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An information diffusion model in social networks with carrier compartment and delay

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Abstract. With the wide applications of the communication networks, the topic of information networks security is getting more and more attention from governments and individuals. This paper is devoted to investigating a malware propagation model with carrier compartment and delay to describe the process of malware propagation in mobile wireless sensor networks. Based on matrix theory for characteristic values, the local stability criterion of equilibrium points is established. Applying the linear approximation method of nonlinear systems, we study the existence of Hopf bifurcation at the equilibrium points. At the same time, we identify some sensitive parameters in the process of malware propagation. Finally, numerical simulations are performed to illustrate the theoretical results.

Keywords: social networks, malware propagation, Hopf bifurcation, local stability.

1 Introduction

In recent years, social networks, as a new platform for information propagation and communication and for establishing wide social relations, have gradually come into focus; see, e.g., [7,11,12,26,30]. Compared with traditional ways of communication, social networks have greatly increased the speed of information transmission and diffusion. Moreover, social networks also play significant roles in social ties and business negotiation, as well as information sharing.

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Since social networking has very important applications, considerable attention has been paid to application and control problems of wireless sensor networks (WSNs). Particularly, communication protocols, hardware design, resource efficiency, battlefield surveillance, and home security have been extensively studied; see, e.g., [5, 16, 23, 27, 29]. In recent years, malicious software has become increasingly of concern to the economy and society; hence, some authors have studied malware wireless sensor networks (MWSNs). They try to understand how the MWSN spreads. In fact, for MWSNs, mobility is the most important features in its wide applications in our everyday life [10]. For example, in an intelligent factory, nodes may be attached to equipment to collect information, such as equipment running condition, efficiency, and maintenance [1]. The purpose of monitoring these conditions is to ensure that the equipment is always running at the highest efficiency. For more applications to MWSNs, see, e.g., [13, 17, 20]. As Khan et al. [13] point out the advantages of MWSNs over static wireless sensor networks include enhanced target tracking, better improved coverage, energy efficiency, and superior channel capacity. Owing to the characteristics of their wide applications, MWSNs are becoming attacked targets [20]. Injecting malware often happens on the Internet, which causes harm to some nodes, especially mobile nodes, such as instability of society, loss of online accounts, loss of data, and network paralysis.

To eliminate and control the damage of malware, we must study reasons for malware occurrence, its degree of damage to economy and society, and the dynamic characteristics of malware propagation. Some interesting results have been obtained for malware propagation. Newman [19] studied a large class of standard epidemiological models that can be solved exactly on a wide variety of networks, and proposed a percolation theory based evaluation of the spread of an epidemic on graphs with given degree distributions. However, the temporal dynamics of epidemic spread were not considered by Newman. Liu et al. [15] considered the spreading behavior of malware across mobile. Based on the theory of complex networks, the spreading threshold that monitors the dynamics of the model was calculated, the properties of malware epidemics were investigated. Wang [21, 22] also obtained the threshold for a kind of malware to propagate, where all the nodes were supposed to be stationary. The above systems are deterministic models based on a system of ordinary differential equations. We note that there exists stochastic perturbation in real world, so some stochastic epidemic models have also been proposed; see, e.g., [2].

On the other hand, delays exist widely in different dynamic system, such as various engineering, biological, and economic systems (see, e.g., [4, 6, 9, 14, 33]). In generally, delays are divided into many categories, such as constant delay (discrete delay), time-varying delay, and distribution delay. For the dynamical behavior analysis of delayed networks systems, different types of time delay, have been taken into account by using a variety of techniques that include the Lyapunov functional method, M-matrix theory, topological degree theory, and techniques of inequality analysis; see, e.g., [18, 24, 25, 28]. In this paper, the delay is the bifurcation parameter. Because a social network usually contains tens of thousands of user nodes, it is inconvenient for us to give a clustering analysis in the current work. We use the concept of network motifs proposed by Benson [3] to search for clusters of an online social network.

Motivated by the above work, the present work will consider a new model with carrier compartment and delay. We use the delay to accomplish the control of the Hopf bifurcation in MWSNs. Our main contributions are summarized as follows.

(i) Based on the clustering analysis of social networks and the SIR model in the epidemic theory, we develop a new malware propagation model with a carrier compartment and delay. To the best of our knowledge, few results have been obtained for the present model. Our new model is more accurate about the actual situation; hence, it will be more important for the applications.

(ii) We study the sensitivity of some parameters, guaranteeing the controllability of the model. Thus, we can control stability and instability by sensitive parameters, which has important implications for practical applications.

(iii) It is nontrivial to establish a unified framework to handle the mathematical techniques (including matrix theory, Hopf analysis, etc.) for overcoming the above difficulties.

The remaining structure of this paper is arranged as follows. In Section 2, we construct a malware propagation model. In Section 3, we study the local stability and Hopf bifurcation of the equilibrium points for the present model. In Section 4, some numerical simulations are presented to illustrate our theoretical results. Finally, conclusions are drawn in Section 5.

2 Modeling a malware propagation model

For the classic malware propagation model, all nodes are divided into three classes depending on their states: susceptible (healthy), infected, and removed (immunized). Recently, Guillén and del Rey [8] studied a new malware propagation model using a carrier compartment. Some devices are not targeted by the malware (for example, IOS devices for Android malware), and they can be denoted as carrier devices. However, the influence of delays cannot be considered in the model of [8]. In malware propagation model used in [8], the densities of infected individuals $I(t)$ are not influenced by discrete delay. In real malware propagation model, $I(t)$ may be influenced by different classes of delays at the same time, including discrete delays. Hence, we add the delay term $I(t-\tau)$ in our model. Based on the above description, our model can be represented as a flow chart (see Fig. 1) or as a set of coupled differential equations (ODES) as follows:

$$
\begin{aligned}
\frac{\mathrm{d}S(t)}{\mathrm{d}t} &= A - \beta S(t)\big[I(t-\tau)+C(t)\big] - \eta S(t) + \delta R(t),\\
\frac{\mathrm{d}C(t)}{\mathrm{d}t} &= \beta S(t)\big[I(t-\tau)+C(t)\big] - \gamma C(t),\\
\frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \beta S(t)\big[I(t-\tau)+C(t)\big] - \varepsilon I(t) - \eta I(t),\\
\frac{\mathrm{d}R(t)}{\mathrm{d}t} &= \gamma C(t) + \varepsilon I(t) + \eta S(t) - \delta R(t)
\end{aligned}
\tag{1}
$$

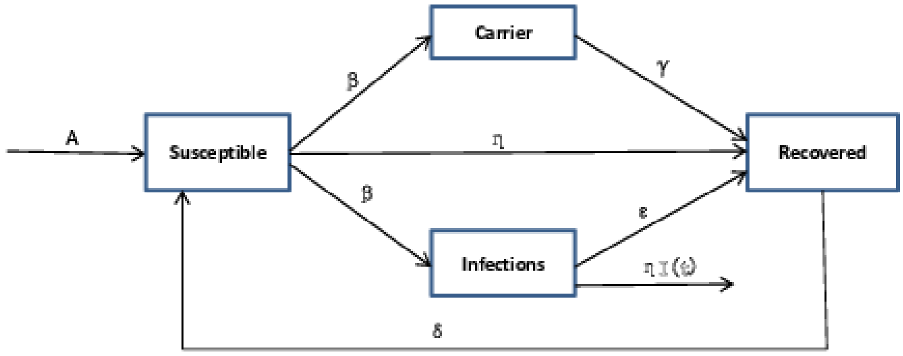

Figure 1. Flow diagram for system (1).

Table 1. Symbols and their meanings of system (1).

Parameters	Notes
A	The number of susceptible nodes
β	The constant contact rate for $S(t)$, $I(t-\tau)$, and $C(t)$
η	The vaccination rate of nodes
δ	The rate constant for nodes becoming susceptible again
γ	The number of new recovered devices from carrier
ε	The rate constant for nodes leaving $I(t)$ for $R(t)$

with initial conditions

$$\begin{aligned} &S(t) \geqslant 0, \quad C(t) \geqslant 0, \quad R(t) \geqslant 0, \quad t \in [0, \infty), \\ &I(t) \geqslant 0, \quad t \in [-\tau, \infty),\ \tau \geqslant 0, \end{aligned} \tag{2}$$

where $C(t)$ is a carrier. For the dynamic relations among $S(t)$, $C(t)$, $I(t)$, and $R(t)$, see Fig. 1.

Remark 1. In [8], considering the special class of carrier devices, the authors studied the following system with carrier:

$$\begin{aligned} \frac{\mathrm{d}S(t)}{\mathrm{d}t} &= \epsilon R(t) - aS(t)\big(I(t) + C(t)\big) - \nu S(t), \\ \frac{\mathrm{d}C(t)}{\mathrm{d}t} &= a(1-\delta)S(t)\big(I(t) + C(t)\big) - b_C C(t), \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} &= a\delta S(t)\big(I(t) + C(t)\big) - b_I I(t), \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} &= b_C C(t) + b_I I(t) + \nu S(t) - \epsilon R(t). \end{aligned}$$

In generally, infectious $I(t)$ is affected by the delay. However, the above system cannot include the influence of delay. In order to reflect the actual situation, in system (1), we add the delay to infectious $I(t)$. Hence, our model is more accurate than the existing ones.

As $S(t)+C(t)+I(t)+R(t)=N$, where N is a positive constant, then system (1) can be written as follows:

$$\begin{aligned}\frac{\mathrm{d}S(t)}{\mathrm{d}t} &= A-\beta S(t)\big[I(t-\tau)+C(t)\big]\\ &\quad -\eta S(t)+\delta\big[N-S(t)-C(t)-I(t)\big],\\ \frac{\mathrm{d}C(t)}{\mathrm{d}t} &= \beta S(t)\big[I(t-\tau)+C(t)\big]-\gamma C(t),\\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \beta S(t)\big[I(t-\tau)+C(t)\big]-\varepsilon I(t)-\eta I(t).\end{aligned} \tag{3}$$

Similar to the analysis of [8], we can verify that the positive cone D is a positive invariant set with respect to (3), where

$$D=\big\{(S,C,I)\in\mathbb{R}^3\colon\ S,C,I>0,\ S+C+I\leqslant N\big\}.$$

In view of SIR epidemic model and characteristic of malware propagation model, we give the following three facts:

(i) When nodes lie in state I, users can immunize their nodes with countermeasures;
(ii) Some part of all recovered nodes go through a temporary immunity with probability δ, others with probability γ;
(iii) For convenience of calculating equilibrium points, the number N of all is a constant, which is independent on time t.

3 Local stability and Hopf bifurcation

In this section, we will discuss the local stability and Hopf bifurcation of system (3) by analyzing the corresponding characteristic equations. Furthermore, we can obtain that the time delay τ is the bifurcation parameter, which is significant for control theory. Obviously, the virus-free equilibrium point of (3) is

$$E_0=(S_0,C_0,I_0)=\left(\frac{A+\delta N}{\eta+\delta},0,0\right).$$

Furthermore, if the following condition hold:

(H1) $\beta(A+\delta N)(\gamma+\varepsilon+\eta)>\varepsilon+\eta$,

then the endemic equilibrium point of (3) is positive and represents as follows:

$$E^*=(S^*,C^*,I^*),$$

where

$$\begin{aligned}S^* &= \frac{\varepsilon+\eta}{\beta(\varepsilon+\eta+\gamma)},\qquad I^*=\frac{\gamma(A+\delta N)(\gamma+\varepsilon+\eta)-\gamma\beta^{-1}(\varepsilon+\eta)}{(\gamma+\varepsilon+\eta)(\gamma\varepsilon+\gamma\eta+\delta\varepsilon+\delta\eta+\delta\gamma)},\\ C^* &= \frac{(A+\delta N)(\varepsilon+\eta)(\gamma+\varepsilon+\eta)-\beta^{-1}(\varepsilon+\eta)^2}{(\gamma+\varepsilon+\eta)(\gamma\varepsilon+\gamma\eta+\delta\varepsilon+\delta\eta+\delta\gamma)}.\end{aligned}$$

Let

$$b = \varepsilon + \eta + \gamma - \beta S_0 - S_0, \qquad k = \frac{\beta(A + \delta N)}{\eta + \delta}.$$

For convenience of proof, we list the following assumptions:

(H2) $b > 2\sqrt{(\varepsilon + \eta - \beta S_0)(\gamma - S_0) - \beta^2 S_0^2}$, $(\varepsilon + \eta - \beta S_0)(\gamma - S_0) > \beta^2 S_0^2$;
(H3) $(\gamma - k)(\varepsilon + \eta) > (1 - \beta^2)k^2 - k > 0$;
(H4) $(\varepsilon + \eta + \gamma)^2 - 2k(\varepsilon + \eta + \gamma) > 4(\gamma - k)(\varepsilon + \eta)$;
(H5) $(\gamma - k)^2(\varepsilon + \eta)^2 < [(1 - \beta^2)k^2 - k]^2$;
(H6) $\beta S^* < \min\{\eta + \delta,\ \eta + \varepsilon\}$;
(H7) $a_2 > 2a_1,\ \ a_1^2 - 2a_0 a_2 - b_1^2 > 0$;
(H8) $a_0^2 > b_2 + b_0$.

3.1 Virus-free equilibrium and its stability

Theorem 1. *Assume that assumption* (H2) *hold, then the virus-free equilibrium point* E_0 *of system* (3) *with* $\tau = 0$ *is locally asymptotically stable.*

Proof. The characteristic equation of system (3) at E_0

$$\det\begin{pmatrix} -\eta - \delta - \lambda & -k - \delta & -\delta \\ 0 & k - \gamma - \lambda & k\mathrm{e}^{-\lambda\tau} \\ 0 & k & -\varepsilon - \eta + k\mathrm{e}^{-\lambda\tau} - \lambda \end{pmatrix} = 0,$$

which is equivalent to

$$(\lambda + \eta + \delta)(\lambda - k + \gamma)\big(\lambda + \varepsilon + \eta - k\mathrm{e}^{-\lambda\tau}\big) - \beta^2 k^2 \mathrm{e}^{-\lambda\tau}(\lambda + \eta + \delta) = 0. \tag{4}$$

Reducing (4), we have

$$(\lambda + \eta + \delta)\big[\lambda^2 + u_1\lambda + u_0 + (v_1\lambda + v_0)\mathrm{e}^{-\lambda\tau}\big] = 0, \tag{5}$$

where

$$u_1 = \varepsilon + \eta + \gamma - k, \qquad u_0 = (\gamma - k)(\varepsilon + \eta),$$
$$v_1 = -k, \qquad v_0 = (1 - \beta^2)\frac{\beta^2(A + \delta N)^2}{(\eta + \delta)^2} - \gamma k.$$

As $\tau = 0$, (4) is changed into

$$(\lambda + \eta + \delta)(\lambda - k + \gamma)(\lambda + \varepsilon + \eta - k) - \beta^2 k^2(\lambda + \eta + \delta) = 0. \tag{6}$$

In view of assumption (H2), (6) has three negative roots:

$$\lambda_1 = -\eta - \delta < 0,$$
$$\lambda_{2,3} = \frac{1}{2}\big[-(\varepsilon + \eta + \gamma - \beta S_0 - S_0) \pm \sqrt{(\varepsilon + \eta + \gamma - \beta S_0 - S_0)^2 + 4\beta^2 S_0^2 - 4(\varepsilon + \eta - \beta S_0)(\gamma - S_0)}\,\big] < 0.$$

Thus, the equilibrium E_0 is locally asymptotically stable as $\tau = 0$. □

Next, we investigate the effect of the delay τ on the stability of the equilibrium E_0.

Theorem 2. *Assume that assumptions* (H3) *and* (H4) *hold, then the virus-free equilibrium point E_0 of system* (3) *with $\tau > 0$ is locally asymptotically stable.*

Proof. By (5) we have

$$\lambda^2 + u_1\lambda + u_0 + (v_1\lambda + v_0)\mathrm{e}^{-\lambda\tau} = 0. \tag{7}$$

Assume that $\mathrm{i}\omega$ ($\omega > 0$) is a root of (7). Then ω satisfies the following equation:

$$-\omega^2 + u_1\omega\mathrm{i} + u_0 + (v_1\omega\mathrm{i} + v_0)(\cos\omega\tau - \mathrm{i}\sin\omega\tau) = 0,$$

which implies that

$$\begin{aligned} v_1\omega\cos\omega\tau - v_0\sin\omega\tau &= u_1\omega, \\ v_0\cos\omega\tau + v_1\omega\sin\omega\tau &= \omega^2 - u_0. \end{aligned} \tag{8}$$

Taking square on both sides of (8) and summing them up, we have

$$\omega^4 + \left(u_1^2 - v_1^2 - 2u_0\right)\omega^2 + u_0^2 - v_0^2 = 0. \tag{9}$$

Set $z = \omega^2$, (9) becomes

$$z^2 + \left(u_1^2 - v_1^2 - 2u_0\right)z + u_0^2 - v_0^2 = 0. \tag{10}$$

Since $u_0^2 - v_0^2 = [(\gamma - k)(\varepsilon + \eta)]^2 - [(1 - \beta^2)k^2 - \gamma k]^2$, in view of (H3), we have

$$u_0^2 - v_0^2 > 0. \tag{11}$$

On the other hand, by (H4) we have

$$\begin{aligned} &\left(u_1^2 - v_1^2 - 2u_0\right)^2 - 4\left(u_0^2 - v_0^2\right) \\ &\quad = \left[(\varepsilon + \eta + \gamma)^2 - 2k(\varepsilon + \eta + \gamma) - 2(\gamma - k)(\varepsilon + \eta)\right]^2 \\ &\qquad - 4\left[(\gamma - k)(\varepsilon + \eta)\right]^2 + 4\left[\left(1 - \beta^2\right)k^2 - \gamma k\right]^2 > 0. \end{aligned} \tag{12}$$

From (11) and (12), (10) has no positive solutions. Moreover, there are no positive solutions for (5) and (7). Thus, the equilibrium E_0 is locally asymptotically stable as $\tau > 0$. □

Theorem 3. *Suppose that assumptions* (H2)–(H5) *hold. Then for system* (3)*, the following statements are true*:

(i) *The equilibrium point E_0 of system* (3) *is locally asymptotically stable as $\tau \in [0, \tau_0)$.*
(ii) *The Hopf bifurcation is τ_0^j. That is, system* (3) *has a branch of periodic solutions bifurcating from the equilibrium E_0 near $\tau = \tau_0^j$, where τ_0^j is defined by* (13).

Proof. From (H4) and (H5), (10) has a unique positive root, denoted by Z_0. Then (9) has a unique positive root $\omega_0 = \sqrt{Z_0}$. By (8) we have

$$\cos\omega_0\tau = \frac{(\omega_0^2 - u_0)v_0 + u_1 v_1 \omega_0^2}{v_0^2 + v_1^2\omega_0^2}.$$

Then denote

$$\tau_0^j = \frac{1}{\omega_0}\left[\arccos\frac{(\omega_0^2 - u_0)v_0 + u_1 v_1 \omega_0^2}{v_0^2 + v_1^2\omega_0^2} + 2j\pi\right], \quad j = 0, 1, 2, \dots. \tag{13}$$

Thus, $\pm\mathrm{i}\omega_0$ is a pair of purely imaginary roots of (7) at $\tau = \tau_0^j$. Furthermore, all the roots of (7) have negative real parts for $\tau \in [0, \tau_0^j)$, and all the roots of (7), except $\pm\mathrm{i}\omega_0$, have negative real parts for $\tau = \tau_0^j$. Using the Hopf bifurcation theorem, we can obtain the following transversality condition:

$$\frac{\mathrm{d}(\mathrm{Re}(\lambda(\tau)))}{\mathrm{d}\tau}\bigg|_{\substack{\tau=\tau_0^j\\ \lambda=\mathrm{i}\omega_0}} > 0.$$

In fact, in view of (H4) and (H5), we have

$$\begin{aligned}
\frac{\mathrm{d}(\mathrm{Re}(\lambda(\tau)))}{\mathrm{d}\tau}\bigg|_{\substack{\tau=\tau_0^j\\ \lambda=\mathrm{i}\omega_0}} &= \frac{1}{\omega_0(v_1^2\omega_0^2 + v_0^2)}\big[2v_1\omega_0^2\sin\omega_0\tau_0^j - u_1 v_1\cos\omega_0\tau_0^j \\
&\quad - v_1^2\omega_0 + 2v_0\omega_0\cos\omega_0\tau_0^j + u_1 v_0\sin\omega_0\tau_0^j\big] \\
&= \frac{1}{(v_1^2\omega_0^2 + v_0^2)^2}\big[2v_1^2\omega_0^4 + \big(2v_0^2 + u_1^2 v_1^2 - v_1^4 - 2u_0 v_1^2\big)\omega_0^2 \\
&\quad + u_1 2v_0^2 - v_1^2 v_0^2 - 2u_0 v_0^2\big] \\
&= \frac{1}{(v_1^2\omega_0^2 + v_0^2)^2}\bigg[\frac{1}{2}v_1^2\big(v_1^2 + 2u_0 - u_1^2\big)^2 \\
&\quad + \frac{1}{2}v_1\big(v_1^2 + 2u_0 - u_1^2\big)\sqrt{\big(v_1^2 + 2u_0 - u_1^2\big)^2 - 4\big(u_0^2 - v_0^2\big)} \\
&\quad + v_0^2\sqrt{\big(v_1^2 + 2u_0 - u_1^2\big)^2 - 4\big(u_0^2 - v_0^2\big)} \\
&\quad + 2v_1^2\big(v_0^2 - u_0^2\big)\bigg] > 0.
\end{aligned}$$

□

3.2 Endemic-equilibrium and its stability

The characteristic equation of system (3) at the positive equilibrium E^* is of the form

$$\det\begin{pmatrix} -\beta I^* - \eta - \delta - \lambda & -\beta S^* - \delta & -\beta S^*\mathrm{e}^{-\lambda\tau} - \delta \\ \beta I^* & \beta S^* - \gamma - \lambda & \beta S^*\mathrm{e}^{-\lambda\tau} \\ \beta I^* & \beta S^* & -\varepsilon - \eta + \beta S^*\mathrm{e}^{-\lambda\tau} - \lambda \end{pmatrix},$$

which is equivalent to

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 + \big(b_2\lambda^2 + b_1\lambda + b_0\big)\mathrm{e}^{-\lambda\tau} = 0, \tag{14}$$

where

$$\begin{aligned}
a_2 &= \varepsilon + \gamma + \beta S^* + 2\eta + \delta + \beta I^*,\\
a_1 &= (\eta + \delta)(\gamma + \beta S^*) + 2\delta\beta I^* + \beta I^*(\gamma + \varepsilon + \eta),\\
a_0 &= (\gamma + \beta S^* + \eta + \delta)(\varepsilon + \eta) + (\eta + \delta)(\gamma + \beta S^*)(\varepsilon + \eta) + \delta\beta\gamma I^*\\
&\quad + \beta I^*(\varepsilon + \eta)(\delta + \gamma),\\
b_2 &= -\beta S^*, \qquad b_1 = -\beta S^*(\eta + \delta + \gamma), \qquad b_0 = -\beta S^*(\eta + \delta)\gamma.
\end{aligned}$$

When $\tau = 0$, (14) becomes

$$\lambda^3 + (a_2 + b_2)\lambda^2 + (a_1 + b_1)\lambda + a_0 + a_0 + b_0 = 0.$$

It is easy to see that

$$\begin{aligned}
a_2 + b_2 &= \varepsilon + \gamma + 2\eta + \delta + \beta I^* > 0,\\
a_1 + b_1 &= \gamma(\eta + \delta - \beta S^*) + 2\delta\beta I^* + \beta I^*(\gamma + \varepsilon + \eta) > 0.
\end{aligned}$$

By assumption (H6) we have

$$\begin{aligned}
a_0 + b_0 &= (\gamma + \beta S^* + \eta + \delta)(\varepsilon + \eta) + (\eta + \delta)\gamma(\varepsilon + \eta - \beta S^*)\\
&\quad + (\eta + \delta)(\varepsilon + \eta)\beta S^* + \delta\beta\gamma I^* + \beta I^*(\varepsilon + \eta)(\delta + \gamma)\\
&> 0.
\end{aligned}$$

Hence, the endemic equilibrium E^* of system (3) is locally asymptotically stable when $\tau = 0$. Based on the above analysis, we have the following theorem.

Theorem 4. *Assume that assumptions* (H1) *and* (H6) *hold, then the endemic equilibrium point E^* of system* (3) *with $\tau = 0$ is locally asymptotically stable.*

If $\mathrm{i}\omega$ ($\omega > 0$) is a solution of (14), separating real and imaginary parts, (14) becomes

$$\begin{aligned}
&-\mathrm{i}\omega^3 - a_2\omega^2 + a_1\mathrm{i}\omega + a_0\\
&\quad + \big[-b_2\omega^2 + b_1\mathrm{i}\omega + \delta + b_0\big](\cos\omega\tau - \mathrm{i}\sin\omega\tau) = 0.
\end{aligned} \tag{15}$$

By (15) we have

$$\begin{aligned}
b_1\omega\cos\omega\tau + (b_2 - b_0)\sin\omega\tau &= \omega^3 - a_1\omega,\\
(-b_2 + b_0)\cos\omega\tau + b_1\omega\sin\omega\tau &= a_2\omega^2 - a_0.
\end{aligned} \tag{16}$$

Taking square on both sides of (16) and summing them up, we obtain

$$\omega^6 + (a_2 - 2a_1)\omega^4 + \big(a_1^2 - 2a_0a_2 - b_1^2\big)\omega^2 + a_0^2 - (b_2 + b_0) = 0. \tag{17}$$

Set $z = \omega^2$, (17) is transformed into the following equation:

$$z^3 + A_1 z^2 + A_2 z + A_3 = 0, \tag{18}$$

where

$$A_1 = a_2 - 2a_1, \qquad A_2 = a_1^2 - 2a_2 a_0 - b_1^2, \qquad A_3 = a_0^2 - b_2 - b_0.$$

From the results of [31] we give the following theorem for the distribution of roots of (18).

Theorem 5.

(i) *If* $a_0^2 < b_2 + b_0$, *then* (18) *has at least one positive root.*

(ii) *If* $a_0^2 \geqslant b_2 + b_0$, *then* (18) *has positive roots if and only if* $\tilde{z} > 0$, $r(\tilde{z}) \leqslant 0$, *where*

$$r(z) = z^3 + A_1 z^2 + A_2 z + A_3, \qquad \tilde{z} = \frac{-A_1 + \sqrt{A_1^2 - 3A_2}}{3}. \tag{19}$$

If (18) has positive real roots Z_1, Z_2, and Z_3, then we have

$$\omega_1 = \sqrt{Z_1}, \qquad \omega_2 = \sqrt{Z_2}, \qquad \omega_3 = \sqrt{Z_3}.$$

By (16) we have

$$\cos \omega_k \tau_k = \frac{b_1 \omega_k^2 (\omega_k^2 - a_1)}{b_1^2 \omega_k^2 + (b_2 - b_0)^2} - \frac{(b_2 - b_0)(a_2 \omega_k^2 - a_0)}{b_1^2 \omega_k^2 + (b_2 - b_0)^2}$$

and

$$\tau_k^j = \frac{1}{\omega_k} \arccos\left(\frac{b_1 \omega_k^2 (\omega_k^2 - a_1)}{b_1^2 \omega_k^2 + (b_2 - -b_0)^2} - \frac{(b_2 - b_0)(a_2 \omega_k^2 - a_0)}{b_1^2 \omega_k^2 + (b_2 - b_0)^2} + 2j\pi \right).$$

where $k = 1, 2, 3$, $j = 0, 1, \dots$.

Thus, $\pm \mathrm{i}\omega$ is a pair of purely imaginary roots of (18). Denote

$$\tau_0 = \tau_{k_0}^0 = \min_{k=1,2,3} \{\tau_k^0\}, \qquad \omega_0 = \omega_{k_0}. \tag{20}$$

Now, we give an important result for the roots of (17).

Theorem 6. *If assumptions* (H1), (H7), *and* (H8) *hold, then* (17) *has no positive real roots.*

Proof. By (H7) we have

$$a_2 - 2a_1 > 0$$

and

$$a_1^2 - 2a_0 a_2 - b_1^2 > 0.$$

By (H8) it is easy to show that

$$a_0^2 - (b_2 + b_0) > 0.$$

According to Descartes's rule of sign, (17) has no positive real roots. □

From the results of [10, 32] we have the following theorem.

Theorem 7. *Let* $\lambda(\tau) = \alpha(\tau) \pm \mathrm{i}\omega(\tau)$ *be the root of* (14) *near* $\tau = \tau_0$ *satisfying* $\alpha(\tau_0) = 0$ *and* $\omega(\tau_0) = \omega_0$*, where* τ_0 *is defined by* (20)*. Assume that* $r'(\omega_0^2) > 0$*, where* $r(z)$ *is defined by* (19)*. Then* $\pm\mathrm{i}\omega_0$ *is a pair of simple purely imaginary roots of* (14) *and satisfies*

$$\frac{\mathrm{d}(\mathrm{Re}(\lambda(\tau)))}{\mathrm{d}\tau}\bigg|_{\substack{\tau=\tau_0\\ \lambda=\pm\mathrm{i}\omega_0}} > 0.$$

Using Theorems 4–7, we have the following theorem.

Theorem 8. *Suppose that* (H1), (H6)–(H8) *hold.*

(i) *If the conditions of Theorem* 4 *hold, then the equilibrium point* E^* *of system* (3) *is locally asymptotically stable as* $\tau \in [0, \tau_0)$*.*
(ii) *If the condition of* (i) *is satisfied and* $r'(\omega_0^2) > 0$*, then system* (3) *undergoes a Hopf bifurcation at* E^* *when* $\tau = \tau_0$*.*

Here τ_0 *and* ω_0 *are defined by* (20)*.*

4 Numerical analysis and discussion

In this section, we simulate and discuss the dynamic characteristics of the proposed malware model with carrier compartment and delay by using simulations in Matlab.

4.1 Sensitivity analysis of the rate constant ε on the number of carrier nodes

To obtain the sensitivity of the rate constant ε in (3), let $N = 1000$, $A = 1.2$, $\beta = 0.5$, $\eta = 0.2$, $\delta = 0.005$, $\gamma = 0.3$ and assign 0.2, 0.3, 0.4 to ε, respectively. Then all the conditions of Theorem 8 hold. For different ε, we can easily obtain the corresponding positive equilibrium points and the region of stability of system (3), see Table 2.

In view of Table 2, we find that the critical value τ_0 increases when the parameter ε increases. Hence the rate constant ε is sensitive for the stability of (3). We can extended range of stability for (3) by using the increase of parameter ε. Without loss of generality, we assign $\tau = 4 < \tau_0$ with the other parameters unchanged. The simulation results are shown in Fig. 2. From Fig. 2 we notice that the number of carrier nodes converges to the positive equilibrium point E^*. When $\tau > \tau_0$, according to Theorem 8, the solutions emerge from the positive equilibrium point E^*, as shown in Fig. 3. From Fig. 3 we notice that the malware propagation goes into periodic oscillation.

Table 2. The dynamical properties of system (3) for different constant ε.

ε	positive equilibrium points	τ_0
0.2	$(0.3429, 13.2562, 9.9422)^\top$	5.9743
0.3	$(0.3750, 16.0714, 9.6428)^\top$	6.0081
0.4	$(0.4000, 15.8266, 7.9133)^\top$	8.2649

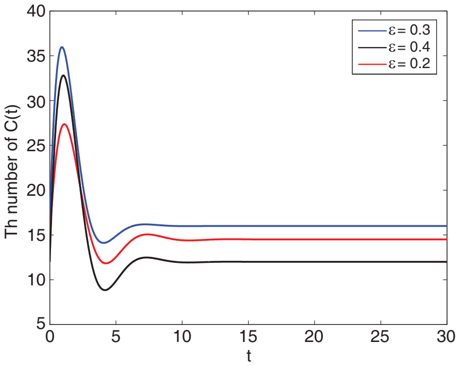

Figure 2. When the rate constant ε is variable, the positive equilibrium point E^* is stable with $\tau = 4 < \tau_0$.

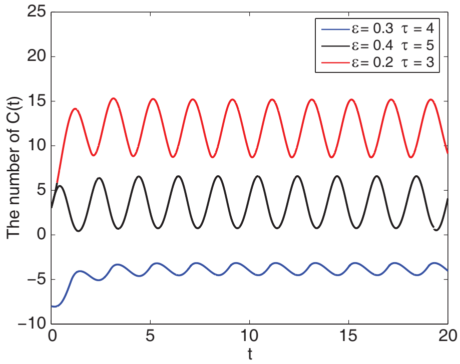

Figure 3. Hopf bifurcation occurs from the equilibrium point E^* when $\tau > \tau_0$ with the rate constant ε varying.

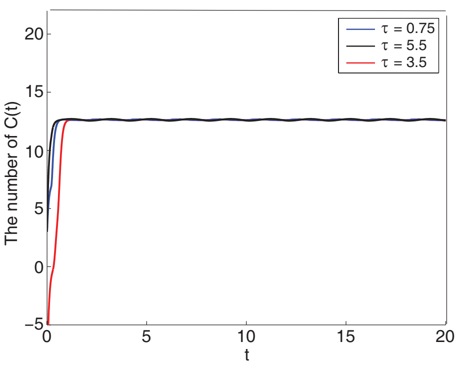

Figure 4. The positive equilibrium point E^* is stable when $\tau < \tau_0$.

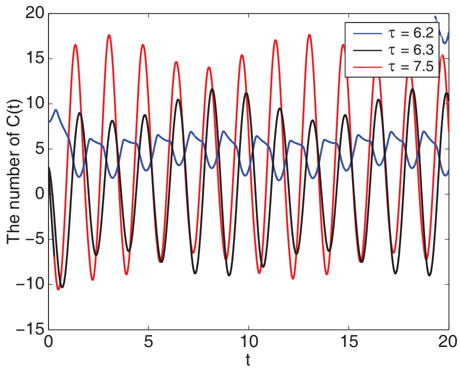

Figure 5. Hopf bifurcation occurs from the positive equilibrium point when $\tau > \tau_0$.

4.2 Influence of delays on the number of carrier nodes

To study the sensitivity of the delay τ in (3), let $N = 800$, $A = 1.16$, $\beta = 0.5$, $\eta = 0.2$, $\delta = 0.005$, $\gamma = 0.3$, $\varepsilon = 0.2$. The positive equilibrium point of (3) is $E^* = (0.3428, 13.0509, 9.7882)^\top$. All the conditions of Theorem 8 hold. According to Theorem 8, the critical value is $\tau_0 = 5.8762$. As the other parameters remained unchange, we assign $0.75, 3.5, 5.5$ to τ. The simulation results can be seen in Fig. 4. From Fig. 4 we notice that the number of carrier nodes converges to the positive equilibrium point E^* of (3). Next, we assign $6.2, 6.3, 7.5$ to τ with $\tau > \tau_0$, and the other parameters remained unchange. According to Theorem 8, the solutions of (11) emerge from the positive equilibrium point E^*, as shown in Fig. 5.

Remark 2. From numerical results we find that the value of delay τ is key to the stability of the above model. In the present paper, we obtain only local stability of system (3). For global stability of (3), because of the existence of the delay in (3), so far we cannot obtain global stability results for (3). We hope that someone will solve the problem in the future. Furthermore, some parameters in (3) has strongly influence the stability interval of the system, which is important for control theory.

5 Conclusions

In this article, we study a new malware spreading model with a carrier compartment and delays. It is noted that the new class of carrier devices is considered (apart from susceptible, infectious and recovered), which is different from the corresponding ones of past work.

By theoretical analysis and numerical simulations we show how time delay affects malware propagation. Also, using delay as a bifurcating parameter, we obtain some conditions for occurrence of Hopf bifurcation. Furthermore, numerical simulations verify the correctness of theoretical analyses. More importantly, we investigate the effect of a variable parameter ε on the scale of a malware spreading model. Simulation results show that there is a great impact on stability interval. However, many important questions about malware spreading remain to be studied, such as optimal control, clustering for complex networks, global stability, and the direction of bifurcation.

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