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Stability analysis for delayed quaternion-valued neural networks via nonlinear measure approach*

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Abstract. In this paper, the existence and stability analysis of the quaternion-valued neural networks (QVNNs) with time delay are considered. Firstly, the QVNNs are equivalently transformed into four real-valued systems. Then, based on the Lyapunov theory, nonlinear measure approach, and inequality technique, some sufficient criteria are derived to ensure the existence and uniqueness of the equilibrium point as well as global stability of delayed QVNNs. In addition, the provided criteria are presented in the form of linear matrix inequality (LMI), which can be easily checked by LMI toolbox in MATLAB. Finally, two simulation examples are demonstrated to verify the effectiveness of obtained results. Moreover, the less conservatism of the obtained results is also showed by two comparison examples.

Keywords: quaternion-valued neural networks (QVNNs), stability, nonlinear measure approach, linear matrix inequality.

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1 Introduction

Quaternion, as a supercomplex number, was discovered by W.R. Hamilton in 1943, and it has been shown that the quaternion is with expansive potential for future development in three-dimensional and four-dimensional data processing. In three-dimensional space, for example, the spatial rotation could be described tersely and efficiently with the quaternion [22]. As its extensively application prospects are revealed, plenty of scholars from different areas, such as quantum mechanics, attitude control, computer, and so forth, show extreme enthusiasm to the quaternion and its applications [2, 14, 15, 35].

Real-valued neural networks (RVNNs) have been successfully applied in secure communication, information processing, engineer optimization, automatic control engineering, and other areas. Correspondingly, numerous meaningful results have been reported [1, 4, 5, 9, 10, 29, 31, 36, 45–47]. In order to ensure the fixed-time synchronization for memristive neural networks, Cao and Li proposed some control strategies to achieve desired performance in [9]. The multistability of delayed competitive neural networks was studied, and some delay-independent criteria were established to ascertain the existence and stability of multi-equilibria [36]. Based on Halanay inequality, Yang et al. provided some sufficient criteria for the stability of discrete neural networks in [46]. However, RVNNs have its own limitations, such as the detection of symmetry problem cannot be resolved by a real-valued neuron, whereas it can be well solved by a complex-valued neuron [23]. In addition, the problem involving with ultrasonic wave, electromagnetic processing, quantum wave can be also well resolved by the complex number. Therefore, the performance of complex-valued neural networks (CVNNs) is more preferable than that of RVNNs in practical application with complex signals, and CVNNs have captured plenty of attentions from different areas [6,16,18,20,23,25,38,41,43,50]. By virtue of Halanay inequality and matrix measure approach, some sufficient criteria were presented to guarantee CVNNs be exponentially stable with different activation functions [16]. Based on the energy minimization method and local inhibition, Zhou and Song have investigated the complete stability of delayed CVNNs [50], and sufficient conditions were given out in the form of LMIs. Bao et al. have discussed the drive-response synchronization of fractional-order CVNNs, criteria were established by designing a delay feedback controller [6]. The Lagrange stability of CVNNs was considered [43], some sufficient conditions were obtained by using Lyapunov theory. Analogously, neural networks along with quaternion should be with better performs and wider applications than both CVNNs and RVNNs due to the quaternion as an extension of plurality. In fact, the three-dimensional and four-dimensional data can be expressed as an entirety, and this is more truly in modeling of practical application, and quaternion-valued neural networks (QVNNs) bring forth at the right time. The states, activation functions, connection weights, and input of QVNNs take values as quaternions, quaternion vectors, and quaternion matrices. Increasing scholars are devoted to investigating dynamical behavior of QVNNs for its extensive application prospects [3, 8, 21, 32, 33, 39, 40, 44]. The quaternionic multilayer perceptrons were employed to predict the chaotic time series, and the better performances along with smaller complexity of quaternionic multilayer perceptrons than those of CVNNs were also verified in [3]. As one of applications of QVNNs, the optimum separation of polarized signals was achieved, and better performance of separation based on QVNNs was shown by simulation results [8]. The problem of color image compression was well resolved by QVNNs along with BP algorithm, while it cannot be done by RVNNs with BP algorithm [21]. The instantaneously trained neural networks were discussed in [39], and obtained results showed that quaternion encoding can greatly reduce the sizes of networks. Some sufficient criteria were given out to ensure the μ-stability of QVNNs [33, 40]. Song et al. have investigated the stability and robust stability for delayed QVNNs by homeomorphic theorem and inequality technique [12, 13]. Based on matrix measure and Halanay inequality, Liu et al. considered the exponential stability of delayed QVNNs, and several criteria were presented in [32, 48]. Several criteria have been provided to ensure the boundedness and periodicity of discrete-time QVNNs [19]. The dissipativity analysis for delayed QVNNs was conducted in [44], and some algebraic criteria ascertaining the global dissipativity were proposed by some analytic techniques.

As is known to us all, the stability is pivotal to various applications of neural networks, and it is the precondition of applications in optimization problems, associative memory, and so on. Therefore, it is not only important but also necessary to study stability of neural networks, and many excellent results have been published [11, 17, 24, 26–28, 30, 34, 37, 42, 49]. The existence and uniqueness of the equilibrium of fractional-order CVNNs was discussed via contracting mapping principle [49], and several delay-dependent conditions were derived. Based on homeomorphism theory and Lyapunov function, the existence and stability of equilibrium for several neural networks were considered in [11, 40, 42]. A novelty approach called as the nonlinear measure approach was employed to discuss the stability of Hopfield neural networks [37]. Since then, the nonlinear measure method was employed to discuss the stability problem of neural networks [17, 26, 28]. Gong et al. have discussed the asymptotic stability of CVNNs, and sufficient criteria were obtained in the form of LMIs by using the nonlinear measure approach [17]. Based on the nonlinear measure approach, some delay-independent conditions were established to ascertain the stability of delayed neural networks [26]. The robust stability of inertial Cohen–Grossberg neural networks was discussed by using the nonlinear measure approach and Halanay inequality [28]. To the best of our knowledge, however, only few results if not none have discussed the stability of delayed QVNNs with nonlinear measure approach, which is one of our motivations to carry out this research.

Motivated by the aforementioned analysis, this paper aims to discuss the stability of delayed QVNNs. Compared to reported results, the main contributions of this manuscript can be summarized as follow:

(i) The existence and uniqueness of the equilibrium point of QVNN are discussed with nonlinear measure approach. Compared to some existing results, our results are with less conservatism.
(ii) The globally exponential stability of QVNNs is discussed by disassembling the QVNNs into four equivalently real-valued systems, and the estimation of exponential convergence can be established with our results.
(iii) The obtained criteria are given out in the shape of LMIs, which can be easily checked by the LMI toolbox in MATLAB.

The rest part is arrayed as follows. Model descriptions and preliminaries are presented in Section 2. In Section 3, the stability analysis for QVNNs is demonstrated via nonlinear measure approach. Two comparison examples are demonstrated to show the validity and less conservatism of our results in Section 4. Section 5 gives out conclusions as well as some future works.

2 Preliminaries

Firstly, some definitions and notations are recapitulated. The quaternion is a kind of super-complex number involving a real part and three imaginary parts i, j, k, and a quaternion x can be denoted as

$$x = x^{(r)} + x^{(i)}i + x^{(j)}j + x^{(k)}k,$$

where $x^{(r)}, x^{(i)}, x^{(j)}, x^{(k)} \in \mathbb{R}$, $\mathbb{R}$ represents the set of real number, and i, j, k satisfy the Hamilton rule, i.e.,

$$i^2 = j^2 = k^2 = ijk = -1,$$
$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j.$$

Obviously, the quaternion is a noncommutative division algebra. The set of quaternion is denoted by $\mathbb{Q}$, i.e., $\mathbb{Q} = \{x^{(r)} + x^{(i)}i + x^{(j)}j + x^{(k)}k \mid x^{(r)}, x^{(i)}, x^{(j)}, x^{(k)} \in \mathbb{R}\}$. $\mathbb{Q}^n$ denotes the n-dimensional quaternion space, and the conjugate of quaternion x is denoted as $\bar{x} = x^{(r)} - x^{(i)}i - x^{(j)}j - x^{(k)}k$. The modulus of $x \in \mathbb{Q}$ is defined as

$$|x| = \sqrt{x\bar{x}} = \sqrt{(x^{(r)})^2 + (x^{(i)})^2 + (x^{(j)})^2 + (x^{(k)})^2},$$

and the norm of quaternion vector $x = (x_1, x_2, \dots, x_n)^{\mathrm{T}}$ is given as

$$\|x\| = \left(\sum_{p=1}^{n} |x_p|^2\right)^{1/2}.$$

$\varphi \in C([-\tau, 0]; \mathbb{Q}^n)$ represents a class of continuous mapping set from $[t_0 - \tau, t_0]$ to $\mathbb{Q}^n$. For $\varphi \in C([t_0 - \tau, t_0]; \mathbb{Q}^n)$, $\|\varphi\| \doteq \sup_{t_0 - \tau \leqslant s \leqslant t_0} |\varphi(s)|$.

In this article, considering the delayed QVNN as follow:

$$\dot{q}(t) = -Cq(t) + Af\big(q(t)\big) + Bf\big(q(t - \tau)\big) + U, \tag{1}$$

where $q(t) \in \mathbb{Q}^n$ is the state vector; $C = \operatorname{diag}\{c_1, c_2, \dots, c_n\} > 0$ denotes the self-feedback matrix; $A, B \in \mathbb{Q}^{n \times n}$ are the link weights matrices; $f(\cdot)$ is the vector activation function; τ is the time delay; $U \in \mathbb{Q}^n$ is an external input vector. The initial condition is given as $q(s) = \varphi(s)$, $s \in [t_0 - \tau, t_0]$.

Since $q(t) = q^{(r)}(t) + iq^{(i)}(t) + jq^{(j)}(t) + kq^{(k)}(t)$, $q^{(r)}(t), q^{(i)}(t), q^{(j)}(t), q^{(k)}(t) \in \mathbb{R}^n$, and denoting $f(t_\tau)) = f(t - \tau)$, then the QVNN (1) can be separated into four

RVNNs as

$$\begin{aligned}
\dot{q}^{(r)}(t) &= -Cq^{(r)}(t) + A^{(r)}f^{(r)}\big(q^{(r)}(t)\big) - A^{(i)}f^{(i)}\big(q^{(i)}(t)\big) - A^{(j)}f^{(j)}\big(q^{(j)}(t)\big)\\
&\quad - A^{(k)}f^{(k)}\big(q^{(k)}(t)\big) + B^{(r)}f^{(r)}\big(q^{(r)}(t_\tau)\big) - B^{(i)}f^{(i)}\big(q^{(i)}(t_\tau)\big)\\
&\quad - B^{(j)}f^{(j)}\big(q^{(j)}(t_\tau)\big) - B^{(k)}f^{(k)}\big(q^{(k)}(t_\tau)\big) + U^{(r)},\\
\dot{q}^{(i)}(t) &= -Cq^{(i)}(t) + A^{(r)}f^{(i)}\big(q^{(i)}(t)\big) + A^{(i)}f^{(r)}\big(q^{(r)}(t)\big) + A^{(j)}f^{(k)}\big(q^{(k)}(t)\big)\\
&\quad - A^{(k)}f^{(j)}\big(q^{(j)}(t)\big) + B^{(r)}f^{(i)}\big(q^{(i)}(t_\tau)\big) + B^{(i)}f^{(r)}\big(q^{(r)}(t_\tau)\big)\\
&\quad + B^{(j)}f^{(k)}\big(q^{(k)}(t_\tau)\big) - B^{(k)}f^{(j)}\big(q^{(j)}(t_\tau)\big) + U^{(i)},\\
\dot{q}^{(j)}(t) &= -Cq^{(j)}(t) + A^{(r)}f^{(j)}\big(q^{(j)}(t)\big) + A^{(j)}f^{(r)}\big(q^{(r)}(t)\big) - A^{(i)}f^{(k)}\big(q^{(k)}(t)\big)\\
&\quad + A^{(k)}f^{(i)}\big(q^{(i)}(t)\big) + B^{(r)}f^{(j)}\big(q^{(j)}(t_\tau)\big) + B^{(j)}f^{(r)}\big(q^{(r)}(t_\tau)\big)\\
&\quad - B^{(i)}f^{(k)}\big(q^{(k)}(t_\tau)\big) + B^{(k)}f^{(i)}\big(q^{(i)}(t_\tau)\big) + U^{(j)},\\
\dot{q}^{(k)}(t) &= -Cq^{(k)}(t) + A^{(r)}f^{(k)}\big(q^{(k)}(t)\big) + A^{(k)}f^{(r)}\big(q^{(r)}(t)\big) + A^{(i)}f^{(j)}\big(q^{(j)}(t)\big)\\
&\quad - A^{(j)}f^{(i)}\big(q^{(i)}(t)\big) + B^{(r)}f^{(k)}\big(q^{(k)}(t_\tau)\big) + B^{(k)}f^{(r)}\big(q^{(r)}(t_\tau)\big)\\
&\quad + B^{(i)}f^{(j)}\big(q^{(j)}(t_\tau)\big) - B^{(j)}f^{(i)}\big(q^{(i)}(t_\tau)\big) + U^{(k)},
\end{aligned}$$

which can be written as

$$\begin{aligned}
\dot{Q}(t) &= -C_1Q(t) + A_1f_1\big(Q(t)\big) + A_2f_2\big(Q(t)\big) + A_3f_3\big(Q(t)\big) + A_4f_4\big(Q(t)\big)\\
&\quad + B_1f_1\big(Q(t-\tau)\big) + B_2f_2\big(Q(t-\tau)\big) + B_3f_3\big(Q(t-\tau)\big)\\
&\quad + B_4f_4\big(Q(t-\tau)\big) + U_1,
\end{aligned}\tag{2}$$

where

$$Q(t) = \big(\big(q^{(r)}(t)\big)^{\mathrm{T}}, \big(q^{(i)}(t)\big)^{\mathrm{T}}, \big(q^{(j)}(t)\big)^{\mathrm{T}}, \big(q^{(k)}(t)\big)^{\mathrm{T}}\big)^{\mathrm{T}}, \qquad C_1 = \operatorname{diag}\{C, C, C, C\},$$

$$\begin{aligned}
A_1 &= \operatorname{diag}\big\{A^{(r)}, A^{(i)}, A^{(j)}, A^{(k)}\big\}, & A_2 &= \operatorname{diag}\big\{-A^{(i)}, A^{(r)}, A^{(k)}, -A^{(j)}\big\},\\
A_3 &= \operatorname{diag}\big\{-A^{(j)}, -A^{(k)}, A^{(r)}, A^{(i)}\big\}, & A_4 &= \operatorname{diag}\big\{-A^{(k)}, A^{(j)}, -A^{(i)}, A^{(r)}\big\},\\
B_1 &= \operatorname{diag}\big\{B^{(r)}, B^{(i)}, B^{(j)}, B^{(k)}\big\}, & B_2 &= \operatorname{diag}\big\{-B^{(i)}, B^{(r)}, B^{(k)}, -B^{(j)}\big\},\\
B_3 &= \operatorname{diag}\big\{-B^{(j)}, -B^{(k)}, B^{(r)}, B^{(i)}\big\}, & B_4 &= \operatorname{diag}\big\{-B^{(k)}, B^{(j)}, -B^{(i)}, B^{(r)}\big\},
\end{aligned}$$

$$\begin{aligned}
f_1\big(Q(t)\big) &= \big(\big(f^{(r)}\big(q^{(r)}(t)\big)\big)^{\mathrm{T}}, \big(f^{(r)}\big(q^{(r)}(t)\big)\big)^{\mathrm{T}}, \big(f^{(r)}\big(q^{(r)}(t)\big)\big)^{\mathrm{T}}, \big(f^{(r)}\big(q^{(r)}(t)\big)\big)^{\mathrm{T}}\big)^{\mathrm{T}},\\
f_2\big(Q(t)\big) &= \big(\big(f^{(i)}\big(q^{(i)}(t)\big)\big)^{\mathrm{T}}, \big(f^{(i)}\big(q^{(i)}(t)\big)\big)^{\mathrm{T}}, \big(f^{(i)}\big(q^{(i)}(t)\big)\big)\big)^{\mathrm{T}}, \big(f^{(i)}\big(q^{(i)}(t)\big)\big)^{\mathrm{T}}\big)^{\mathrm{T}},\\
f_3\big(Q(t)\big) &= \big(\big(f^{(j)}\big(q^{(j)}(t)\big)\big)^{\mathrm{T}}, \big(f^{(j)}\big(q^{(j)}(t)\big)\big)^{\mathrm{T}}, \big(f^{(j)}\big(q^{(j)}(t)\big)\big)\big)^{\mathrm{T}}, \big(f^{(j)}\big(q^{(j)}(t)\big)\big)^{\mathrm{T}}\big)^{\mathrm{T}},\\
f_4\big(Q(t)\big) &= \big(\big(f^{(k)}\big(q^{(k)}(t)\big)\big)^{\mathrm{T}}, \big(f^{(k)}\big(q^{(k)}(t)\big)\big)^{\mathrm{T}}, \big(f^{(k)}\big(q^{(k)}(t)\big)\big)\big)^{\mathrm{T}}, \big(f^{(k)}\big(q^{(k)}(t)\big)\big)^{\mathrm{T}}\big)^{\mathrm{T}},\\
U_1 &= \big(\big(U^{(r)}\big)^{\mathrm{T}}, \big(U^{(i)}\big)^{\mathrm{T}}, \big(U^{(j)}\big)^{\mathrm{T}}, \big(U^{(k)}\big)^{\mathrm{T}}\big)^{\mathrm{T}}.
\end{aligned}$$

An equilibrium point of QVNNs $\dot{q}(t) = -Cq(t) + Af(q(t)) + Bf(q(t-\tau)) + U$ is a constant quaternion vector $\hat{q}$ satisfying $-C\hat{q} + Af(\hat{q}) + Bf(\hat{q}) + U = 0$. Obviously, the QVNN (1) shares the same equilibrium point and identical dynamics characters with system (2) by regarding $q(t) = q^{(r)}(t) + iq^{(i)}(t) + jq^{(j)}(t) + kq^{(k)}(t)$ as an vector $Q(t) = ((q^{(r)}(t))^{\mathrm{T}}, (q^{(i)}(t))^{\mathrm{T}}, (q^{(j)}(t))^{\mathrm{T}}, (q^{(k)}(t))^{\mathrm{T}})^{\mathrm{T}}$. Therefore, one can investigate the existence and uniqueness of the equilibrium of system (2) instead of system (1).

Assumption H. Let $q = q^{(r)} + iq^{(i)} + jq^{(j)} + kq^{(k)},\ q^{(r)}, q^{(i)}, q^{(j)}, q^{(k)} \in \mathbb{R}^n$, the activation function $f_p(q)$ is of the following form:

$$f_p(q) = f_p^{(r)}\big(q^{(r)}\big) + if_p^{(i)}\big(q^{(i)}\big) + jf_p^{(j)}\big(q^{(j)}\big) + kf_p^{(k)}\big(q^{(k)}\big), \quad p = 1, 2, \dots, n,$$

where the continuous function $f_p^{(d)}\colon \mathbb{R} \to \mathbb{R}$. There exist positive constants $l_p^{(d)}$ such that

$$\big|f_p^{(d)}(x) - f_p^{(d)}(y)\big| \leqslant l_p^{(d)}|x-y|, \quad d = r, i, j, k;\ p = 1, 2, \dots, n,$$

and let

$$\begin{aligned}
\Gamma_1 &= \operatorname{diag}\big\{\big(l_1^{(r)}\big)^2, \big(l_2^{(r)}\big)^2, \dots, \big(l_n^{(r)}\big)^2\big\},\\
\Gamma_2 &= \operatorname{diag}\big\{\big(l_1^{(i)}\big)^2, \big(l_2^{(i)}\big)^2, \dots, \big(l_n^{(i)}\big)^2\big\},\\
\Gamma_3 &= \operatorname{diag}\big\{\big(l_1^{(j)}\big)^2, \big(l_2^{(j)}\big)^2, \dots, \big(l_n^{(j)}\big)^2\big\},\\
\Gamma_4 &= \operatorname{diag}\big\{\big(l_1^{(k)}\big)^2, \big(l_2^{(k)}\big)^2, \dots, \big(l_n^{(k)}\big)^2\big\}.
\end{aligned}$$

Definition 1. (See [32].) The unique equilibrium point $\tilde{Q}$ of QVNN (1) is said to be globally exponentially stable if there exist two positive constants M, α such that

$$\big|Q(t) - \tilde{Q}\big| \leqslant M\mathrm{e}^{-\alpha(t-t_0)}, \quad t \geqslant t_0.$$

Definition 2. (See [26].) Let Ω be an open set of $\mathbb{R}^n$, and $G : \Omega \to \mathbb{R}^n$ is an operator. The constant

$$m_\Omega \doteq \sup_{x,y\in\Omega,\, x\neq y} \frac{\langle G(x) - G(y),\, x-y\rangle}{\|x-y\|_2^2}$$

is said to be the nonlinear measure of G on Ω with the Euclidean norm $\|\cdot\|_2$.

Lemma 1. (*See* [26].) *If $m_\Omega(G) \leqslant 0$, then G is an injective mapping on Ω. Moreover, if $\Omega = \mathbb{R}^n$, then G is a homeomorphism of $\mathbb{R}^n$.*

Lemma 2. (*See* [7].) *The LMI $S = \left(\begin{smallmatrix} S_{11} & S_{12}\\ S_{12}^{\mathrm{T}} & S_{22}\end{smallmatrix}\right) < 0$ with $S_{11} = S_{11}^{\mathrm{T}}$, $S_{22} = S_{22}^{\mathrm{T}}$ is equivalent to one of the following conditions*:

(i) $S_{22} < 0$, $S_{11} - S_{12}S_{22}^{-1}S_{12}^{\mathrm{T}} < 0$;
(ii) $S_{11} < 0$, $S_{22} - S_{12}^{\mathrm{T}}S_{11}^{-1}S_{12} < 0$.

Lemma 3. *Let $x, y \in \mathbb{R}^n$, then, for any matrix $S > 0$, $2x^{\mathrm{T}}y \leqslant x^{\mathrm{T}}Sx + y^{\mathrm{T}}S^{-1}y$.*

3 Main results

Theorem 1. *If Assumption* H *holds and there exist appropriate dimension positive defined matrix P and diagonal matrices $S_p > 0$, $p = 1, 2, \dots, 8$, such that $\varXi < 0$, then QVNN* (1) *has a unique equilibrium point, and it is globally asymptotically stable, where*

$$\varXi = \begin{pmatrix} \varPi_{11} & PA_1 & PA_2 & PA_3 & PA_4 & PB_1 & PB_2 & PB_3 & PB_4 \\ A_1^{\mathrm{T}}P & \varPi_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_2^{\mathrm{T}}P & 0 & \varPi_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\ A_3^{\mathrm{T}}P & 0 & 0 & \varPi_{44} & 0 & 0 & 0 & 0 & 0 \\ A_4^{\mathrm{T}}P & 0 & 0 & 0 & \varPi_{55} & 0 & 0 & 0 & 0 \\ B_1^{\mathrm{T}}P & 0 & 0 & 0 & 0 & \varPi_{66} & 0 & 0 & 0 \\ B_2^{\mathrm{T}}P & 0 & 0 & 0 & 0 & 0 & \varPi_{77} & 0 & 0 \\ B_3^{\mathrm{T}}P & 0 & 0 & 0 & 0 & 0 & 0 & \varPi_{88} & 0 \\ B_4^{\mathrm{T}}P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varPi_{99} \end{pmatrix},$$

$$\varPi_{11} = -PC_1 - C_1P + 4\varOmega_1 + 4\varOmega_2 + 4\varOmega_3 + 4\varOmega_4 + 4\varOmega_5 + 4\varOmega_6 + 4\varOmega_7 + 4\varOmega_8,$$

$$\varPi_{p+1,p+1} = -I_4 \otimes S_p, \quad p = 1, 2, \dots, 8;$$

$$\varOmega_p = \begin{pmatrix} S_p\varGamma_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \varOmega_{4+p} = \begin{pmatrix} S_{4+p}\varGamma_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad p = 1, 2, 3, 4.$$

Proof. Consider the operator $G : \mathbb{R}^{4n} \to \mathbb{R}^{4n}$ as follows:

$$\begin{aligned} G(Q) &= -C_1Q + A_1f_1(Q) + A_2f_2(Q) + A_3f_3(Q) + A_4f_4(Q) \\ &\quad + B_1f_1(Q) + B_2f_2(Q) + B_3f_3(Q) + B_4f_4(Q) + U_1, \quad Q \in \mathbb{R}^{4n}. \end{aligned}$$

Then we introduce the following differential system:

$$\dot{H}(t) = PG\big(H(t)\big). \tag{3}$$

Obviously, system (2) shares the same equilibrium point with system (3). Therefore, one can discuss the existence and uniqueness of the equilibrium point of system (3) instead of (2). According to Lemma 1, the existence and uniqueness can be ensured by proving $m_\varOmega(PG) \leqslant 0$ for two different vectors $Q = ((q^{(r)})^{\mathrm{T}}, (q^{(i)})^{\mathrm{T}}, (q^{(j)})^{\mathrm{T}}, (q^{(k)})^{\mathrm{T}})^{\mathrm{T}}$ and $H = ((h^{(r)})^{\mathrm{T}}, (h^{(i)})^{\mathrm{T}}, (h^{(j)})^{\mathrm{T}}, (h^{(k)})^{\mathrm{T}})^{\mathrm{T}}$. Considering the inner product $\langle PG(Q) - PG(H), Q - H\rangle$, one can get

$$\begin{aligned} &\big\langle PG(Q) - PG(H), Q - H\big\rangle \\ &\quad = (Q - H)^{\mathrm{T}}P\big(G(Q) - G(H)\big) \\ &\quad = (Q - H)^{\mathrm{T}}P\big(-C_1(Q - H) + A_1\big(f_1(Q) - f_1(H)\big) + A_2\big(f_2(Q) - f_2(H)\big) \\ &\qquad + A_3\big(f_3(Q) - f_3(H)\big) + A_4\big(f_4(Q) - f_4(H)\big) + B_1\big(f_1(Q) - f_1(H)\big) \\ &\qquad + B_2\big(f_2(Q) - f_2(H)\big) + B_3\big(f_3(Q) - f_3(H)\big) + B_4\big(f_4(Q) - f_4(H)\big)\big) \end{aligned}$$

$$
\begin{aligned}
&= -(Q-H)^{\mathrm T} PC_1(Q-H) + (Q-H)^{\mathrm T} PA_1\big(f_1(Q)-f_1(H)\big) \\
&\quad + (Q-H)^{\mathrm T} PA_2\big(f_2(Q)-f_2(H)\big) + (Q-H)^{\mathrm T} PA_3\big(f_3(Q)-f_3(H)\big) \\
&\quad + (Q-H)^{\mathrm T} PA_4\big(f_4(Q)-f_4(H)\big) + (Q-H)^{\mathrm T} PB_1\big(f_1(Q)-f_1(H)\big) \\
&\quad + (Q-H)^{\mathrm T} PB_2\big(f_2(Q)-f_2(H)\big) + (Q-H)^{\mathrm T} PB_3\big(f_3(Q)-f_3(H)\big) \\
&\quad + (Q-H)^{\mathrm T} PB_4\big(f_4(Q)-f_4(H)\big). \qquad (4)
\end{aligned}
$$

According to Lemma 3 and Assumption H, the following inequality holds with a positive diagonal matrix $S_1 \in \mathbb{R}^{n\times n}$:

$$
\begin{aligned}
&2(Q-H)^{\mathrm T} PA_1\big(f_1(Q)-f_1(H)\big) \\
&\quad \leqslant (Q-H)^{\mathrm T} PA_1\big(I_4 \otimes S_1^{-1}\big)A_1^{\mathrm T} P(Q-H) + 4\sum_{p=1}^{n} s_{1p}\big(f_p^{(r)}\big(Q_p^{(r)}\big) - f_p^{(r)}\big(H_p^{(r)}\big)\big)^2 \\
&\quad \leqslant (Q-H)^{\mathrm T} PA_1\big(I_4 \otimes S_1^{-1}\big)A_1^{\mathrm T} P(Q-H) + 4\sum_{p=1}^{n} s_{1p}\big(l_p^{(r)}\big)^2\big|Q_p^{(r)} - H_p^{(r)}\big|^2 \\
&\quad = (Q-H)^{\mathrm T} PA_1\big(I_4 \otimes S_1^{-1}\big)A_1^{\mathrm T} P(Q-H) + 4(Q-H)^{\mathrm T}\varOmega_1(Q-H). \qquad (5)
\end{aligned}
$$

Similarly, the following inequalities are also true with positive diagonal matrices $S_2, S_3, S_4 \in \mathbb{R}^{n\times n}$:

$$
\begin{aligned}
&2(Q-H)^{\mathrm T} PA_2\big(f_2(Q)-f_2(H)\big) \\
&\quad \leqslant (Q-H)^{\mathrm T} PA_2\big(I_4 \otimes S_2^{-1}\big)A_2^{\mathrm T} P(Q-H) + 4(Q-H)^{\mathrm T}\varOmega_2(Q-H), \qquad (6) \\
&2(Q-H)^{\mathrm T} PA_3\big(f_3(Q)-f_3(H)\big) \\
&\quad \leqslant (Q-H)^{\mathrm T} PA_3\big(I_4 \otimes S_3^{-1}\big)A_3^{\mathrm T} P(Q-H) + 4(Q-H)^{\mathrm T}\varOmega_3(Q-H), \qquad (7) \\
&2(Q-H)^{\mathrm T} PA_4\big(f_4(Q)-f_4(H)\big) \\
&\quad \leqslant (Q-H)^{\mathrm T} PA_4\big(I_4 \otimes S_4^{-1}\big)A_4^{\mathrm T} P(Q-H) + 4(Q-H)^{\mathrm T}\varOmega_4(Q-H), \qquad (8) \\
&2(Q-H)^{\mathrm T} PB_1\big(f_1(Q)-f_1(H)\big) \\
&\quad \leqslant (Q-H)^{\mathrm T} PB_1\big(I_4 \otimes S_2^{-1}\big)B_1^{\mathrm T} P(Q-H) + 4(Q-H)^{\mathrm T}\varOmega_5(Q-H), \qquad (9) \\
&2(Q-H)^{\mathrm T} PB_2\big(f_2(Q)-f_2(H)\big) \\
&\quad \leqslant (Q-H)^{\mathrm T} PB_2\big(I_4 \otimes S_2^{-1}\big)B_2^{\mathrm T} P(Q-H) + 4(Q-H)^{\mathrm T}\varOmega_6(Q-H), \qquad (10) \\
&2(Q-H)^{\mathrm T} PB_3\big(f_3(Q)-f_3(H)\big) \\
&\quad \leqslant (Q-H)^{\mathrm T} PB_3\big(I_4 \otimes S_3^{-1}\big)B_3^{\mathrm T} P(Q-H) + 4(Q-H)^{\mathrm T}\varOmega_7(Q-H), \qquad (11) \\
&2(Q-H)^{\mathrm T} PB_4\big(f_4(Q)-f_4(H)\big) \\
&\quad \leqslant (Q-H)^{\mathrm T} PB_4(I_4 \otimes S_4^{-1})B_4^{\mathrm T} P(Q-H) + 4(Q-H)^{\mathrm T}\varOmega_8(Q-H). \qquad (12)
\end{aligned}
$$

Combining (4)–(12), one can easily get

$$\begin{aligned}&\big\langle PG(Q) - PG(H),\, Q - H\big\rangle\\&\quad\leqslant \frac{1}{2}(Q-H)^{\mathrm{T}}\big(-PC_1 - C_1P + 4\Omega_1 + 4\Omega_2 + 4\Omega_3 + 4\Omega_4 + 4\Omega_5 + 4\Omega_6 + 4\Omega_7\\&\qquad + 4\Omega_8 + PA_1\big(I_4\otimes S_1^{-1}\big)A_1^{\mathrm{T}}P + PA_2\big(I_4\otimes S_2^{-1}\big)A_2^{\mathrm{T}}P + PA_3\big(I_4\otimes S_3^{-1}\big)A_3^{\mathrm{T}}P\\&\qquad + PA_4\big(I_4\otimes S_4^{-1}\big)A_4^{\mathrm{T}}P + PB_1\big(I_4\otimes S_5^{-1}\big)B_1^{\mathrm{T}}P + PB_2\big(I_4\otimes S_6^{-1}\big)B_2^{\mathrm{T}}P\\&\qquad + PB_3\big(I_4\otimes S_7^{-1}\big)B_3^{\mathrm{T}}P + PB_4\big(I_4\otimes S_8^{-1}\big)B_4^{\mathrm{T}}P\big)(Q-H).\end{aligned}$$

In view of $\Xi < 0$, the following inequality can be derived by employing Lemma 2 continuously:

$$\big\langle PG(Q) - PG(H),\, Q - H\big\rangle < 0 \quad \text{for } Q \neq H.$$

According to the definition of nonlinear measure, one gets $m_{\mathbb{R}^{4n}}(PG) < 0$. Correspondingly, the differential system (3) has a unique equilibrium point, which is ascertained by the Lemma 1. Therefore, the existence and uniqueness of the equilibrium point of QVNN (1) can also be guaranteed.

Without loss of generality, the unique equilibrium point can be denoted as $\tilde{Q} = (\tilde{q}^{(r)}, \tilde{q}^{(i)}, \tilde{q}^{(j)}, \tilde{q}^{(k)})$. Resorting to the substitution $x(t) = Q(t) - \tilde{Q}$, the following system can be obtained from (2):

$$\begin{aligned}\dot{x}(t) &= -C_1x(t) + A_1\tilde{f}_1\big(x(t)\big) + A_2\tilde{f}_2\big(x(t)\big) + A_3\tilde{f}_3\big(x(t)\big) + A_4\tilde{f}_4\big(x(t)\big)\\&\quad + B_1\tilde{f}_1\big(x(t-\tau)\big) + B_2\tilde{f}_2\big(x(t-\tau)\big) + B_3\tilde{f}_3\big(x(t-\tau)\big) + B_4\tilde{f}_4\big(x(t-\tau)\big), \end{aligned}\tag{13}$$

where

$$\begin{aligned}&\tilde{f}_1\big(x(t)\big) = f_1\big(x(t)+\tilde{Q}\big) - f_1(\tilde{Q}), \qquad \tilde{f}_2\big(x(t)\big) = f_2\big(x(t)+\tilde{Q}\big) - f_2(\tilde{Q}),\\&\tilde{f}_3\big(x(t)\big) = f_3\big(x(t)+\tilde{Q}\big) - f_3(\tilde{Q}), \qquad \tilde{f}_4\big(x(t)\big) = f_4\big(x(t)+\tilde{Q}\big) - f_4(\tilde{Q}),\\&\tilde{f}_1\big(x(t-\tau)\big) = f_1\big(x(t-\tau)+\tilde{Q}\big) - f_1(\tilde{Q}),\\&\tilde{f}_2\big(x(t-\tau)\big) = f_2\big(x(t-\tau)+\tilde{Q}\big) - f_2(\tilde{Q}),\\&\tilde{f}_3\big(x(t-\tau)\big) = f_3\big(x(t-\tau)+\tilde{Q}\big) - f_3(\tilde{Q}),\\&\tilde{f}_4\big(x(t-\tau)\big) = f_4\big(x(t-\tau)+\tilde{Q}\big) - f_4(\tilde{Q}).\end{aligned}$$

Considering the Lyapunov functional as

$$\begin{aligned}V\big(x(t)\big) &= x^{\mathrm{T}}(t)Px(t) + \int_{t-\tau}^{t} \tilde{f}_1^{\mathrm{T}}\big(x(s)\big)(I_4\otimes S_5)\tilde{f}_1\big(x(s)\big)\,\mathrm{d}s\\&\quad + \int_{t-\tau}^{t} \tilde{f}_2^{\mathrm{T}}\big(x(s)\big)(I_4\otimes S_6)\tilde{f}_2\big(x(s)\big)\,\mathrm{d}s + \int_{t-\tau}^{t} \tilde{f}_3^{\mathrm{T}}\big(x(s)\big)(I_4\otimes S_7)\tilde{f}_3\big(x(s)\big)\,\mathrm{d}s\\&\quad + \int_{t-\tau}^{t} \tilde{f}_4^{\mathrm{T}}\big(x(s)\big)(I_4\otimes S_8)\tilde{f}_4\big(x(s)\big)\,\mathrm{d}s.\end{aligned}$$

Calculating the derivative of $V(x(t))$, one can derive

$$\begin{aligned}
&\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\bigg|_{(13)} \\
&= 2x^{\mathrm{T}}(t)P\big(-C_1x(t) + A_1\tilde{f}_1\big(x(t)\big) + A_2\tilde{f}_2\big(x(t)\big) + A_3\tilde{f}_3\big(x(t)\big) \\
&\quad + A_4\tilde{f}_4\big(x(t)\big) + B_1\tilde{f}_1\big(x(t-\tau)\big) + B_2\tilde{f}_2\big(x(t-\tau)\big) + B_3\tilde{f}_3\big(x(t-\tau)\big) \\
&\quad + B_4\tilde{f}_4\big(x(t-\tau)\big)\big) + \tilde{f}_1^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_5)\tilde{f}_1\big(x(t)\big) \\
&\quad - \tilde{f}_1^{\mathrm{T}}\big(x(t-\tau)\big)(I_4\otimes S_5)\tilde{f}_1\big(x(t-\tau)\big) + \tilde{f}_2^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_6)\tilde{f}_2\big(x(t)\big) \\
&\quad - \tilde{f}_2^{\mathrm{T}}\big(x(t-\tau)\big)(I_4\otimes S_6)\tilde{f}_2\big(x(t-\tau)\big) + \tilde{f}_3^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_7)\tilde{f}_3\big(x(t)\big) \\
&\quad - \tilde{f}_3^{\mathrm{T}}\big(x(t-\tau)\big)(I_4\otimes S_7)\tilde{f}_3\big(x(t-\tau)\big) + \tilde{f}_4^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_8)\tilde{f}_4\big(x(t)\big) \\
&\quad - \tilde{f}_4^{\mathrm{T}}\big(x(t-\tau)\big)(I_4\otimes S_8)\tilde{f}_4\big(x(t-\tau)\big) \\
&\leqslant -2x^{\mathrm{T}}(t)PC_1x(t) + x^{\mathrm{T}}(t)PA_1\big(I_4\otimes S_1^{-1}\big)A_1^{\mathrm{T}}Px(t) \\
&\quad + \tilde{f}_1^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_1)\tilde{f}_1\big(x(t)\big) + x^{\mathrm{T}}(t)PA_2\big(I_4\otimes S_2^{-1}\big)A_2^{\mathrm{T}}Px(t) \\
&\quad + \tilde{f}_2^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_2)\tilde{f}_2\big(x(t)\big) + x^{\mathrm{T}}(t)PA_3\big(I_4\otimes S_3^{-1}\big)A_3^{\mathrm{T}}Px(t) \\
&\quad + \tilde{f}_3^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_3)\tilde{f}_3\big(x(t)\big) + x^{\mathrm{T}}(t)PA_4\big(I_4\otimes S_4^{-1}\big)A_4^{\mathrm{T}}Px(t) \\
&\quad + \tilde{f}_4^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_4)\tilde{f}_4\big(x(t)\big) + x^{\mathrm{T}}(t)PB_1\big(I_4\otimes S_5^{-1}\big)B_1^{\mathrm{T}}Px(t) \\
&\quad + x^{\mathrm{T}}(t)PB_2\big(I_4\otimes S_6^{-1}\big)B_2^{\mathrm{T}}Px(t) + x^{\mathrm{T}}(t)PB_3\big(I_4\otimes S_7^{-1}\big)B_3^{\mathrm{T}}Px(t) \\
&\quad + x^{\mathrm{T}}(t)PB_4\big(I_4\otimes S_8^{-1}\big)B_4^{\mathrm{T}}Px(t) + \tilde{f}_1^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_5)\tilde{f}_1\big(x(t)\big) \\
&\quad + \tilde{f}_2^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_6)\tilde{f}_2\big(x(t)\big) + \tilde{f}_3^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_7)\tilde{f}_3\big(x(t)\big) \\
&\quad + \tilde{f}_4^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_8)\tilde{f}_4\big(x(t)\big). \qquad (14)
\end{aligned}$$

Based on Assumption H, the following inequalities can be obtained:

$$\begin{aligned}
\tilde{f}_1^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_1)\tilde{f}_1\big(x(t)\big) &\leqslant x^{\mathrm{T}}(t)\Omega_1x(t), \\
\tilde{f}_2^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_2)\tilde{f}_2\big(x(t)\big) &\leqslant x^{\mathrm{T}}(t)\Omega_2x(t), \\
\tilde{f}_3^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_3)\tilde{f}_3\big(x(t)\big) &\leqslant x^{\mathrm{T}}(t)\Omega_3x(t), \\
\tilde{f}_4^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_4)\tilde{f}_4\big(x(t)\big) &\leqslant x^{\mathrm{T}}(t)\Omega_4x(t), \\
\tilde{f}_1^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_5)\tilde{f}_1\big(x(t)\big) &\leqslant x^{\mathrm{T}}(t)\Omega_5x(t), \\
\tilde{f}_2^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_6)\tilde{f}_2\big(x(t)\big) &\leqslant x^{\mathrm{T}}(t)\Omega_6x(t), \\
\tilde{f}_3^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_7)\tilde{f}_3\big(x(t)\big) &\leqslant x^{\mathrm{T}}(t)\Omega_7x(t), \\
\tilde{f}_4^{\mathrm{T}}\big(x(t)\big)(I_4\otimes S_8)\tilde{f}_4\big(x(t)\big) &\leqslant x^{\mathrm{T}}(t)\Omega_8x(t).
\end{aligned} \qquad (15)$$

Substituting (15) to (14), one gets

$$\begin{aligned}\dot V\big(x(t)\big) \leqslant x^{\mathrm T}(t)\big(&-PC_1 - C_1P + 4\Omega_1 + 4\Omega_2\\&+ 4\Omega_3 + 4\Omega_4 + 4\Omega_5 + 4\Omega_6 + 4\Omega_7 + 4\Omega_8\\&+ PA_1\big(I_4 \otimes S_1^{-1}\big)A_1^{\mathrm T}P + PA_2\big(I_4 \otimes S_2^{-1}\big)A_2^{\mathrm T}P\\&+ PA_3\big(I_4 \otimes S_3^{-1}\big)A_3^{\mathrm T}P + PA_4\big(I_4 \otimes S_4^{-1}\big)A_4^{\mathrm T}P\\&+ PB_1\big(I_4 \otimes S_5^{-1}\big)B_1^{\mathrm T}P + PB_2\big(I_4 \otimes S_6^{-1}\big)B_2^{\mathrm T}P\\&+ PB_3\big(I_4 \otimes S_7^{-1}\big)B_3^{\mathrm T}P + PB_4\big(I_4 \otimes S_8^{-1}\big)B_4^{\mathrm T}P\big)x(t).\end{aligned}$$

According to $\varXi < 0$ and Lemma 2, one yields $\dot V(x(t)) < 0$ for all $x(t) \neq 0$. $\dot V(x(t)) = 0$ if and only if $x(t) = 0$, and $\|V(x(t))\| \to \infty$ as $\|x(t)\| \to \infty$. Hence, the equilibrium point of system (2) is globally asymptotically stable. Correspondingly, the equilibrium point of QVNN (1) is globally asymptotically stable. □

Theorem 2. *If Assumption* H *holds and there exist appropriate dimension positive matrix P, diagonal matrices $S_p > 0$, $p = 1, 2, \dots, 8$, and positive constant α such that $\widehat{\varXi} < 0$, then QVNN* (1) *has a unique equilibrium point, and the equilibrium point of system* (2) *is globally exponential stable, where*

$$\widehat{\varXi} = \begin{pmatrix} \widehat{\varPi}_{11} & PA_1 & PA_2 & PA_3 & PA_4 & PB_1 & PB_2 & PB_3 & PB_4\\ A_1^{\mathrm T}P & \varPi_{22} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ A_2^{\mathrm T}P & 0 & \varPi_{33} & 0 & 0 & 0 & 0 & 0 & 0\\ A_3^{\mathrm T}P & 0 & 0 & \varPi_{44} & 0 & 0 & 0 & 0 & 0\\ A_4^{\mathrm T}P & 0 & 0 & 0 & \varPi_{55} & 0 & 0 & 0 & 0\\ B_1^{\mathrm T}P & 0 & 0 & 0 & 0 & \varPi_{66} & 0 & 0 & 0\\ B_2^{\mathrm T}P & 0 & 0 & 0 & 0 & 0 & \varPi_{77} & 0 & 0\\ B_3^{\mathrm T}P & 0 & 0 & 0 & 0 & 0 & 0 & \varPi_{88} & 0\\ B_4^{\mathrm T}P & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \varPi_{99} \end{pmatrix},$$

$$\begin{aligned}\widehat{\varPi}_{11} = \alpha P - PC_1 - C_1P + 4\Omega_1 + 4\Omega_2 + 4\Omega_3 + 4\mathrm{e}^{\alpha\tau}\Omega_4\\ + 4\Omega_5 + 4\mathrm{e}^{\alpha\tau}\Omega_6 + 4\mathrm{e}^{\alpha\tau}\Omega_7 + 4\mathrm{e}^{\alpha\tau}\Omega_8,\end{aligned}$$

$$\varPi_{p+1,p+1} = -I_4 \otimes S_p, \quad p = 1, 2, \dots, 8;$$

$$\Omega_p = \begin{pmatrix} S_p\varGamma_p & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Omega_{4+p} = \begin{pmatrix} S_{4+p}\varGamma_p & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad p = 1, 2, 3, 4.$$

Proof. According to Lemma 2, the following inequality can be derived with the condition $\widehat{\varXi} < 0$:

$$\begin{aligned}&\alpha P - PC_1 - PC_1 + 4\Omega_1 + 4\Omega_2 + 4\Omega_3\\&\quad + 4\Omega_4 + 4\mathrm{e}^{\alpha\tau}\Omega_5 + 4\mathrm{e}^{\alpha\tau}\Omega_6 + 4\mathrm{e}^{\alpha\tau}\Omega_7 + 4\mathrm{e}^{\alpha\tau}\Omega_8\end{aligned}$$

$$+ PA_1\big(I_4 \otimes S_1^{-1}\big)A_1^{\mathrm{T}}P + PA_2\big(I_4 \otimes S_2^{-1}\big)A_2^{\mathrm{T}}P + PA_3\big(I_4 \otimes S_3^{-1}\big)A_3^{\mathrm{T}}P$$
$$+ PA_4\big(I_4 \otimes S_4^{-1}\big)A_4^{\mathrm{T}}P + PB_1\big(I_4 \otimes S_5^{-1}\big)B_1^{\mathrm{T}}P + PB_2\big(I_4 \otimes S_6^{-1}\big)B_2^{\mathrm{T}}P$$
$$+ PB_3\big(I_4 \otimes S_7^{-1}\big)B_3^{\mathrm{T}}P + PB_4\big(I_4 \otimes S_8^{-1}\big)B_4^{\mathrm{T}}P < 0. \tag{16}$$

Let $\mathrm{e}^{\alpha\tau} = 1 + \beta$. Obviously, $\beta > 0$ as $\alpha > 0$. Correspondingly, inequality (16) can be rewritten as

$$\begin{aligned}
&\alpha P - PC_1 - C_1P + 4\Omega_1 + 4\Omega_2 + 4\Omega_3 + 4\Omega_4 \\
&\quad + 4(1+\beta)\Omega_5 + 4(1+\beta)\Omega_6 + 4(1+\beta)\Omega_7 + 4(1+\beta)\Omega_8 \\
&\quad + PA_1\big(I_4 \otimes S_1^{-1}\big)A_1^{\mathrm{T}}P + PA_2\big(I_4 \otimes S_2^{-1}\big)A_2^{\mathrm{T}}P + PA_3\big(I_4 \otimes S_3^{-1}\big)A_3^{\mathrm{T}}P \\
&\quad + PA_4\big(I_4 \otimes S_4^{-1}\big)A_4^{\mathrm{T}}P + PB_1\big(I_4 \otimes S_5^{-1}\big)B_1^{\mathrm{T}}P + PB_2\big(I_4 \otimes S_6^{-1}\big)B_2^{\mathrm{T}}P \\
&\quad + PB_3\big(I_4 \otimes S_7^{-1}\big)B_3^{\mathrm{T}}P + PB_4\big(I_4 \otimes S_8^{-1}\big)B_4^{\mathrm{T}}P < 0.
\end{aligned} \tag{17}$$

$\Xi < 0$ can be ascertained by inequality (17) and Lemma 2 for $\alpha > 0$, $P > 0$, $\Omega_i \geqslant 0$, $i = 5, 6, 7, 8$. Consequently, the conditions in Theorem 1 are satisfied, which implies that the existence and uniqueness of the equilibrium point can be ascertained.

Next, the exponential stability will be investigated with the following Lyapunov functional:

$$\begin{aligned}
V(t) = \mathrm{e}^{\alpha t}x^{\mathrm{T}}(t)Px(t) &+ \int_{t-\tau}^{t} \tilde{f}_1^{\mathrm{T}}\big(x(s)\big)(I_4 \otimes S_5)\tilde{f}_1\big(x(s)\big)\mathrm{e}^{\alpha(s+\tau)}\,\mathrm{d}s \\
&+ \int_{t-\tau}^{t} \tilde{f}_2^{\mathrm{T}}\big(x(s)\big)(I_4 \otimes S_6)\tilde{f}_2\big(x(s)\big)\mathrm{e}^{\alpha(s+\tau)}\,\mathrm{d}s \\
&+ \int_{t-\tau}^{t} \tilde{f}_3^{\mathrm{T}}\big(x(s)\big)(I_4 \otimes S_7)\tilde{f}_3\big(x(s)\big)\mathrm{e}^{\alpha(s+\tau)}\,\mathrm{d}s \\
&+ \int_{t-\tau}^{t} \tilde{f}_4^{\mathrm{T}}\big(x(s)\big)(I_4 \otimes S_8)\tilde{f}_4\big(x(s)\big)\mathrm{e}^{\alpha(s+\tau)}\,\mathrm{d}s.
\end{aligned}$$

Calculating the derivative of $V(x(t))$, one can derive

$$\begin{aligned}
&\frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\bigg|_{(13)} \\
&\quad = \alpha\mathrm{e}^{\alpha t}x^{\mathrm{T}}(t)Px(t) + 2\mathrm{e}^{\alpha t}x^{\mathrm{T}}(t)P\big(-C_1x(t) + A_1\tilde{f}_1\big(x(t)\big) + A_2\tilde{f}_2\big(x(t)\big) \\
&\qquad + A_3\tilde{f}_3\big(x(t)\big) + A_4\tilde{f}_4\big(x(t)\big) + B_1\tilde{f}_1\big(x(t-\tau)\big) + B_2\tilde{f}_2\big(x(t-\tau)\big) \\
&\qquad + B_3\tilde{f}_3\big(x(t-\tau)\big) + B_4\tilde{f}_4\big(x(t-\tau)\big)\big) + \mathrm{e}^{\alpha(t+\tau)}\tilde{f}_1^{\mathrm{T}}\big(x(t)\big)(I_4 \otimes S_5)\tilde{f}_1\big(x(t)\big) \\
&\qquad - \mathrm{e}^{\alpha t}\tilde{f}_1\big(x(t-\tau)\big)(I_4 \otimes S_5)\tilde{f}_1\big(x(t-\tau)\big) + \mathrm{e}^{\alpha(t+\tau)}\tilde{f}_2\big(x(t)\big)(I_4 \otimes S_6)\tilde{f}_2\big(x(t)\big)
\end{aligned}$$

$$
\begin{aligned}
&- \mathrm{e}^{\alpha t} \tilde{f}_2^{\mathrm{T}}\big(x(t-\tau)\big)(I_4 \otimes S_6)\tilde{f}_2\big(x(t-\tau)\big) + \mathrm{e}^{\alpha(t+\tau)} \tilde{f}_3^{\mathrm{T}}\big(x(t)\big)(I_4 \otimes S_7)\tilde{f}_3\big(x(t)\big) \\
&- \mathrm{e}^{\alpha t} \tilde{f}_3^{\mathrm{T}}\big(x(t-\tau)\big)(I_4 \otimes S_7)\tilde{f}_3\big(x(t-\tau)\big) + \mathrm{e}^{\alpha(t+\tau)} \tilde{f}_4^{\mathrm{T}}\big(x(t)\big)(I_4 \otimes S_8)\tilde{f}_4\big(x(t)\big) \\
&- \mathrm{e}^{\alpha t} \tilde{f}_4^{\mathrm{T}}\big(x(t-\tau)\big)(I_4 \otimes S_8)\tilde{f}_4\big(x(t-\tau)\big).
\end{aligned}
$$

Similar to the proof of Theorem 1, the following inequality can be established with $\hat{\Xi} < 0$:

$$
\begin{aligned}
\dot{V}\big(x(t)\big) \leqslant{}& \mathrm{e}^{\alpha t} x^{\mathrm{T}}(t)\big(\alpha P - PC_1 - C_1 P + 4\Omega_1 + 4\Omega_2 + 4\Omega_3 \\
&+ 4\Omega_4 + 4\mathrm{e}^{\alpha\tau}\Omega_5 + 4\mathrm{e}^{\alpha\tau}\Omega_6 + 4\mathrm{e}^{\alpha\tau}\Omega_7 + 4\mathrm{e}^{\alpha\tau}\Omega_8 \\
&+ PA_1\big(I_4 \otimes S_1^{-1}\big)A_1^{\mathrm{T}} P + PA_2\big(I_4 \otimes S_2^{-1}\big)A_2^{\mathrm{T}} P \\
&+ PA_3\big(I_4 \otimes S_3^{-1}\big)A_3^{\mathrm{T}} P + PA_4\big(I_4 \otimes S_4^{-1}\big)A_4^{\mathrm{T}} P \\
&+ PB_1\big(I_4 \otimes S_5^{-1}\big)B_1^{\mathrm{T}} P + PB_2\big(I_4 \otimes S_6^{-1}\big)B_2^{\mathrm{T}} P \\
&+ PB_3\big(I_4 \otimes S_7^{-1}\big)B_3^{\mathrm{T}} P + PB_4\big(I_4 \otimes S_8^{-1}\big)B_4^{\mathrm{T}} P\big)x(t) \\
<{}& 0.
\end{aligned}
$$

By some sample calculations, one gets

$$
\mathrm{e}^{\alpha t} x^{\mathrm{T}}(t)Px(t) \leqslant V\big(x(t)\big) \leqslant \sup_{t_0-\tau \leqslant s \leqslant t_0} V\big(x(s)\big),
$$

which can result in $x^{\mathrm{T}}(t)x(t) \leqslant M\mathrm{e}^{-\alpha t}$, where $M = (\sup_{t_0-\tau\leqslant s\leqslant t_0} V(x(s)))/\lambda_{\min}(P)$. Following from the definition of exponential stability, the equilibrium point of system (2) is globally exponentially stable. Consequently, the equilibrium point of QVNN (1) is globally exponentially stable. □

Remark 1. Different from some of the existing results [11, 40, 42, 49], in which the existence and uniqueness for the equilibrium point were investigated based on the contracting mapping principle or homeomorphism theory, the nonlinear measure approach is developed to discuss the existence and uniqueness for the equilibrium point of QVNNs. The criteria are derived by employing the linear matrix inequality, and they can be easily verified via the LMI toolbox in MATLAB. Furthermore, the globally exponential stability of QVNNs can be reached with the proposal criteria.

Remark 2. The stability is one of key factors to a control system, and it is pivotal to the various applications of neural networks. The stability of RVNNs and CVNNs have been discussed extensively [17, 28, 37, 49]. The QVNNs are with better performance and better application than both CVNNs and RVNNs in deal with the problem involving three-dimensional and four-dimensional data. However, there few published literatures concerning the dynamical behavior of QVNNs [32, 33, 44]. Therefore, it is necessary to consider the stability of QVNNs.

4 Illustrative examples

In this section, two numerical examples will be given out to show our theoretical results to be correct and effective.

Example 1. Considering the following QVNN with the same parameters of Example 1 in [32]:

$$\dot{q}(t) = -Cq(t) + Af\big(q(t)\big) + Bg\big(q(t-\tau)\big) + U, \tag{18}$$

where

$$A = \begin{pmatrix} -0.1 - 0.5i - 0.1j - 0.5k & 0.4 + 0.2i - 0.4j + 0.2k \\ -0.3 + 0.1i - 0.1j + 0.1k & -0.6 - 0.1i - 0.1j - 0.1k \end{pmatrix},$$

$$B = \begin{pmatrix} -0.1 - 0.4i - 0.1j - 0.4k & -0.4 - 0.1i - 0.4j - 0.1k \\ -0.1 - 0.1i - 0.1j - 0.1k & -0.1 - 0.2i - 0.1j - 0.2k \end{pmatrix},$$

$$C = \begin{pmatrix} 7 & 0 \\ 0 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 - i - j - k \\ -2 - i - 2j - k \end{pmatrix},$$

$$f\big(q(t)\big) = g\big(q(t)\big) = \tanh\big(q(t)\big), \qquad \tau = 3.$$

Let $\alpha = 3.9563$, and by using the LMI toolbox in MATLAB, the solutions of $\hat{\Xi} < 0$ can be resolved as following:

$$P = \begin{pmatrix} 0.6508 & -0.0022 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0022 & 0.1264 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.6568 & -0.0021 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.0021 & 0.1914 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.6517 & -0.0028 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.0028 & 0.1919 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.6569 & -0.0026 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.0026 & 0.4929 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} 3.9102 & 0 \\ 0 & 3.9102 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 4.0256 & 0 \\ 0 & 4.0256 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 4.0264 & 0 \\ 0 & 4.0264 \end{pmatrix}, \qquad S_4 = \begin{pmatrix} 3.9102 & 0 \\ 0 & 3.9102 \end{pmatrix},$$

$$S_5 = \begin{pmatrix} 3.8561 & 0 \\ 0 & 3.8561 \end{pmatrix}, \qquad S_6 = \begin{pmatrix} 3.7954 & 0 \\ 0 & 33.7954 \end{pmatrix},$$

$$S_7 = \begin{pmatrix} 3.8561 & 0 \\ 0 & 3.8561 \end{pmatrix}, \qquad S_8 = \begin{pmatrix} 3.7954 & 0 \\ 0 & 3.7954 \end{pmatrix}.$$

Therefore, the conditions of Theorem 1 are satisfied, the unique equilibrium point QVNN (18) is globally exponentially stable.

Example 2. Considering the QVNN as follow:

$$\dot{q}(t) = -Cq(t) + Af\big(q(t)\big) + Bg\big(q(t-\tau)\big) + U, \tag{19}$$

where

$$A = \begin{pmatrix} 1.2 + 0.5i - 0.4j + 0.5k & 0.4 + 1.2i + 0.4j + 1.2k \\ 1.3 + 0.3i + 0.6j + 0.7k & 0.6 + 0.5i - 0.6j - 0.6k \end{pmatrix},$$

$$B = \begin{pmatrix} 1.1 - 0.5i - 0.1j - 0.4k & -0.4 + 0.5i + 1.4j + 0.5k \\ -1.0 + 0.9i - 1.0j + 1.2k & 1.1 - 1.2i - 1.1j + 1.2k \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} 2 - 1.2i + j + k \\ 1 + i - 1.5j - 1.2k \end{pmatrix},$$

$$f\big(q(t)\big) = g\big(q(t)\big) = \tanh\big(q(t)\big), \qquad \tau = 2.7.$$

Let $\alpha = 3.0$, and by using the LMI toolbox in MATLAB, the solutions of $\hat{\Xi} < 0$ can be resolved as following:

$$P = \begin{pmatrix} 1.7168 & -0.0097 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0097 & 1.2134 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.5342 & -0.0487 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.0487 & 1.3234 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.9546 & 1.3166 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.2198 & 0.1919 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.7092 & 0.1608 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.1608 & 1.4286 \end{pmatrix},$$

$$S_1 = \begin{pmatrix} 20.4497 & 0 \\ 0 & 20.4497 \end{pmatrix}, \qquad S_2 = \begin{pmatrix} 20.5259 & 0 \\ 0 & 20.5259 \end{pmatrix},$$

$$S_3 = \begin{pmatrix} 20.6120 & 0 \\ 0 & 20.6120 \end{pmatrix}, \qquad S_4 = \begin{pmatrix} 20.4497 & 0 \\ 0 & 20.4497 \end{pmatrix},$$

$$S_5 = \begin{pmatrix} 20.7924 & 0 \\ 0 & 20.7924 \end{pmatrix}, \qquad S_6 = \begin{pmatrix} 3.7954 & 0 \\ 0 & 33.7954 \end{pmatrix},$$

$$S_7 = \begin{pmatrix} 20.7409 & 0 \\ 0 & 20.7409 \end{pmatrix}, \qquad S_8 = \begin{pmatrix} 20.6519 & 0 \\ 0 & 20.6519 \end{pmatrix},$$

which shows that the conditions of Theorem 2 are satisfied, the unique equilibrium point QVNN (19) is globally exponentially stable. The simulation results are shown in Fig. 1.

Remark 3. Example 1 can be solved by both results obtained in [32] and our results. However, the exponent convergence index $\bar{\omega}$ obtained by Theorem 1 in [32] is less than 0.3156, while the exponent convergence index can take the value as 3.9563, which is 12.5 times that 0.3156. The simulation results [32, Fig. 1] also showed the fast convergence rate. On the other hand, it is easy to check that parameters in Example 2 do not meet the conditions of Theorem 1 in [32], which means that Theorem 1 in [32] is invalid to Example 2. From this point, our results are with less conservatism than that of [32], and the obtained results will be with better applications.

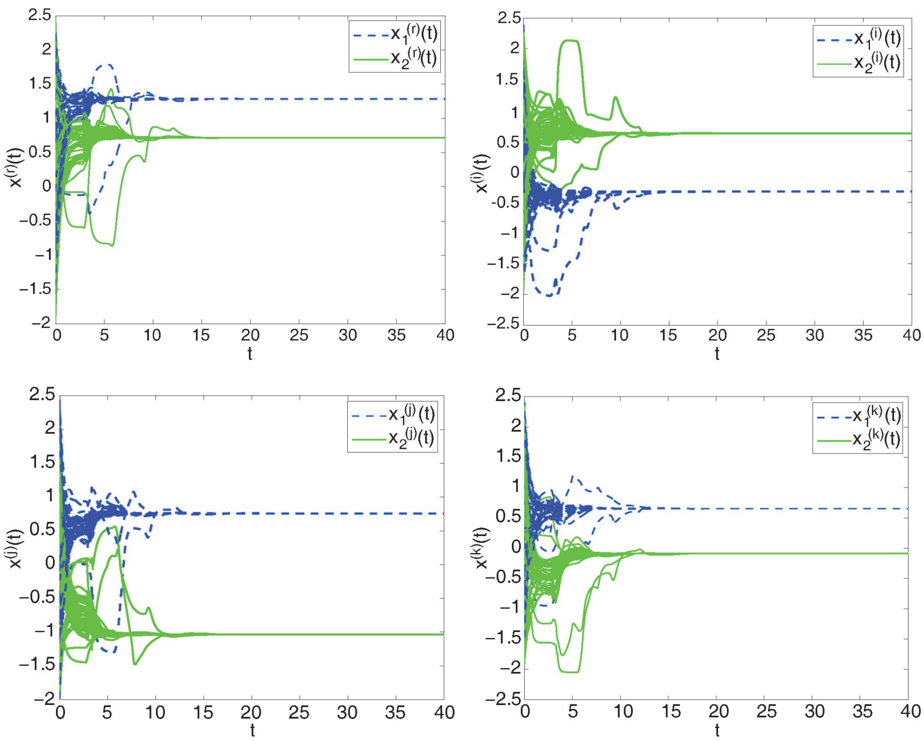

Figure 1. The state trajectories of $x^{(r)}(t)$, $x^{(i)}(t)$, $x^{(j)}(t)$, $x^{(k)}(t)$ of QVNN (19) with 50 random initial conditions.

5 Conclusions

The existence and stability analysis for the equilibrium point of the QVNNs are discussed in this paper. The QVNNs is equivalently transformed into four real-valued systems to resolve the difficulty of noncommutativity for the multiplication of quaternion, and then some sufficient conditions are established by virtue of Lyapunov functional, nonlinear measure approach, and inequality technique. The proposal criteria can be easily verified by LMI toolbox in MATLAB. Furthermore, the effectiveness our results are checked by two numerical examples. In addition, the reported results concerning the dynamical of QVNNs are rare, and lots of interesting topics are still open. In the near future, we will consider the synchronization and stabilization of QVNNs.

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