applications of the Markov jump systems can be found in economic systems, modeling production system, network control systems, and communication systems [4, 18, 25, 27, 31]. Therefore, much attention has been attracted to dynamical behaviors of Markovian NNs. In [50], based on tracker information, authors studied synchronization of discrete-time NNs with delays and Markov jump topologies. Synchronization of Markovian coupled NNs with nonidentical node-delays and random coupling strengths was investigated in [49]. Delay-dependent exponential stability of recurrent NNs with Markovian jumping parameters and proportional delays was studied in [57]. However, to the best of our knowledge, result on finite-time synchronization of Markovian NNs with proportional delays and discontinuous activation functions is seldom.

Inspired by the above discussions, this paper aims to investigate finite-time synchronization of NNs with both proportional delays, markovian switching, and DAFs in the framework of Filippov solution. The main contributions are: (i) New state feedback controllers with and without the sign function are designed, which can overcome the effects of both the uncertainties of Filippov solutions and unbounded proportional delays simultaneously; (ii) Based on newly designed Lyapunov–Krasovskii functionals and 1-norm-based analysis techniques, sufficient conditions are derived to guarantee the finite-time synchronization of the considered system; (iii) Without inducing any free parameters,

the uncertainties of Markovian switching is dealt with by using M -matrix technique, by which the settling time is determined explicitly; (iv) It is shown that controllers with sign function can reduce the control gains while those without the sign function can overcome chattering phenomenon.

The rest of this paper is organized as follows. In Section 2, NNs with proportional delays, Markovian switching, and DAFs are presented. Some necessary assumptions, definitions are also given in this section. In Section 3, finite-time synchronization for the presented model is studied. Then, in Section 4, simulation examples are given to show the effectiveness of the theoretical results. Conclusions and future research interests are given in Section 5.

Notations. The notations are quite standard. Throughout this paper, \mathbb{R}^n denotes the set of $n \times 1$ real vectors. The superscript T stands for matrix or vector transposition. I_n represents the identity matrix of n dimension. $\mathbf{1}_n$ is a column vector with all n elements being 1. $\|\cdot\|_1$ is the 1-norm of a vector or a matrix, i.e., $\|x\|_1 = \sum_{k=1}^n |x_k|$ for $x = (x_1, x_2, \dots, x_n)^{\text{T}} \in \mathbb{R}^n$. $A = (a_{kl})_{n \times m} \in \mathbb{R}^{n \times m}$ represents the set of $n \times m$ real matrices. $\overline{\text{co}}[E]$ is the closure of the convex hull of the set $E \in \mathbb{R}^n$. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 1}, \mathbf{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t \geq 1}$ and satisfy the usual conditions (i.e., the filtration contains all \mathbf{P} -null sets and is right-continuous). Denote by $L_{\mathcal{F}_0}^p([p, 1]; \mathbb{C}^n)$ the family of all \mathcal{F}_0 -measurable $C([p, 1]; \mathbb{C}^n)$ -valued random variables $\zeta = \{\zeta(s) : p \leq s \leq 1\}$ such that $\sup_{p \leq s \leq 1} \mathbf{E}\{\|\zeta(s)\|\} < \infty$, where $\mathbf{E}\{\cdot\}$ stands for the mathematical expectation with respect to the given probability measure \mathbf{P} .

2 Model description and some preliminaries

Let $\{\gamma(t), t \geq 1\}$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 1}, \mathbf{P})$ taking values in a finite state space $\mathcal{N} = \{1, 2, \dots, m\}$ with generator $\Gamma = (\pi_{ij}), (i, j) \in \mathcal{N}$, given by

$$P_{ij}(\Delta t) = \mathbf{P}\{\gamma(t + \Delta t) = j : \gamma(t) = i\} = \begin{cases} \pi_{ij}\Delta t + O(\Delta t) & \text{if } i \neq j, \\ 1 + \pi_{ii}\Delta t + O(\Delta t) & \text{if } i = j, \end{cases}$$

where $\Delta t > 0$ and $\lim_{\Delta t \rightarrow 0} (O(\Delta t)/\Delta t) = 0$. Here $\pi_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\pi_{ii} = -\sum_{j=1, j \neq i}^m \pi_{ij}$.

Consider discontinuous NN model with Markovian switching and proportional delays as follows:

$$\dot{x}(t) = -C(\gamma(t))x(t) + A(\gamma(t))f(x(t)) + B(\gamma(t))f(x(qt)) + J(\gamma(t)), \quad (1)$$

where $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^{\text{T}} \in \mathbb{R}^n$ represents the state vector of NN at time $t \geq 1$; $\{\gamma(t), t \geq 1\}$ is the continuous-time Markov process describing the evolution of the mode; $C(\gamma(t)) = \text{diag}(c_1(\gamma(t)), c_2(\gamma(t)), \dots, c_n(\gamma(t)))$ in which $c_k(\gamma(t)) > 0$, $k = 1, 2, \dots, n$; $A(\gamma(t)) = (a_{kl}(\gamma(t)))_{n \times n}$ and $B(\gamma(t)) = (b_{kl}(\gamma(t)))_{n \times n}$ are the nondelayed weight matrix and time-delayed weight matrix, respectively; $f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)), \dots, f_n(x_n(t)))^{\text{T}}$ is the activation function representing the output

of the NN; the constant q is a proportional delay factor of the NN and satisfies $0 < q < 1$ and $qt = t - (1 - q)t$ in which $(1 - q)t = \tau(t)$ is a time-varying continuous function that satisfies $(1 - q)t \rightarrow +\infty$ as $t \rightarrow +\infty$; $J(\gamma(t)) = (J_1(\gamma(t)), J_2(\gamma(t)), \dots, J_n(\gamma(t)))^T \in \mathbb{R}^n$ is an external input vector.

Throughout this paper, the following assumptions are useful.

- (H1) For every $k = 1, 2, \dots, n$, $f_k : \mathbb{R} \rightarrow \mathbb{R}$ is continuous except on a countable set of isolate points $\{\rho_k^i\}$, where there exist finite right and left limits $f_k^+(\rho_k^i)$ and $f_k^-(\rho_k^i)$, respectively. Moreover, $f_k(\cdot)$ has at most a finite number of jump discontinuities in every compact interval of \mathbb{R} .
- (H2) For each $k = 1, 2, \dots, n$, there exist nonnegative constants h_k and p_k such that $\sup |\xi_k - \eta_k| \leq h_k|u - v| + p_k$ for all $u, v \in \mathbb{R}$, where $\xi_k \in K[f_k(u)]$ and $\eta_k \in K[f_k(v)]$, $K[f_k(u)] = [\min\{f_k^-(u), f_k^+(u)\}, \max\{f_k^-(u), f_k^+(u)\}]$.
- (H3) In this paper, we assume that the generator Γ is irreducible.

As for differential equation with discontinuous right-hand side, its solution does not exist in conventional sense. In order to obtain our main results, some basic definitions and useful lemmas are needed.

Definition 1. (See [13].) The Filippov set-valued map of $f(x)$ at $x \in \mathbb{R}^n$ is defined as follows:

$$F(x) = \bigcap_{\delta > 0} \bigcap_{\mu(\Omega)=0} \overline{\text{co}}[f(B(x, \delta) \setminus \Omega)],$$

where $B(x, \delta) = \{y: \|y - x\| \leq \delta\}$, and $\mu(\Omega)$ is the Lebesgue measure of set Ω .

Definition 2. (See [24].) A function $x : [q, T] \rightarrow \mathbb{R}^n$, $T \in (1, +\infty]$, is a Filippov solution of NN (1) on $[q, T]$ if

- (i) $x(t)$ is continuous on $[q, T]$ and absolutely continuous on $[1, T]$;
- (ii) There exists a measurable function $\varpi(t) = (\varpi_1(t), \varpi_2(t), \dots, \varpi_n(t))^T : [q, T] \rightarrow \mathbb{R}^n$ such that, for almost all (a.a.) $t \in [q, T]$, $\varpi(t) \in K[f(x(t))]$ and

$$\dot{x}(t) = -C(\gamma(t))x(t) + A(\gamma(t))\varpi(t) + B(\gamma(t))\varpi(qt) + J(\gamma(t)),$$

$$\text{where } K[f(x(t))] = (K[f_1(x_1(t))], \dots, K[f_n(x_n(t))])^T : \mathbb{R}^n \rightarrow \mathbb{R}^n.$$

Definition 3. For any continuous function $\varphi : [q, 1] \rightarrow \mathbb{R}^n$ and measure selection $\psi : [q, 1] \rightarrow \mathbb{R}^n$ such that $\psi(s) \in K[f(\varphi(s))]$ for a.a. $s \in [q, 1]$ by an initial value problem associated to (1) with initial condition (φ, ψ) , one means the following problem: find a couple of functions $[x, \varpi] : [q, T] \rightarrow \mathbb{R}^n \times \mathbb{R}^n$ such that x is a solution of (1) on $[q, T]$ for some $T > 1$, ϖ is an output associated to x , and

$$\begin{aligned} \dot{x}(t) &= -C(\gamma(t))x(t) + A(\gamma(t))\varpi(t) + B(\gamma(t))\varpi(qt) \\ &\quad + J(\gamma(t)) \quad \text{for a.a. } t \in [q, T], \\ x(s) &= \varphi(s) \quad \text{for all } s \in [q, 1], \\ \varpi(s) &= \psi(s) \quad \text{for a.a. } s \in [q, 1]. \end{aligned} \tag{2}$$

By similar methods given in [51], it is not difficult to derive the following result.

Lemma 1. *Suppose that (H1)–(H2) are satisfied. Then any IVP for (1) has at least a local solution $[x, \varpi]$ defined on $[1, T]$ for some $T \in (1, \infty]$.*

Definition 4. (See [10].) Function $V(x) : \mathbb{R}^n \rightarrow \mathbb{R}$ is C-regular if $V(x)$ is:

- (i) regular in \mathbb{R}^n ;
- (ii) positive definite, i.e., $V(x) > 0$ for $x \neq 0$ and $V(0) = 0$;
- (iii) radially unbounded, i.e., $V(x) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$.

Note that a C-regular Lyapunov function $V(x)$ is not necessarily differentiable. Let $V : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function. The Clarke's generalized gradient of V at $x \in \mathbb{R}^n$ is defined by $\partial V(x) = \overline{\text{co}}[\lim_{x_k \rightarrow x} \nabla V(x_k) : x_k \notin \Omega \cup \tilde{\mathcal{N}}]$, where $\Omega \in \mathbb{R}^n$ is set with Lebesgue measure zero, where ∇V does not exist, and $\tilde{\mathcal{N}} \in \mathbb{R}^n$ is an arbitrary set with measure zero.

Lemma 2 [Chain rule]. (See [10].) *If $V(x(t)) : \mathbb{R}^n \rightarrow \mathbb{R}$ is C-regular and $x(t)$ is absolutely continuous on any compact subinterval of $[0, +\infty)$, then $x(t)$ and $V(x(t)) : [0, +\infty) \rightarrow \mathbb{R}$ are differentiable for a.a. $t \in [0, +\infty)$, and*

$$\frac{d}{dt}(V(x(t))) = \varpi(t)\dot{x}(t) \quad \text{for all } \varpi(t) \in \partial V(x(t)),$$

where $\partial V(x(t))$ is Clarke generalized gradient of V at $x(t)$.

Consider the NN (1) as drive system. The controlled response NN is given by

$$\begin{aligned} \dot{y}(t) = & -C(\gamma(t))y(t) + A(\gamma(t))f(y(t)) + B(\gamma(t))f(y(qt)) \\ & + U(\gamma(t), t) + J(\gamma(t)), \end{aligned} \quad (3)$$

where $y(t) = (y_1(t), y_2(t), \dots, y_n(t))^T \in \mathbb{R}^n$ represents the state vector; $f(y(t)) = (f_1(y_1(t)), f_2(y_2(t)), \dots, f_n(y_n(t)))^T \in \mathbb{R}^n$ is the activation function; $U(\gamma(t), t) = (U_1(\gamma(t), t), U_2(\gamma(t), t), \dots, U_n(\gamma(t), t))^T$ is the controller to be designed, the other parameters are the same as those defined in system (1).

According to Definition 3 and Lemma 1, the IVP of NN (3) is presented as follows:

$$\begin{aligned} \dot{y}(t) = & -C(\gamma(t))y(t) + A(\gamma(t))\delta(t) + B(\gamma(t))\delta(qt) \\ & + U(\gamma(t), t) + J(\gamma(t)) \quad \text{for a.a. } t \in [q, T], \\ y(s) = & \tilde{\varphi}(s) \quad \text{for all } s \in [q, 1], \\ \delta(s) = & \tilde{\psi}(s) \quad \text{for a.a. } s \in [q, 1]. \end{aligned} \quad (4)$$

Recall that the Markov process $\{\gamma(t), t \geq 1\}$ takes values in the finite space $\mathcal{N} = \{1, 2, \dots, m\}$. For simplicity, denote $C(\gamma(t)) = C_i$, $A(\gamma(t)) = A_i$, $B(\gamma(t)) = B_i$, $U(\gamma(t), t) = U_i(t)$ when $\gamma(t) = i$, $i \in \mathcal{N}$.

Let $z(t) = y(t) - x(t)$. Substituting (2) from (4) yields the following error system:

$$\dot{z}(t) = -C_i z(t) + A_i \beta(t) + B_i \beta(qt) + U_i(t), \quad (5)$$

where $\beta(t) = \delta(t) - \varpi(t)$, $z(s) = \tilde{\varphi}(s) - \varphi(s)$, $s \in [q, 1]$.

Definition 5. The NN (3) is said to be synchronized with (1) in finite time if there exists a constant $t_1 > 0$ such that $\lim_{t \rightarrow t_1} \mathbf{E}\{\|z(t)\|_1\} = 0$ and $\mathbf{E}\{\|z(t)\|_1\} \equiv 0$ for $t > t_1$, where t_1 is called the settling time.

Lemma 3. (See [14].) If $W = (w_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$ with $w_{ij} \leq 0$ ($i \neq j$), then the following statements are equivalent:

- (i) W is an M -matrix.
- (ii) W^{-1} exists, and all the elements of W^{-1} are nonnegative.
- (iii) All the eigenvalues of W have positive real parts.

Lemma 4. (See [52].) Let $W = (w_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$ with $w_{ij} \leq 0$ ($i \neq j$), $\sum_{j=1}^n w_{ij} = 0$, $i, j = 1, 2, \dots, n$. If W is irreducible, then, for any $\xi > 0$, $W + \xi I_n$ is a nonsingular M -matrix.

Lemma 5. (See [26].) Let $V(x(t), \gamma(t), t > 0) = V(x(t), i, t)$ be a stochastic positive candidate, and define its weak infinitesimal operator as

$$\mathcal{L}V(x(t), i, t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} [\mathbf{E}\{V(x(t + \Delta t), \gamma(t + \Delta t), t + \Delta t) \mid x(t), \gamma(t) = i\} - V(x(t), i, t)].$$

3 Main results

In this section, two types of state feedback controllers are designed. Based on Lyapunov stability theorem and 1-norm technique, general criteria for finite-time synchronization of discontinuous NNs with proportional delays and Markovian switching are obtained by rigorous mathematical proofs.

First, consider the following state feedback controller:

$$U_{k,i}(t) = -\lambda_{k,i} z_k(t) - \chi_{k,i} \text{sign}(z_k(t)), \quad k = 1, 2, \dots, n, \quad i \in \mathcal{N}, \quad (6)$$

where $\lambda_{k,i}$ and $\chi_{k,i}$ are positive constants to be determined.

Theorem 1. Suppose that (H1) and (H2) are satisfied. Then the NN (3) can be synchronized with (1) in finite time under controller (6) if the following conditions are satisfied:

$$\begin{aligned} \lambda_{k,i} &> -c_{k,i} + \sum_{l=1}^n |a_{lk,i}| h_k + \sum_{l=1}^n \frac{|b_{lk}| h_k}{q} = \varepsilon_{k,i}, \\ \chi_{k,i} &\geq \sum_{l=1}^n (|a_{kl,i}| + |b_{kl,i}|) p_l + 1. \end{aligned} \quad (7)$$

Moreover, the settling time is estimated as

$$t_1 \leq \frac{q}{Q} \left(q \sum_{k=1}^n |z_k(1)| + \sum_{k=1}^n \sum_{l=1}^n h_l |b_{kl}| \int_0^1 |z_l(s)| ds \right),$$

where $\sum_{l=1}^n h_l |b_{kl}| = \max\{\sum_{l=1}^n h_l |b_{kl,i}|, i \in \mathcal{N}\}$, $Q = \min\{Q_i, i \in \mathcal{N}\}$, $(Q_1, Q_2, \dots, Q_m)^T = (-\Gamma - \xi)^{-1} \mathbf{1}_m / \nu$, $\xi = \text{diag}(\xi_1, \xi_2, \dots, \xi_m)$; $\xi_i = \max\{\varepsilon_{k,i} - \lambda_{k,i}, k = 1, 2, \dots, n, i = 1, 2, \dots, m\}$, ν be the maximum of the row sums of $(-\Gamma - \xi)^{-1}$.

Proof. Since $\{z(t), \gamma(t), t \geq 1\}$ is not a Markov process, in order to cast our model into the framework for a Markov system, let us define a new Markov process $\{z_t, \gamma(t), t \geq 1\}$ according to $z_t(s) = z(t + s)$, $s \in [q, 1]$. Then $\{z_t, \gamma(t), t \geq 1\}$ is a Markov process [11].

By (7), one can get that $\xi_i = \max\{\varepsilon_{k,i} - \lambda_{k,i}, k = 1, 2, \dots, n, i = 1, 2, \dots, m\} < 0$. From (H3), due to Γ is irreducible, then $-\Gamma$ is also irreducible. According to Lemma 4, $-\Gamma - \xi$ is a nonsingular M -matrix. By Lemma 3, the following process is same as those in [52]. Therefore, one can obtain that all the elements of $(Q_1, Q_2, \dots, Q_m)^T = (-\Gamma - \xi)^{-1} \mathbf{1}_m / \nu$ are positive, $\max(Q_i, i \in \mathcal{N}) = 1$, and

$$\sum_{j \in \mathcal{N}} \pi_{ij} Q_j + Q_i \xi_i = -\frac{1}{\nu} < 0. \tag{8}$$

Consider Markovian switching Lyapunov–Krasovskii functional as follows:

$$V(z_t, i, t) = Q_i \sum_{k=1}^n |z_k(t)| + \sum_{k=1}^n \sum_{l=1}^n \frac{h_l |b_{kl}|}{q} \int_{qt}^t |z_l(s)| ds.$$

By Lemma 5, \mathcal{L} be weak infinitesimal operator of the random process $\{z_t, i, t\}$, then according to Lemma 2, differentiating $V(z_t, i, t)$ along the solutions of (5) and considering controller (6), one obtains that

$$\begin{aligned} \mathcal{L}V(z_t, i, t) &= Q_i \sum_{k=1}^n \text{sign}(z_k(t)) \dot{z}_k(t) + \sum_{k=1}^n \sum_{l=1}^n \frac{h_l |b_{kl}|}{q} |z_l(t)| \\ &\quad - \sum_{k=1}^n \sum_{l=1}^n h_l |b_{kl}| |z_l(qt)| + \sum_{j \in \mathcal{N}} \pi_{ij} Q_j \sum_{k=1}^n |z_k(t)| \\ &= Q_i \sum_{k=1}^n \left\{ -c_{k,i} |z_k(t)| + \text{sign}(z_k(t)) \left[\sum_{l=1}^n a_{kl,i} \beta_l(t) \right. \right. \\ &\quad \left. \left. + \sum_{l=1}^n b_{kl,i} \beta_l(qt) - \lambda_{k,i} z_k(t) - \chi_{k,i} \text{sign}(z_k(t)) \right] \right\} \\ &\quad + \sum_{k=1}^n \sum_{l=1}^n \frac{h_l |b_{kl}|}{q} |z_l(t)| - \sum_{i=k}^n \sum_{l=1}^n h_l |b_{kl}| |z_l(qt)| \\ &\quad + \sum_{j \in \mathcal{N}} \pi_{ij} Q_j \sum_{k=1}^n |z_k(t)|. \tag{9} \end{aligned}$$

By (H2), one obtains that

$$\begin{aligned} & \text{sign}(z_k(t)) \sum_{l=1}^n a_{kl,i} \beta_l(t) + \text{sign}(z_k(t)) \sum_{l=1}^n b_{kl,i} \beta_l(qt) \\ & \leq \sum_{l=1}^n |a_{kl,i}| h_l |z_l(t)| + \sum_{l=1}^n |b_{kl,i}| h_l |z_l(qt)| + \sum_{l=1}^n (|a_{kl,i}| + |b_{kl,i}|) p_l \alpha_k, \end{aligned} \quad (10)$$

where $\alpha_k = 1$ if $z_k(t) \neq 0$, otherwise $\alpha_k = 0$.

Substituting (10) into (9) yields

$$\begin{aligned} & \mathcal{L}V(z_t, i, t) \\ & \leq Q_i \sum_{k=1}^n \left\{ -c_{k,i} |z_k(t)| + \sum_{l=1}^n |a_{kl,i}| h_l |z_l(t)| + \sum_{l=1}^n |b_{kl,i}| h_l |z_l(qt)| \right. \\ & \quad \left. + \sum_{l=1}^n (|a_{kl,i}| p_l + |b_{kl,i}| p_l) \alpha_k - \lambda_{k,i} |z_k(t)| - \chi_{k,i} \alpha_k \right\} \\ & \quad + \sum_{k=1}^n \sum_{l=1}^n \frac{h_l |b_{kl,i}|}{q} |z_l(t)| - \sum_{k=1}^n \sum_{l=1}^n h_l |b_{kl,i}| |z_l(qt)| + \sum_{j \in \mathcal{N}} \pi_{ij} Q_j \sum_{k=1}^n |z_k(t)| \\ & \leq \sum_{k=1}^n \left[\sum_{j \in \mathcal{N}} \pi_{ij} Q_j + Q_i \left(-c_{k,i} + \sum_{l=1}^n |a_{lk,i}| h_k + \sum_{l=1}^n \frac{|b_{lk,i}| h_k}{q} - \lambda_{k,i} \right) \right] |z_k(t)| \\ & \quad + Q_i \sum_{k=1}^n \left[\sum_{l=1}^n (|a_{kl,i}| p_l + |b_{kl,i}| p_l) - \chi_{k,i} \right] \alpha_k \\ & \leq \sum_{k=1}^n \left(\sum_{j \in \mathcal{N}} \pi_{ij} Q_j + Q_i \xi_i \right) |z_k(t)| \\ & \quad + Q_i \sum_{k=1}^n \left[\sum_{l=1}^n (|a_{kl,i}| + |b_{kl,i}|) p_l - \chi_{k,i} \right] \alpha_k, \end{aligned} \quad (11)$$

where $\max\{Q_i, i \in \mathcal{N}\} = 1$ has been used.

By (7) and (8), it is followed from (11) that, when $z_k(t) \neq 0$,

$$\mathcal{L}V(z_t, i, t) \leq -Q_i \sum_{k=1}^n \alpha_k \leq -\underline{Q}, \quad (12)$$

where $\underline{Q} = \min\{Q_i, i \in \mathcal{N}\}$.

According to the arbitrariness of $i \in \mathcal{N}$, one can obtain from (12) that

$$\frac{d}{dt} \mathbf{E}\{V(z_t, \gamma(t), t)\} \leq -\underline{Q}. \quad (13)$$

Integrating both sides of inequality (13) from 1 to t gets the following inequality:

$$\mathbf{E}\{V(z_t, \gamma(t), t)\} - V(z_1, \gamma(1), 1) \leq -Qt. \quad (14)$$

Now we prove that there exists an instant $\tilde{t} \in (1, +\infty)$ such that $\mathbf{E}\{V(z_{\tilde{t}}, \gamma(\tilde{t}), \tilde{t})\} = 0$. Otherwise, suppose that $\mathbf{E}\{V(z_t, \gamma(t), t)\} > 0$ for all $t > 1$, i.e., $\mathbf{E}\{Q_i \sum_{k=1}^n |z_k(t)|\} > 0$ or $\mathbf{E}\{\sum_{k=1}^n \sum_{l=1}^n (h_l |b_{kl}|/q) \int_{qt}^t |z_l(s)| ds\} > 0$. Since $\mathbf{E}\{\sum_{k=1}^n \sum_{l=1}^n (h_l |b_{kl}|/q) \times \int_{qt}^t |z_l(s)| ds\} > 0$ for $t > 1$ implies $\mathbf{E}\{Q_i \sum_{k=1}^n |z_k(t)|\} > 0$, we only discuss

$$\mathbf{E}\left\{\sum_{k=1}^n \sum_{l=1}^n \frac{h_l |b_{kl}|}{q} \int_{qt}^t |z_l(s)| ds\right\} > 0.$$

When $\mathbf{E}\{\sum_{k=1}^n \sum_{l=1}^n (h_l |b_{kl}|/q) \int_{qt}^t |z_l(s)| ds\} > 0$, which means that there exists at least one $l_0 \in \{1, 2, \dots, n\}$ such that $\mathbf{E}\{\int_{qt}^t |z_{l_0}(s)| ds\} > 0$, then there exists $\hat{t} \in (qt, t)$ such that $\mathbf{E}\{|z_{l_0}(\hat{t})|\} > 0$. From the arbitrariness of $t > 1$ one has that there exists a least one $l_0 \in \{1, 2, \dots, n\}$ such that $\mathbf{E}\{|z_{l_0}(t)|\} > 0$ for all $t > 1$, which means inequality (14). Therefore, $\lim_{t \rightarrow +\infty} \mathbf{E}\{V(z_t, \gamma(t), t)\} = -\infty$. This contradicts to the fact that $\mathbf{E}\{V(z_t, \gamma(t), t)\} \geq 0$. Therefore, by the above discussions, there exists an instant $\tilde{t} \in (1, +\infty)$ such that

$$\lim_{t \rightarrow \tilde{t}} \mathbf{E}\{V(z_t, \gamma(t), t)\} = 0 \quad \text{and} \quad \mathbf{E}\{V(z_t, \gamma(t), t)\} \equiv 0 \quad \forall t \geq \tilde{t}. \quad (15)$$

By (14) and (15), one can get that

$$\tilde{t} \leq \frac{V(z_1, \gamma(1), 1)}{Q}.$$

Note that (15) also means that $\lim_{t \rightarrow t_1} \mathbf{E}\{\|z(t)\|_1\} = 0$ and $\mathbf{E}\{\|z(t_1)\|_1\} \equiv 0$ for all $t \geq t_1$, where $t_1 = q\tilde{t}$. In fact, seeing from the process of calculability of this paper, we have proved that $\mathbf{E}\{V(z_{\tilde{t}}, \gamma(\tilde{t}), \tilde{t})\} = 0$, i.e., $\mathbf{E}\{Q_i \sum_{k=1}^n |z_k(\tilde{t})|\} = 0$ and $\mathbf{E}\{\sum_{k=1}^n \sum_{l=1}^n (h_l |b_{kl}|/q) \int_{q\tilde{t}}^{\tilde{t}} |z_l(s)| ds\} = 0$. Furthermore, $\int_{q\tilde{t}}^{\tilde{t}} |z_l(s)| ds = 0$. Suppose that $|z_l(q\tilde{t})| > 0$, on the basis of the continuity of $|z_l(s)|$, there exists an arbitrary small interval $[q\tilde{t}, a) \subset [q\tilde{t}, \tilde{t}]$ such that $|z_l(s)| > 0$ for $s \in [q\tilde{t}, a)$, i.e., $\int_{q\tilde{t}}^a |z_l(s)| ds > 0$. However, $\int_{q\tilde{t}}^{\tilde{t}} |z_l(s)| ds = 0$, there is a contradiction. Therefore, we can obtain $|z_l(q\tilde{t})| = 0$, and then the synchronization time can correct to $t_1 = q\tilde{t}$. According to Definition 5, the neural network (3) is synchronized with (1) in t_1 . The proof is completed. \square

Remark 1. In Theorem 1, simple controller is designed. Recently, finite-time synchronization of NNs with proportional delays was investigated in [38] by designing controllers with the sign function (see [38, (3.2) and (3.3)]). Compared with the controllers in [38], controller (6) in this paper is much more simple. It is well known that simple controllers are easy to be implemented and save energy in practice. Specially, the results in [38] were

obtained in the framework of classical finite-time stability theorem in [53]. However, without using the classical finite-time stability theorem, novel techniques are developed in this paper by using the concept of 1-norm. Hence, Theorem 1 essentially improves the corresponding ones in [38].

It is well known that controller with sign function leads to chattering phenomenon, which can induce undesirable effects such as abrasion of machine equipment. In order to eliminate the chattering, state feedback controller without the sign function is designed as follows:

$$U_{k,i}(t) = \begin{cases} -\lambda_{k,i}z_k(t) - \chi_{k,i}\frac{z_k(t)}{\|z(t)\|_1}, & \|z(t)\|_1 \neq 0, \\ 0, & \|z(t)\|_1 = 0, \end{cases} \quad (16)$$

where $k \in 1, 2, \dots, n, i \in \mathcal{N}, \lambda_{k,i}$ and $\chi_{k,i}$ are positive constants to be determined.

Theorem 2. Suppose that (H1) and (H2) are satisfied. NN (3) can be synchronized with (1) in finite time under controller (16) if the following conditions are satisfied: $\lambda_{k,i} > \varepsilon_{k,i}, \chi_{k,i} \geq \sum_{k=1}^n \sum_{l=1}^n |a_{kl,i}|p_l + |b_{kl,i}|p_l + 1 = \hat{\chi}_i$. Moreover, the settling time is estimated as $t_{\leq} (q/Q)(q \sum_{k=1}^n |z_k(1)| + \sum_{k=1}^n \sum_{l=1}^n h_l |b_{kl}| \int_q^1 |z_l(s)| ds)$, where $\varepsilon_{k,i}$ and \underline{Q} are given in Theorem 1.

Proof. Considering Markovian switching Lyapunov–Krasovskii functional is same as in Theorem 1. According to Lemma 2, using similar procedure as that given in Theorem 1, one obtains that

$$\begin{aligned} & \mathcal{L}V(z_t, i, t) \\ & \leq Q_i \left[\sum_{k=1}^n \sum_{l=1}^n (|a_{kl,i}|p_l + |b_{kl,i}|p_l) - \sum_{k=1}^n \text{sign}(z_k(t)) \chi_{k,i} \frac{z_k(t)}{\|z(t)\|_1} \right], \end{aligned} \quad (17)$$

where $\|z(t)\|_1 \neq 0$.

From (17) one has

$$- \sum_{k=1}^n \text{sign}(z_k(t)) \chi_{k,i} \frac{z_k(t)}{\|z(t)\|_1} = - \sum_{k=1}^n \chi_{k,i} \frac{|z_k(t)|}{\|z(t)\|_1} \leq -\hat{\chi}_i,$$

where $\hat{\chi}_i = \min\{\chi_{k,i}, k = 1, 2, \dots, n, i = 1, 2, \dots, m\}$.

According to the conditions in Theorem 2, one can get that

$$\mathcal{L}V(z_t, i, t) \leq -Q_i \leq -\underline{Q}.$$

The following analysis method is same as those in Theorem 1. The settling time is $t_{\leq} (q/Q)V(z_1, \gamma(1), 1)$. Thus, the neural network (3) can be synchronized with (1) in settling time t_1 . The proof is completed. \square

Remark 2. In Theorems 1 and 2, by designing two kinds of controllers, the synchronization errors are controlled to convergent to zero in a finite settling time. Recently, finite-time stabilization of NNs with proportional delays was considered in [15] and [23].

However, the techniques in [15] and [23] can only guarantee that the solutions of the NNs to be bounded in a given time. Hence, results in [15] and [23] cannot be applied to the problem in this paper. Moreover, the effect of Markovian parameters on the dynamical behaviors of NNs was not considered in [15] and [23]. In this sense, results of this paper extend the ones in [15] and [23] essentially.

Remark 3. Considering the fact that sign function may induce chattering in practice, controller (16) is also designed, where the sign function has been removed. In the literature, it seems that the sign function is indispensable for finite-time control to zero [22, 44, 51, 52]. Theorem 2 shows that the finite-time synchronization can still be realized without inducing any chattering phenomenon. Therefore, Theorem 2 improves the results in [22, 44, 51, 52] essentially.

Remark 4. It is not difficult to find from Theorems 1 and 2 that controller (6) with sign function can reduce the control gains $\chi_{k,i}$ effectively but induce chattering, while the controller (16) without sign function can overcome chattering phenomenon but the control gains $\chi_{k,i}$ are larger than those in (6). Therefore, controllers (6) and (16) have advantage and disadvantage, respectively.

4 Numerical examples

In this section, numerical simulations are given to demonstrate the effectiveness of the above theoretical analysis. In simulations, step length is taken as 0.0001.

Consider Markovian NNs with proportional delays and discontinuous activations as follows:

$$\dot{x}(t) = -C(\gamma(t))x(t) + A(\gamma(t))f(x(t)) + B(\gamma(t))f(x(qt)) + J(\gamma(t)), \quad (18)$$

where $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^2$, $\gamma(t) = i = 1, 2$, $q = 0.6$, $J_1 = (-1.2, 1.5)^T$, $J_2 = (-1, 1.8)^T$,

$$C_1 = \begin{pmatrix} 3 & 0 \\ 0 & 3.5 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 1.2 & 8 \\ -6 & 3.8 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 1.2 & -1.2 \\ -4.5 & 4.6 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 3.5 & 0 \\ 0 & 1.5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1.7 & 8.5 \\ -5.3 & 3.5 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 1.5 & -1.5 \\ -6.8 & 6.8 \end{pmatrix},$$

$f(x(t)) = (f_1(x_1(t)), f_2(x_2(t)))^T$ with

$$f_k(x_k(t)) = \begin{cases} \tanh(x_k(t)) + 0.3, & x_k(t) > 0, \\ \tanh(x_k(t)) - 0.6, & x_k(t) < 0, \end{cases} \quad k = 1, 2.$$

It is easy to check that the activation function f is discontinuous at $x = 0$ and satisfies conditions (H1) and (H2) with $h_k = 1$ and $p_k = 0.9$, $k = 1, 2$.

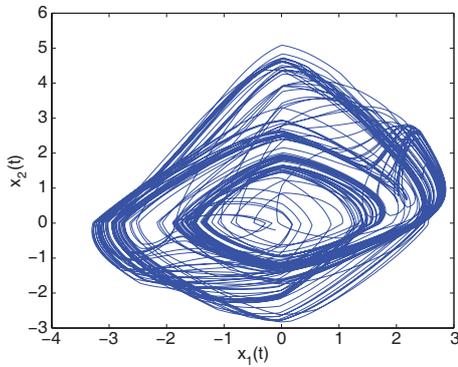


Figure 1. Chaotic-like trajectory for NN mode 1 of (18) with initial value $x(t) = (0.1, -0.2)^T$, $t \in [0.6, 1]$.

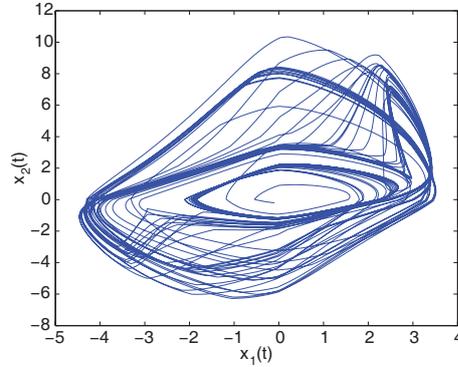


Figure 2. Chaotic-like trajectory for NN mode 2 of (18) with initial value $x(t) = (0.1, -0.2)^T$, $t \in [0.6, 1]$.

The generator matrix Γ is assumed to be

$$\Gamma = \begin{pmatrix} -8 & 8 \\ 6 & -6 \end{pmatrix}. \tag{19}$$

Figures 1 and 2 show the chaotic-like trajectories of mode 1 and mode 2, where $x(t) = (0.1, -0.2)^T$, $t \in [0.6, 1]$.

The controlled response system is given by

$$\begin{aligned} \dot{y}(t) = & -C(\gamma(t))y(t) + A(\gamma(t))f(y(t)) + B(\gamma(t))f(y(qt)) \\ & + U(\gamma(t), t) + J(\gamma(t)), \end{aligned} \tag{20}$$

where $y(t) = (y_1(t), y_2(t))^T$, $\gamma(t) = i = 1, 2$, the other parameters are the same as those defined in system (18).

Now consider finite-time synchronization between Markovian NNs (18) and (20). The following example is to verify Theorems 1 and 2.

Example 1. First, we verify Theorem 1. By simple computation, we obtain that, when $\lambda_{1,1} > \varepsilon_{1,1} = 13.7$, $\lambda_{2,1} > \varepsilon_{2,1} = 17.9667$, $\lambda_{1,2} > \varepsilon_{1,2} = 17.3333$, $\lambda_{2,2} > \varepsilon_{2,2} = 24.3333$, $\chi_{1,1} \geq 11.44$, $\chi_{2,1} \geq 18.01$, $\chi_{1,2} \geq 12.88$, and $\chi_{2,2} \geq 21.16$, system (20) can be synchronized with (18) in finite time under the state feedback controller (6) according to Theorem 1. Take the initial conditions as $x(t) = (0.1, -0.2)^T$, $y(t) = (-0.5, 0.3)^T$, $t \in [0.6, 1]$. Choosing $\lambda_{1,1} = 13.8$, $\lambda_{2,1} = 18$, $\lambda_{1,2} = 17.4$, $\lambda_{2,2} = 24.5$, $\chi_{1,1} = 11.44$, $\chi_{2,1} = 18.01$, $\chi_{1,2} = 12.88$, $\chi_{2,2} = 21.16$, one can obtain that $\xi = -\text{diag}(0.0333, 0.0667)$, and it follows from $(Q_1, Q_2, \dots, Q_m)^T = (1/\nu)(-\Gamma - \xi)^{-1}\mathbf{1}_m$ that $\underline{Q} = 0.9976$. Moreover, the settling time is estimated as $t_1 = 4.3224$. Figure 3 shows the Markovian chain generated by probability transition matrix corresponding to generator (19) with $\gamma(1) = 1$. Figure 4 describes synchronization error $z(t) = y(t) - x(t)$ between systems (20) and (18) under the feedback controller (6), from which one can see that the synchronization is achieved before the settling time $t_1 = 4.3224$.

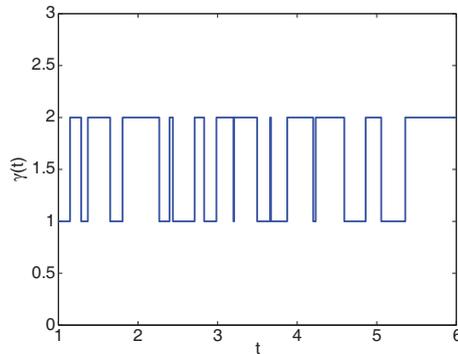


Figure 3. Markov chain generated by probability transition matrix corresponding to the generator (19) with $\Delta t = 0.0001, \gamma(1) = 1$.

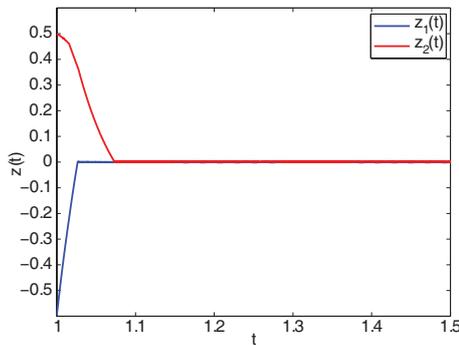


Figure 4. Synchronization error between systems (18) and (20) under the feedback controller (6).

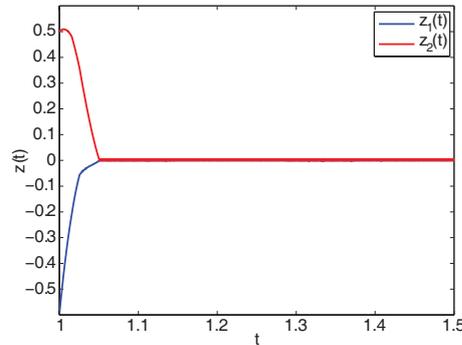


Figure 5. Synchronization error between systems (18) and (20) under the feedback controller (16).

Next, we verify Theorem 2. Consider controller (16). Through computation, one can get that

$$\hat{\chi}_1 = \sum_{k=1}^2 \sum_{l=1}^2 |a_{kl,1}| p_l + |b_{kl,1}| p_l + 1 = 28.45,$$

$$\hat{\chi}_2 = \sum_{k=1}^2 \sum_{l=1}^2 |a_{kl,2}| p_l + |b_{kl,2}| p_l + 1 = 32.04.$$

According to Theorem 2, if $\lambda_{1,1} = 13.8, \lambda_{2,1} = 18, \lambda_{1,2} = 17.4, \lambda_{2,2} = 24.5, \chi_{1,1} = \chi_{2,1} = 28.45, \chi_{1,2} = \chi_{2,2} = 30.24$, the other parameters are the same as those in Fig. 4. Then finite-time synchronization of Markovian NNs (20) and (18) can be realized within $t_1 = 4.3224$ under the state feedback controller (16). Figure 5 describes synchronization error $z(t) = y(t) - x(t)$ between Markovian NNs (18) and (20) under the feedback controller (16), which verifies the effectiveness of Theorem 2.

Remark 5. Note that the chattering in Fig. 4 is not obvious since the time-step for the simulations is very small. From Figs. 4 and 5 one can see that, by using the two controllers (6) and (16), respectively, the synchronization between (18) and (20) is realized. Moreover, when the control gains $\lambda_{k,i}$ in (6) and (16) are taking the same values, one can get from Example 1 that (20) can be synchronized with (18) within the same settling time. Therefore, in practical applications, we can choose feedback controls with or without sign function according to the actual demand.

5 Conclusions

In this paper, finite-time synchronization of Markovian NNs with discontinuous activation functions and proportional delays via state feedback control has been studied. New 1-norm-based Lyapunov–Krasovskii functional has been constructed to deal with proportional delays. New 1-norm analytical techniques have been developed to cope with the difficulties induced by discontinuous activation functions, Markovian switching, as well as proportional delays simultaneously. Based on the framework of Filippov solution, Lyapunov functional method, and M -matrix method, sufficient conditions have been derived to guarantee that the considered NNs can be realized synchronization in a settling time. Numerical simulations have been provided to show the effectiveness of our new theoretical results.

One the other hand, the obtained results are general. In particular, they can be applied in dynamical systems when considering the effects of uncertain environmental factors. In recent years, control theory in biological or engineering systems has developed rapidly [33, 39, 53], which represents for us a research direction that maybe we can apply a mass of dynamical models to nonlinear biological or engineering systems.

As we all know, impulsive effects exist extensively in neural networks. For instance, in implementation of mechanical engineering, the state of the networks is subject to instantaneous perturbations and change suddenly at certain instants, which may be caused by switching phenomenon, frequency change, or other sudden noise. Therefore, in the future research field, thinking over dynamical system with impulsive delays and Markovian switching may be significant in the same way.

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