Nonlinear Analysis: Modelling and Control, Vol. 23, No. 4, 583–598 ISSN 1392-5113
https://doi.org/10.15388/NA.2018.4.8

Coexistence of generalized synchronization and inverse generalized synchronization between chaotic and hyperchaotic systems

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Received: December 3, 2017 / **Revised:** March 25, 2018 / **Published online:** June 15, 2018

Abstract. In this paper, we present new schemes to synchronize different dimensional chaotic and hyperchaotic systems. Based on coexistence of generalized synchronization (GS) and inverse generalized synchronization (IGS), a new type of hybrid chaos synchronization is constructed. Using Lyapunov stability theory and stability theory of linear continuous-time systems, some sufficient conditions are derived to prove the coexistence of generalized synchronization and inverse generalized synchronization between 3D master chaotic system and 4D slave hyperchaotic system. Finally, two numerical examples are illustrated with the aim to show the effectiveness of the approaches developed herein.

Keywords: hyperchaotic systems, generalized synchronization, inverse generalized synchronization, hybrid synchronization, Lyapunov stability.

1 Introduction

The study of nonlinear behaviour, such as stability, control and synchronization of dynamical systems, has become an interdisciplinary research. As a result, several research works have been published on this subject by researchers of different disciplines [5, 6, 9, 34]. Due to the importance and interdisciplinary nature of nonlinear dynamical systems, its applications have been discovered in field studies like mathematics, physics, chemistry,

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engineering, economics, neuroscience and many more [7, 8, 10, 11, 13, 33, 37]. In search of the stable, fast and robust synchronization methods, such as linear and nonlinear feedback controllers, active control, passive control, backstepping, sliding mode control and many more, have been developed [1–3, 12, 39, 41]. Similarly, several types of synchronization have been introduced [4, 14, 17–19, 24, 25, 40]. One of the most exciting synchronization types is the *generalized synchronization* (GS). This type of synchronization is characterized by the existence of a functional relationship ϕ between the state $Y(t)$ of the slave system and the state $X(t)$ of the master system, so that $Y(t) = \phi(X(t))$ after a transient time [32]. Different types of synchronization, such as complete synchronization, anti-synchronization, projective synchronization and function projective synchronization, can be achieved from the generalized synchronization scheme depending on the choice of ϕ. A variation is represented by the *inverse generalized synchronization* (IGS), where the synchronization condition becomes $X(t) = \varphi(Y(t))$ after a transient time. However, since the type is still relatively new, IGS has been applied with successfully in continuous-time systems as well as discrete-time systems [27, 28]. Another interesting type of synchronization is the *hybrid synchronization*, where complete synchronization and anti-synchronization coexist. Faster and more secure digital communication could be achieved using the hybrid synchronization [38].

Recently, Ouannas et al. extended the concept of hybrid synchronization to achieve the coexistence of more than two types of synchronization between different dimensional chaotic (hyperchaotic) systems in discrete-time systems, integer-order differential systems and fractional-order differential systems [15, 16, 20–22, 26, 29]. They also reported for the first time the coexistence of generalized synchronization (GS) and inverse generalized synchronization (IGS) for the fractional-order systems with different dimensions [23]. Meanwhile, to the best of our knowledge, the investigation of coexistence of GS and IGS for integer-order dynamical systems of different dimensions is not yet explored. The present research work focuses on coexistence of GS and IGS between chaotic and hyperchaotic systems.

Motivated by the importance of generalized synchronization and hybrid synchronization in enhancing security and fast information transmission, new schemes of coexistence based on GS and IGS is developed for integer-order chaotic and hyperchaotic systems in this paper. The paper is organized as follow: Section 2 gives some definitions related to generalized and inverse generalized synchronization. Sections 3 and 4 give the basic mathematical background of the coexistence of generalized and inverse generalized synchronization in 3D and 4D, respectively. Section 5 presents some numerical examples, while Section 6 presents discussions. Section 7 concludes the paper.

2 Definitions GS and IGS

We consider the following master and slave systems:

$$\dot{X}(t) = F\bigl(X(t)\bigr), \tag{1}$$

$$\dot{Y}(t) = G\bigl(Y(t)\bigr) + U, \tag{2}$$

where $X(t) = (x_i(t))_{1\leqslant i\leqslant n}$, $Y(t) = (y_i(t))_{1\leqslant i\leqslant m}$ are the states of the master system and the slave system, respectively, $F:\mathbb{R}^n \to \mathbb{R}^n$, $G:\mathbb{R}^m \to \mathbb{R}^m$ and $U = (u_i)_{1\leqslant i\leqslant m}$ is a vector controller.

Definition 1. The master systems (1) and the slave system (2) are in generalized synchronization if there exists a controller $U = (u_i)_{1\leqslant i\leqslant m}$ and a differentiable function $\phi:\mathbb{R}n \to \mathbb{R}^m$ such that

$$\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|Y(t) - \phi(X(t))\| = 0.$$

Definition 2. The master systems (1) and the slave system (2) are in inverse generalized synchronization if there exists a controller $U = (u_i)_{1\leqslant i\leqslant m}$ and a differentiable function $\varphi:\mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{t\to+\infty}\|e(t)\| = \lim_{t\to+\infty}\|X(t) - \varphi(Y(t))\| = 0.$$

3 Coexistence of GS and IGS in 3D

3.1 Master–slave systems description

Here we assume that the master systems can be considered as

$$\dot{x}_i(t) = \sum_{j=1}^{3} a_{ij}x_j(t) + f_i(X(t)), \quad i = 1,2,3, \tag{3}$$

where $X(t) = (x_i(t))_{1\leqslant i\leqslant 3}$ is the state of the master system (3), $(a_{ij}) \in \mathbb{R}^{3\times 3}$, $f_i:\mathbb{R}^3 \to \mathbb{R}$, $i = 1,2,3$. Also, consider the slave system as

$$\dot{y}_i(t) = g_i(Y(t)) + u_i, \quad i = 1,2,3,4, \tag{4}$$

where $Y(t) = (y_i(t))_{1\leqslant i\leqslant 4}$ is the state of the slave system (4), $g_i:\mathbb{R}^4 \to \mathbb{R}$, and u_i, $i = 1,2,3,4$, are controllers.

3.2 Problem formulation

Definition 3. We say that GS and IGS coexist in the synchronization of the master system (3) and the slave system (4) if there exist controllers u_i, $i = 1,2,3,4$, and differentiable functions $\phi_1,\phi_2:\mathbb{R}^3 \to \mathbb{R}$, $\varphi:\mathbb{R}^4 \to \mathbb{R}$ such that the synchronization errors

$$\begin{aligned} e_1(t) &= y_1(t) - \phi_1(X(t)),\\ e_2(t) &= x_2(t) - \varphi(Y(t)),\\ e_3(t) &= y_3(t) - \phi_2(X(t)) \end{aligned} \tag{5}$$

satisfy $\lim_{t\to+\infty}\|e_i(t)\| = 0$, $i = 1,2,3$.

3.3 Analytical results

The error system (5) can be derived as follows:

$$\dot e_1(t) = \dot y_1(t) - \sum_{j=1}^{3} \frac{\partial \phi_1}{\partial x_j} \dot x_j(t),$$
$$\dot e_2(t) = \dot x_2(t) - \sum_{j=1}^{4} \frac{\partial \varphi}{\partial y_j} \dot y_j(t), \tag{6}$$
$$\dot e_3(t) = \dot y_3(t) - \sum_{j=1}^{3} \frac{\partial \phi_2}{\partial x_j} \dot x_j(t).$$

Furthermore, the error system (6) can be written as

$$\dot e_1(t) = u_1 + R_1, \qquad \dot e_2(t) = -\sum_{j=1}^{4} \frac{\partial \varphi}{\partial y_j} u_j + R_2, \qquad \dot e_3(t) = u_3 + R_3, \tag{7}$$

where

$$R_1 = g_1\big(Y(t)\big) - \sum_{i=1}^{3} \frac{\partial \phi_1}{\partial x_i} \left(\sum_{j=1}^{3} a_{ij} x_j(t) + f_i\big(X(t)\big) \right),$$
$$R_2 = \sum_{j=1}^{3} a_{2j} x_j(t) + f_2\big(X(t)\big) - \sum_{j=1}^{4} \frac{\partial \varphi}{\partial y_j} g_j\big(Y(t)\big),$$
$$R_3 = g_3\big(Y(t)\big) - \sum_{i=1}^{3} \frac{\partial \phi_2}{\partial x_j} \left(\sum_{j=1}^{3} a_{ij} x_j(t) + f_i\big(X(t)\big) \right).$$

To achieve synchronization between the master system (3) and the slave system (4), we assume that $\partial\varphi/\partial y_2 \neq 0$ and the controllers u_i, $i = 1, 2, 3, 4$, are constructed as follows:

$$\begin{aligned}
u_1 &= \sum_{j=1}^{3} (a_{1j} - c_{1j}) e_j(t) - R_1, \\
u_2 &= -\frac{\frac{\partial \varphi}{\partial y_1}}{\frac{\partial \varphi}{\partial y_2}} \left(\sum_{j=1}^{3} (a_{1j} - c_{1j}) e_j(t) - R_1 \right) - \frac{1}{\frac{\partial \varphi}{\partial y_2}} \left(\sum_{j=1}^{3} (a_{2j} - c_{2j}) e_j(t) - R_2 \right) \\
&\quad - \frac{\frac{\partial \varphi}{\partial y_3}}{\frac{\partial \varphi}{\partial y_2}} \left(\sum_{j=1}^{3} (a_{3j} - c_{3j}) e_j(t) - R_3 \right), \\
u_3 &= \sum_{j=1}^{3} (a_{3j} - c_{3j}) e_j(t) - R_3, \\
u_4 &= 0,
\end{aligned} \tag{8}$$

where $(c_{ij})_{3\times 3}$ are control constants. By substituting the control law (8) into Eq. (7), the error system can be described as

$$\dot{e}_i(t) = \sum_{j=1}^{3}(a_{ij} - c_{ij})e_j(t), \quad i = 1, 2, 3,$$

or in the compact form

$$\dot{e}(t) = (A - C)e(t), \tag{9}$$

where $e(t) = (e_i(t))_{1\leqslant i\leqslant 3}$, $A = (a_{ij})_{3\times 3}$, $C = (c_{ij})_{3\times 3}$. Constructing the candidate Lyapunov function in the form $V(e(t)) = e^{\mathrm{T}}(t)e(t)$, we obtain

$$\begin{aligned}\dot{V}\big(e(t)\big) &= \dot{e}^{\mathrm{T}}(t)e(t) + e^{\mathrm{T}}(t)\dot{e}(t)\\ &= e^{\mathrm{T}}(t)(A - C)^{\mathrm{T}}e(t) + e^{\mathrm{T}}(t)(A - C)e(t)\\ &= e^{\mathrm{T}}(t)[(A - C)^{\mathrm{T}} + (A - C)]e(t).\end{aligned}$$

If the control matrix C is chosen such that $(A - C)^{\mathrm{T}} + (A - C)$ is a negative definite matrix, we get $\dot{V}(e(t)) < 0$. Thus, from the Lyapunov stability theory the zero solution of the error system (9) is globally asymptotically stable, i.e,

$$\lim_{t\to\infty} \big\|e_i(t)\big\| = 0, \quad i = 1, 2, 3.$$

Therefore, systems (3) and (4) are globally synchronized.

Theorem 1. *There exists a suitable control matrix* $C = (c_{ij})_{3\times 3}$ *to realize the coexistence of GS and IGS between the master system* (3) *and the slave system* (4) *under the control law* (8).

4 Coexistence of IGS and GS 4D

4.1 Master–slave systems description

Now, the master system and the slave system can be described in the following forms:

$$\dot{x}_i(t) = f_i\big(X(t)\big), \quad i = 1, 2, 3, \tag{10}$$

$$\dot{y}_i(t) = \sum_{j=1}^{4} b_{ij}y_j(t) + g_i\big(Y(t)\big) + u_i, \quad i = 1, 2, 3, 4, \tag{11}$$

where $X(t) = (x_i(t))_{1\leqslant i\leqslant 3}$, $Y(t) = (y_i(t))_{1\leqslant i\leqslant 4}$ are the states of the master system and the slave system, respectively, $f_i : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, $(b_{ij}) \in \mathbb{R}^{4\times 4}$, $g_i : \mathbb{R}^4 \to \mathbb{R}$, $i = 1, 2, 3, 4$, are nonlinear functions, and u_i, $i = 1, 2, 3, 4$, are controllers.

4.2 Problem formulation

Definition 4. We say that IGS and GS coexist in the synchronization of the master system (10) and the slave system (11) if there exist controllers u_i, $i = 1, 2, 3, 4$, and differentiable functions $\chi_1, \chi_2 : \mathbb{R}^4 \to \mathbb{R}$, $\psi_1, \psi_2 : \mathbb{R}^3 \to \mathbb{R}$ such that the synchronization errors

$$\begin{aligned} e_1(t) &= x_1(t) - \chi_1\big(Y(t)\big), \qquad & e_2(t) &= y_2(t) - \psi_1\big(X(t)\big), \\ e_3(t) &= x_3(t) - \chi_2\big(Y(t)\big), \qquad & e_4(t) &= y_4(t) - \psi_2\big(X(t)\big) \end{aligned} \tag{12}$$

satisfy $\lim_{t\to+\infty} \|e_i(t)\| = 0$, $i = 1, 2, 3, 4$.

4.3 Analytical results

The error system (12) can be described as follows:

$$\begin{aligned} \dot{e}_1(t) &= -\sum_{j=1}^{4} \frac{\partial \chi_1}{\partial y_j} u_j + T_1, \qquad & \dot{e}_2(t) &= u_2 + T_2, \\ \dot{e}_3(t) &= -\sum_{j=1}^{4} \frac{\partial \chi_2}{\partial y_j} u_j + T_3, \qquad & \dot{e}_4(t) &= u_4 + T_4, \end{aligned} \tag{13}$$

where

$$\begin{aligned} T_1 &= f_1\big(X(t)\big) - \sum_{i=1}^{4} \frac{\partial \chi_1}{\partial y_i} \left(\sum_{j=1}^{4} b_{ij} y_j(t) + g_i\big(Y(t)\big) \right), \\ T_2 &= \sum_{j=1}^{4} b_{2j} y_j(t) + g_2\big(Y(t)\big) - \sum_{j=1}^{3} \frac{\partial \psi_1}{\partial x_j} f_j\big(X(t)\big), \\ T_3 &= f_3\big(X(t)\big) - \sum_{i=1}^{4} \frac{\partial \chi_2}{\partial y_i} \left(\sum_{j=1}^{4} b_{ij} y_j(t) + g_i\big(Y(t)\big) \right), \\ T_4 &= \sum_{j=1}^{4} b_{4j} y_j(t) + g_4\big(Y(t)\big) - \sum_{j=1}^{3} \frac{\partial \psi_2}{\partial x_j} f_j\big(X(t)\big). \end{aligned}$$

In this case, we assume that $(\partial\chi_1/\partial y_3)(\partial\chi_2/\partial y_1) - (\partial\chi_1/\partial y_1)(\partial\chi_2/\partial y_3) \neq 0$. Then the controllers u_i, $i = 1, 2, 3, 4$, are designed as follows:

$$\begin{aligned} u_1 &= \sum_{i=1}^{4} \alpha_i \left(\sum_{j=1}^{4} (b_{ij} - l_{ij}) e_j(t) - T_i \right), \\ u_2 &= \sum_{j=1}^{4} (b_{2j} - l_{2j}) e_j(t) - T_2, \\ u_3 &= \sum_{i=1}^{4} \beta_i \left(\sum_{j=1}^{4} (b_{ij} - l_{ij}) e_j(t) - T_i \right), \\ u_4 &= \sum_{j=1}^{4} (b_{4j} - l_{4j}) e_j(t) - T_4. \end{aligned} \tag{14}$$

where $(l_{ij})_{4\times4}$ are control constants, and

$$\alpha_1 = \frac{\frac{\partial\chi_2}{\partial y_3}}{\frac{\partial\chi_1}{\partial y_3}\frac{\partial\chi_2}{\partial y_1} - \frac{\partial\chi_1}{\partial y_1}\frac{\partial\chi_2}{\partial y_3}}, \qquad \alpha_2 = \frac{\frac{\partial\chi_2}{\partial y_3}\frac{\partial\chi_1}{\partial y_2} - \frac{\partial\chi_1}{\partial y_3}\frac{\partial\chi_2}{\partial y_2}}{\frac{\partial\chi_1}{\partial y_3}\frac{\partial\chi_2}{\partial y_1} - \frac{\partial\chi_1}{\partial y_1}\frac{\partial\chi_2}{\partial y_3}},$$

$$\alpha_3 = \frac{-\frac{\partial\chi_1}{\partial y_3}}{\frac{\partial\chi_1}{\partial y_3}\frac{\partial\chi_2}{\partial y_1} - \frac{\partial\chi_1}{\partial y_1}\frac{\partial\chi_2}{\partial y_3}}, \qquad \alpha_4 = \frac{\frac{\partial\chi_2}{\partial y_3}\frac{\partial\chi_1}{\partial y_4} - \frac{\partial\chi_1}{\partial y_3}\frac{\partial\chi_2}{\partial y_4}}{\frac{\partial\chi_1}{\partial y_3}\frac{\partial\chi_2}{\partial y_1} - \frac{\partial\chi_1}{\partial y_1}\frac{\partial\chi_2}{\partial y_3}},$$

$$\beta_1 = \frac{-\frac{\partial\chi_2}{\partial y_1}}{\frac{\partial\chi_1}{\partial y_3}\frac{\partial\chi_2}{\partial y_1} - \frac{\partial\chi_1}{\partial y_1}\frac{\partial\chi_2}{\partial y_3}}, \qquad \beta_2 = \frac{\frac{\partial\chi_1}{\partial y_1}\frac{\partial\chi_2}{\partial y_2} - \frac{\partial\chi_1}{\partial y_2}\frac{\partial\chi_2}{\partial y_1}}{\frac{\partial\chi_1}{\partial y_3}\frac{\partial\chi_2}{\partial y_1} - \frac{\partial\chi_1}{\partial y_1}\frac{\partial\chi_2}{\partial y_3}},$$

$$\beta_3 = \frac{\frac{\partial\chi_1}{\partial y_1}}{\frac{\partial\chi_1}{\partial y_3}\frac{\partial\chi_2}{\partial y_1} - \frac{\partial\chi_1}{\partial y_1}\frac{\partial\chi_2}{\partial y_3}} \qquad \beta_4 = \frac{\frac{\partial\chi_1}{\partial y_1}\frac{\partial\chi_2}{\partial y_4} - \frac{\partial\chi_1}{\partial y_4}\frac{\partial\chi_2}{\partial y_1}}{\frac{\partial\chi_1}{\partial y_3}\frac{\partial\chi_2}{\partial y_1} - \frac{\partial\chi_1}{\partial y_1}\frac{\partial\chi_2}{\partial y_3}}.$$

By using the control law (14), the error system (13) can be described as

$$\dot{e}(t) = \sum_{j=1}^{4}(b_{1j} - l_{1j})e_j(t)$$

or

$$\dot{e}(t) = (B - L)e(t), \tag{15}$$

where $e(t) = (e_i(t))_{1\leqslant i\leqslant 4}$, $B = (b_{ij})_{4\times4}$, $L = (l_{ij})_{4\times4}$ is a control constant matrix. In this case, the control matrix L is selected such that all eigenvalues of $B - L$ have negative real parts. Thus, by asymptotic stability of linear continuous-time systems, all solutions of error system (15) go to zero as $t \to \infty$. Therefore, systems (10) and (11) are globally synchronized.

Theorem 2. *There exists a suitable control matrix $L = (l_{ij})_{4\times4}$ to realize the coexistence of IGS and GS between the master system* (10) *and the slave system* (11) *under the control law* (14).

5 Numerical examples

Example 1. As the master system, we consider a new 3D chaotic system proposed by Pham et al. [30]

$$\begin{aligned}\dot{x}_1 &= -ax_3,\\ \dot{x}_2 &= bx_1x_3 - cx_3^3,\\ \dot{x}_3 &= x_1^4 + x_2^4 - k - dx_1x_3.\end{aligned} \tag{16}$$

When the parameter values are taken as $a = 0.1$, $b = 3$, $c = 2.2$, $d = 0.2$ and $k = 0.81$, system (16) exhibits chaotic behavior with equilibria located on the rounded square loop as shown in Fig. 1.

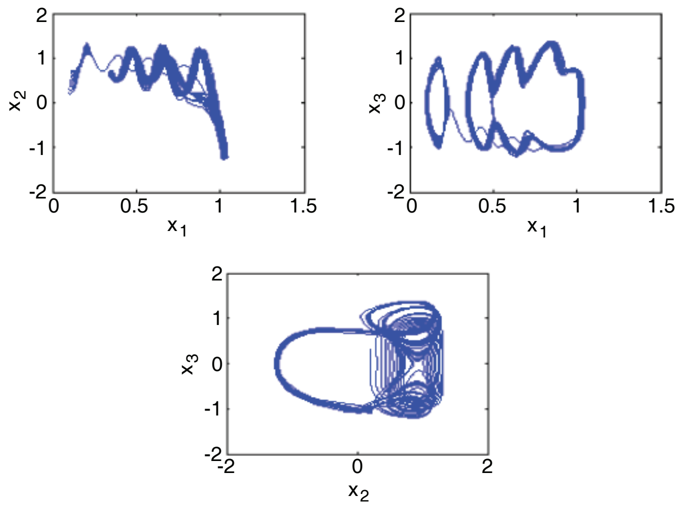

Figure 1. Chaotic attractors of the master system (16) in 2D.

As the slave system, we consider a novel 4D hyperchaotic system introduced by Vaidyanathan [35]

$$\begin{aligned} \dot{y}_1 &= \alpha(y_2 - y_1) + y_2 y_3 + y_4 + u_1, \\ \dot{y}_2 &= -\gamma y_1 y_3 + \delta y_2 + u_2, \\ \dot{y}_3 &= y_1 y_2 - \beta + u_3, \\ \dot{y}_4 &= -\epsilon(y_1 + y_2) + u_4. \end{aligned} \tag{17}$$

System (17) with $u_1 = u_2 = u_3 = u_4 = 0$ exhibits a strange hyperchaotic attractor for the parameter values $\alpha = 60$, $\beta = 27$, $\gamma = 160$, $\delta = 0.3$ and $\epsilon = 2.8$. System (17) does not have any equilibrium points. Hence, system (17) has hidden attractors. When the initial conditions are taken as $(y_1(0), y_2(0), y_3(0), y_4(0)) = (1.2, 0.4, 0.3, 1.4)$, Fig. 2 shows the 3D projection of the novel hyperchaotic system.

According to Definition 3, the error system between the master system (16) and the slave system (17) is defined as

$$e_1 = y_1 - \phi_1(x_1, x_2, x_3), \qquad e_2 = x_2 - \varphi(y_1, y_2, y_3, y_4),$$
$$e_3 = y_3 - \phi_2(x_1, x_2, x_3),$$

where $\phi_1(x_1, x_2, x_3) = x_1 + x_2 + x_3$, $\varphi(y_1, y_2, y_3, y_4) = y_1 + 2y_2 + 3y_3 + y_4^2$ and $\phi_2(x_1, x_2, x_3) = x_1 x_2 + x_3$. So, $\partial\varphi/\partial y_1 = 1$, $\partial\varphi/\partial y_2 = 2$ and $\partial\varphi/\partial y_3 = 3$.

Using the notations presented in Section 3, the linear part A and nonlinear part f of the master system (16) are given as follows:

$$A = \begin{pmatrix} 0 & 0 & -0.1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 \\ b x_1 x_3 - c x_3^3 \\ x_1^4 + x_2^4 - k - d x_1 x_3 \end{pmatrix}.$$

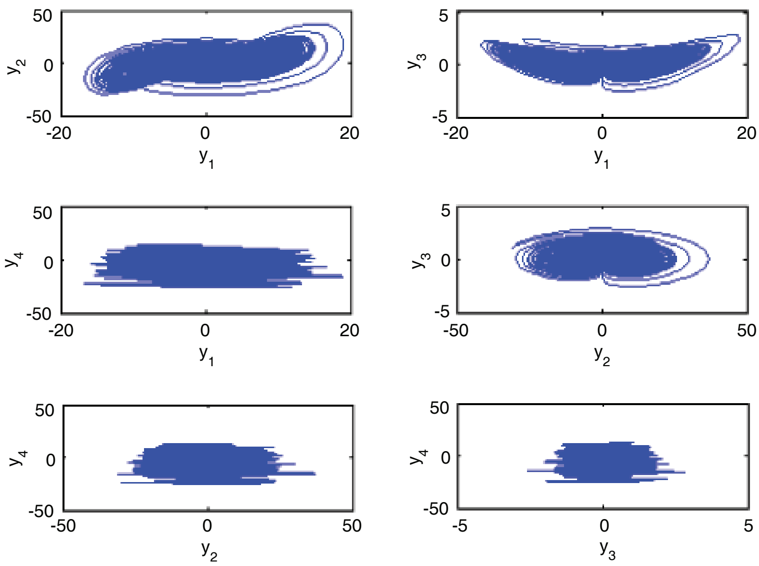

Figure 2. Chaotic attractors of the slave system (17) with $u_1 = u_2 = u_3 = u_4 = 0$ in 2D.

Then the control matrix C is chosen as

$$C = \begin{pmatrix} 1 & 0 & -0.1 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Based on Eq. (8), the controllers u_1, u_2, u_3 and u_4 are designed as follows:

$$\begin{aligned} u_1 &= -e_1 - R_1, \\ u_2 &= \frac{1}{2}e_1 + \frac{1}{2}R_1 + \frac{1}{2}e_2 + \frac{1}{2}R_2 + \frac{9}{2}e_3 + \frac{3}{2}R_3, \\ u_3 &= -3e_3 - R_3, \\ u_4 &= 0, \end{aligned}$$

where

$$\begin{aligned} R_1 &= \alpha(y_2 - y_1) + y_2y_3 + y_4 + ax_3 + (d-b)x_1x_3 + cx_3^3 - x_1^4 - x_2^4 + k, \\ R_2 &= bx_1x_3 - cx_3^3 - (\alpha + 2\delta)y_2 + \alpha y_1 - y_4 + 2\gamma y_1y_3 + 3\beta - y_2(3y_1 + y_3) \\ &\quad + 2y_4\epsilon(y_1 + y_2), \\ R_3 &= y_1y_2 - \beta + ax_2x_3 - bx_1^2x_3 + cx_3^3 - x_1^4 - x_2^4 + k + dx_1x_3. \end{aligned}$$

It is easy to show that $(A - C)^{\mathrm{T}} + (A - C)$ is a negative definite matrix. Therefore, systems (16) and (17) are globally synchronized in 3D. The error system can be described as follows:

$$\dot{e}_1 = -e_1, \qquad \dot{e}_2 = -2e_2, \qquad \dot{e}_3 = -3e_2.$$

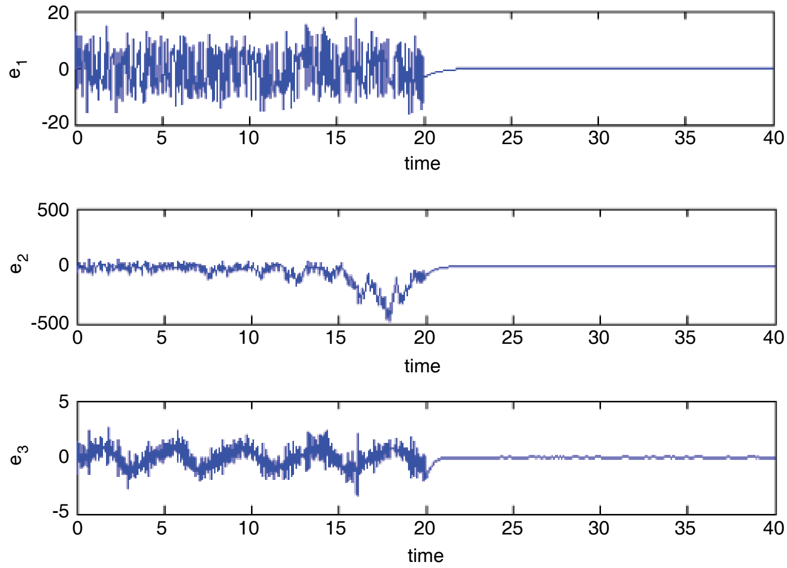

Figure 3. Time evolution of the synchronization errors e_1, e_2 and e_3 between systems (16) and (17).

Figure 3 displays the time evolution of the errors e_1, e_2 and e_3 between the master system (16) and the slave system (17).

Example 2. In this example, the master system is defined by the following new 3D system [31]:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 + x_3x_2,\\ \dot{x}_3 &= -x_1 - ax_1x_2 - bx_1x_3 + c.\end{aligned} \tag{18}$$

When $(a, b, c) = (15, 1, -0.001)$ and the initial conditions $(x_1(0), x_2(0), x_3(0)) = (0, 0.5, 0.5)$, system (18) exhibits a chaotic attractor with no equilibria as shown in Fig. 4.

The salve system is described by

$$\begin{aligned}\dot{y}_1 &= \alpha(y_2 - y_1) + \gamma y_4 + u_1,\\ \dot{y}_2 &= -y_1y_3 - y_2 + \gamma y_4 + u_2,\\ \dot{y}_3 &= y_1y_2 - y_3 - \beta + u_3,\\ \dot{y}_4 &= -\delta(y_1 + y_2) + u_4.\end{aligned} \tag{19}$$

System (19) exhibits a strange hyperchaotic attractor for the parameter values $\alpha = 4$, $\beta = 20$, $\gamma = 0.2$ and $\delta = 0.5$ [36]. System (19) does not have any equilibrium points. Hence, system has hidden attractors. When the initial conditions are taken as $(y_1, y_2, y_3, y_4) = (0.4, 0.4, 0.4, 0.4)$, Fig. 5 shows the 3D projection of the novel hyperchaotic system.

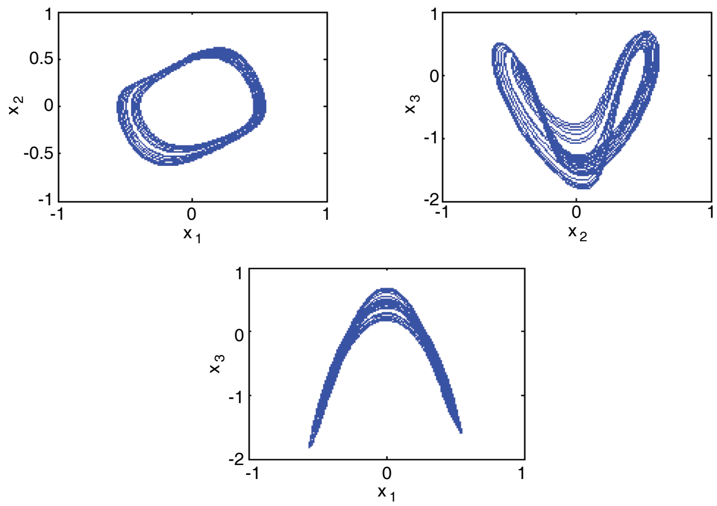

Figure 4. Chaotic attractors of the master system (18) in 2D.

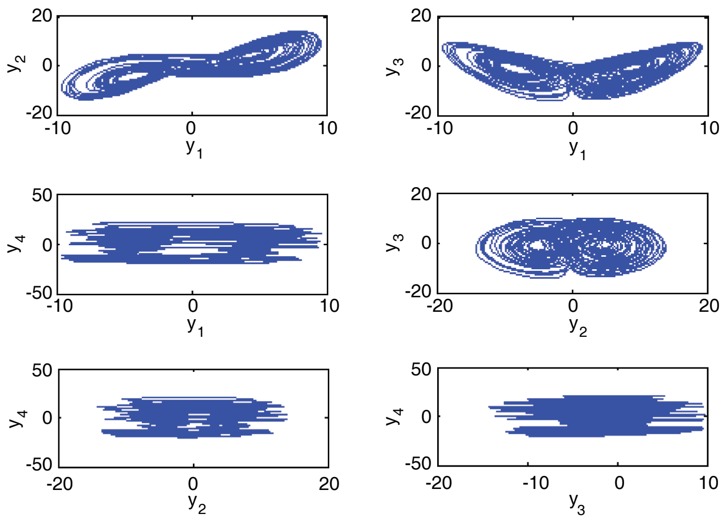

Figure 5. Chaotic attractors of the master system (19) with $u_1 = u_2 = u_3 = u_4 = 0$ in 2D.

According to Definition 4, the error system between the master system (18) and the slave system (19) is described by

$$
\begin{aligned}
e_1 &= x_1 - \chi_1(y_1, y_2, y_3, y_4),\\
e_2 &= y_2 - \psi_1(x_1, x_2, x_3),\\
e_3 &= x_3 - \chi_2(y_1, y_2, y_3, y_4),\\
e_4 &= y_4 - \psi_2(x_1, x_2, x_3),
\end{aligned}
\tag{20}
$$

where $\chi_1(y_1,y_2,y_3,y_4) = y_1 + y_3 + y_2y_4$, $\psi_1(x_1,x_2,x_3) = x_1 + x_2 + x_3$, $\chi_2(y_1,y_2, y_3,y_4) = (y_1^3 - y_3 + y_2y_4)/3$ and $\psi_2(x_1,x_2,x_3) = x_1 + x_2x_3$. So, $(\partial\chi_1/\partial y_3(\partial\chi_2/\partial y_1) - (\partial\chi_1/\partial y_1)(\partial\chi_2/\partial y_3) = y_1^2 + 1$.

Based on the notations used in Section 4, the linear part B and nonlinear part g of the slave system (20) are given as follows:

$$B = \begin{pmatrix} -4 & 4 & 0 & 0.2 \\ 0 & -1 & 0 & 0.2 \\ 0 & 0 & -1 & 0 \\ -0.5 & -0.5 & 0 & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 \\ -y_1y_3 \\ y_1y_2 - \beta \\ 0 \end{pmatrix}.$$

Then the control matrix L is selected as

$$L = \begin{pmatrix} 0 & 4 & 0 & 0.2 \\ 0 & 0 & 0 & 0.2 \\ 0 & 0 & 0 & 0 \\ -0.5 & -0.5 & 0 & 3 \end{pmatrix}.$$

According to Eq. (14), the controllers u_1, u_2, u_3 and u_4 are designed as follows:

$$\begin{aligned} u_1 &= \frac{1}{y_1^2 + 1}\big[(4e_1 + T_1) + 2y_4(e_2 + T_2) + (e_3 + T_3) + 2y_2(3e_4 + T_4)\big], \\ u_2 &= -e_2 - T_2, \\ u_3 &= \frac{4y_1^2}{y_1^2 + 1}(e_1 + T_1) + y_4(e_2 + T_2) - \frac{1}{y_1^2 + 1}(e_3 + T_3) + 3y_2(e_4 + T_4), \\ u_4 &= -3e_4 - T_4, \end{aligned}$$

where

$$\begin{aligned} T_1 &= x_2 - \alpha(y_2 - y_1) + y_4(-\gamma + y_1y_3 + y_2 - \gamma y_4) - y_1y_2 + y_3 + \beta \\ &\quad + \delta y_2(y_1 + y_2), \\ T_2 &= -y_1y_3 - y_2 + \gamma y_4 + x_1(2 + bx_3) - x_2(1 + x_3) + ax_1x_2 - c, \\ T_3 &= -x_1 - ax_1x_2 - bx_1x_3 + c - y_1^2\big(\alpha(y_2 - y_1) + \gamma y_4\big) \\ &\quad - y_4(-y_1y_3 - y_2 + \gamma y_4) + (y_1y_2 - y_3 - \beta) - y_2\big(-\delta(y_1 + y_2)\big), \\ T_4 &= -\delta(y_1 + y_2) - x_2\big(x_3^2 + c\big) + x_2x_1(1 + ax_2 + bx_3) + x_1x_3. \end{aligned}$$

We can show that all eigenvalues of $B - L$ have negative real parts. Therefore, systems (18) and (19) are globally synchronized in 4D. The error system can be written as follows:

$$\dot{e}_1 = -4e_1, \qquad \dot{e}_2 = -e_2, \qquad \dot{e}_3 = -e_3, \qquad \dot{e}_4 = -3e_4,$$

and the numerical results have been plotted in Fig. 6.

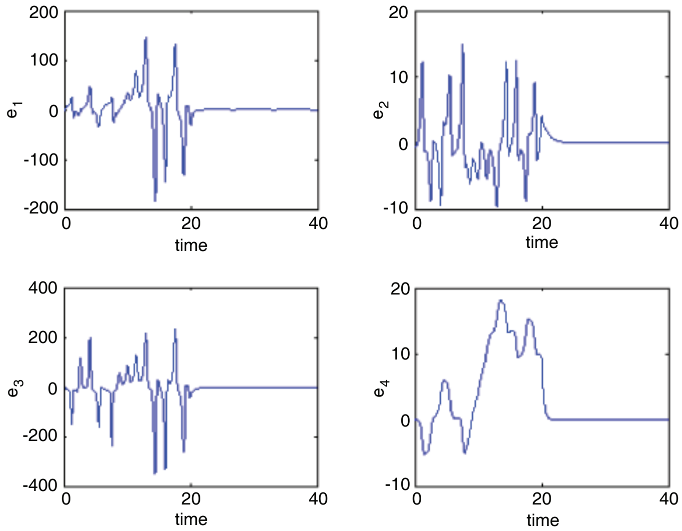

Figure 6. Time evolution of synchronization errors e_1, e_2, e_3 and e_4 between systems (18) and (19).

6 Discussion

The aim of this section is to highlight the novelty introduced by the present approach. First of all, it is worth analyzing the results achieved so far in the literature on the same topic. It is important to note that most relevant studies dedicated to the coexistence of synchronization types between two chaotic systems are related to discrete-time systems [15, 20, 26] and fractional-order systems [16, 22, 23, 29]. We would stress that the topic related to the coexistence of generalized synchronization (GS) and inverse generalized synchronization (IGS) between integer-order systems with different dimensions is almost unexplored. For example, in [23] the authors have presented a robust method, based on stability theory of linear fractional-order systems, to achieve the coexistence of GS and IGS for fractional-order systems only in dimension of the slave system. In our work, the coexistence of GS and IGS has been guaranteed in different dimension by using two approaches: stability theory of linear integer-order systems and Lyapunov stability method. Another schemes of coexistence for integer-order systems have been developed in [21]. However, the method in [21] can only be applied to systems with identical dimensions, and the coexistence of GS and IGS has been not analyzed. Our approach is more general than that in [21] since it can be applied to different systems in different dimensions (see Theorems 1 and 2, respectively), and the scheme of coexistence of GS and IGS can be considered as a generalization of many different schemes.

Based on previous considerations, it should be clear that the methods proposed herein provide a contribution to the topic related to the coexistence of some synchronization types since it guarantees the coexistence of two different generalized type of synchronization for nonidentical systems with different dimensions. This increased complexity related to both the number of synchronization types and the capability to synchronize chaotic dynamics with hyperchaotic ones and provides a deeper insight into the synchronization phenomena between systems described by integer-order differential systems with different dimensions.

7 Conclusions and future works

In this paper, we have presented new approaches to study the coexistence of generalized synchronization (GS) and inverse generalized synchronization (IGS) between chaotic and hyperchaotic systems characterized by different dimensions. The approach, which can be applied to a wide class of chaotic/hyperchaotic systems with different dimensions, is based on two new theorems involving Lyapunov method and stability theory of linear systems and holds for any differentiable scaling function making the tool a reliable approach to achieve general types of chaos synchronization. All the numerical examples reported through the paper have clearly highlighted the capability of the proposed approach in successfully achieving the coexistence of GS and IGS between chaotic and hyperchaotic systems in different dimensions.

As a concluding remark, we would like to highlight that the basic idea of the present paper, the combination of two different synchronization types in order to create a novel synchronization scheme, can be further generalized. This can be achieved by considering two different synchronization types as "building blocks" to obtain several new synchronization schemes using the technique developed in this paper. Consequently, the approach illustrated here can be considered as a "methodology" to create new synchronization schemes starting from two well-established synchronization types. Further developments and extended analysis related to the application of the proposed approach in chaotic communication devices and secure communication systems would be expected to be implemented. Also, new complex schemes of synchronization will be investigated in future works.

Acknowledgment. The authors acknowledge Prof. Guan Rong Chen, Department of Electronic Engineering, City University of Hong Kong for suggesting many helpful references.

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