Nonlinear Analysis: Modelling and Control, Vol. 23, No. 2, 251–268 ISSN 1392-5113
https://doi.org/10.15388/NA.2018.2.7

Optimal control for a higher-order nonlinear parabolic equation describing crystal surface growth*

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Received: May 28, 2017 / **Revised:** November 28, 2017 / **Published online:** February 12, 2018

Abstract. In this paper, we shall study the optimal control of the initial-boundary value problem of a higher-order nonlinear parabolic equation describing crystal surface growth. The existence and uniqueness of weak solutions to the problem are given. According to the variational method, optimal control theories and distributed parameter system control theories, we can deduce that the norm of the solution is related to the control item and initial value in the special Hilbert space. The optimal control of the problem is given, the existence of optimal solution is proved and the optimality system is established.

Keywords: optimal control, higher-order nonlinear parabolic equation, optimal solution, optimality condition.

1 Introduction

The field of optimal control was born in the 1950s with the discovery of the maximum principle as a result of a competition in military affairs in the early days of the cold war. It lies at the forefront of the creative interplay of mathematics, engineering and computer science. Modern optimal control theories and applied models are not only represented by ODE, but also by PDE, especially nonlinear parabolic equation.

In past decades, many papers have already been published to study the control problems of nonlinear parabolic equations. In [17], Yong and Zheng considered the feedback stabilization and optimal control of the Cahn–Hilliard equation in a bounded domain with smooth boundary. In the papers wrote by Ryu and Yagi [13, 14], the optimal control problems of Keller–Segel equations and adsorbate-induced phase transition model were

*This research was supported by the Natural Science Foundation of Jiangsu Province of China for Young Scholar (grant Nos. BK20140130, BK20170172) and China Postdoctoral Science Foundation (grant No. 2017M611684).

considered. Their techniques are based on the energy estimates and the compact method. They established various a priori estimates for the solutions of equations to show that the classical compact method described systematically by Lions (see [8]) is available. Recently, by using an approximate problems, Zheng [21] derived the optimality conditions for an optimal control of multidimensional modified Swift–Hohenberg equation. There is much literature concerned with the optimal control problem for parabolic equations. For more recent result, we refer the reader to [1, 15, 20] and the references therein.

In the study of molecular beam epitaxy, the hight $H(x,t)$ of the surface above the substrate plane satisfies a continuity equation

$$\frac{\partial}{\partial t}H + \nabla \cdot J_{\mathrm{surface}}\{H\} = F, \tag{1}$$

where F is the incident mass flux out of the molecular beam. In general, the systematic current J_{surface} depends on the whole surface configuration. Keeping only the most important terms in a gradient expansion, subtracting the mean height $H = Fu$ and using appropriately rescaled units of height, distance and time [12], equation (1) attains the dimensionless form

$$\frac{\partial u}{\partial t} = -\Delta^2 u - \nabla \cdot \bigl[f\bigl(\nabla u^2\bigr)\nabla u\bigr]. \tag{2}$$

In equation (2), the linear term describes relaxation through adatom diffusion driven by the surface free energy [9], while the second nonlinear term models the nonequilibrium current [6]. Assuming in-plane symmetry, it follows that the nonequilibrium current is (anti)parallel to the local tilt ∇u with a magnitude $f(\nabla u^2)$ depending only on the magnitude of the title. Within a Burton–Cabrera–Frank-type theory [7], for small tilts, the current is proportional to $|\nabla u|$, and the opposite limit is proportional to $|\nabla u|^{-1}$. This suggests the interpolation formula $f(s^2) = 1/(1+s^2)$ (see [5, 11]). Hence, we obtain the following equation:

$$\frac{\partial u}{\partial t} + a\Delta^2 u + \mu\nabla \cdot \left(\frac{\nabla u}{1+|\nabla u|^2}\right) = 0, \quad (x,t) \in \Omega \times (0,T), \tag{3}$$

where a and μ are positive constants, $\Omega \subset \mathbb{R}^2$ is a bounded domain.

During the past years, some investigations of equation (3) were studied. It was Rost and Krug [11] who studied the unstable epitaxy on singular surfaces using equation (3) with a prescribed slope dependent surface current. In their paper, they derived scaling relations for the late stage of growth, where power law coarsening of the mound morphology is observed. In [10], in the limit of weak desorption, Pierre-Louis et al. derived equation (3) for a vicinal surface growing in the step flow mode. This limit turned out to be singular, and nonlinearities of arbitrary order need to be taken into account. Fujimura and Yagi [2, 3] studied the well-posedness of the solution for equation (3). In their papers, the uniqueness local solutions and the global solutions were obtained. A dynamical system determined from the initial-boundary value problem of the model equation was constructed, and the asymptotic behavior of trajectories of the dynamical system was also considered. In [4], Grasselli, Mola and Yagi proved that equation (3) possesses a global as well as an exponential attractor. In addition, if the boundary is smooth enough, they showed that every trajectory converges to a single equilibrium by means of a suitable Lojasiewicz–Simon inequality. Recently, Zhao and Liu [19] studied the long time behavior of equation (3) with periodic boundary conditions. Based on the iteration technique for regularity estimates and the classical existence theorem of global attractors, they proved that the equation possesses a global attractor on some affine space of H^k $(0 \leqslant k < +\infty)$. Latterly, in [18], Zhao and Cao studied the optimal control problem for equation (3) in 1D case.

In this article, we consider the optimal control problem for two-dimensional equation (3) together with the initial and boundary conditions. Suppose that $T > 0$, $Q_0 \subseteq Q = \Omega \times (0,T)$, $C \in \mathcal{L}(W(0,T;V),S)$ is an operator, which is called the observer, S is a real Hilbert space of observations. We are concerned with the distributed optimal control problem

$$\min J(u,w) = \frac{1}{2}\|Cu - z_d\|_S^2 + \frac{\delta}{2}\|w\|^2_{L^2(Q_0)}, \tag{4}$$

subject to the

$$\begin{aligned} &\frac{\partial u}{\partial t} + a\Delta^2 u + \mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) = Bw, \quad (x,t)\in\Omega\times(0,T),\\ &\frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = \frac{\partial\Delta u}{\partial n}\bigg|_{\partial\Omega} = 0, \qquad u(x,0) = u_0(x) \quad \forall x\in\Omega. \end{aligned} \tag{5}$$

The control target is to match the given desired state z_d in L^2-sense by adjusting the body force w in a control volume $Q_0 \subseteq Q = \Omega\times(0,T)$ in the L^2-sense.

Now, we introduce some notations that will be used throughout this paper. Let $V = H_E^2(\Omega) = \{y\colon y \in H^2(\Omega),\ \partial y/\partial n|_{\partial\Omega} = 0\}$, $H = L^2(\Omega)$, let V^* and H^* be dual spaces of V and H. Then we obtain

$$V \hookrightarrow H = H^* \hookrightarrow V^*.$$

Clearly, each embedding is dense.

The extension operator $B \in \mathcal{L}(L^2(Q_0), L^2(0,T;H))$, which is called the controller, is introduced as

$$Bq = \begin{cases} q, & q\in Q_0,\\ 0, & q \in Q\setminus Q_0. \end{cases}$$

We supply H with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, and define the spaces

$$W(0,T;X) = \bigl\{y\colon y\in L^2(0,T;X),\ y_t\in L^2(0,T;X^*)\bigr\},$$
$$W(0,T;X,Z) = \bigl\{y\colon y\in L^\infty(0,T;X),\ y_t\in L^2(0,T;Z)\bigr\},$$

and

$$\overline{W}(0,T;X,Z) = \bigl\{y\colon y\in L^2(0,T;X),\ y_t\in L^2(0,T;Z)\bigr\},$$

which are Hilbert spaces endowed with common inner product.

In the following, the letters c, c_i $(i = 1,2,\dots)$ will always denote positive constants different in various occurrences.

2 Existence and uniqueness of the weak solutions

In this section, we prove the existence and uniqueness of weak solution for problem (5), where $Bw \in L^2(0,T;H)$ and a control $w \in L^2(Q_0)$.

Definition 1. For all $\eta \in V$, $t \in (0,T)$, a function $u(x,t) \in W(0,T;V)$ is called a weak solution to problem (5) if

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}u, \eta\right) + a(\Delta u, \Delta\eta) - \mu\left(\frac{\nabla u}{1+|\nabla u|^2}, \nabla\eta\right) = (Bw, \eta) \quad \forall \eta \in V.$$

We shall give Theorem 1 on the existence and uniqueness of weak solution to problem (5) and prove it.

Theorem 1. *Assume that* $u_0 \in V$, $Bw \in L^2(0,T;H)$, *then problem* (5) *admits a unique weak solution* $u(x,t) \in W(0,T;V)$.

Proof. Galerkin method is applied to the proof.

Denote $\mathrm{A} = a\Delta^2$ as a differential operator, let $\{y_i\}_{i=1}^{\infty}$ denote the eigenfunctions of the operator A. For $n \in \mathbb{N}$, define

$$V_n = \mathrm{span}\{y_1, y_2, \dots, y_n\} \subset V.$$

Suppose $u_n(x,t) = \sum_{j=1}^{n} u_{nj}(t)y_j(x)$ require $u_n(0,\cdot) \to u_0$ in H holds true. By analyzing the limiting behavior of sequences of smooth function $\{u_n\}$, we can prove the existence of a weak solution to problem (5).

Performing the Galerkin procedure for problem (5), we obtain

$$\begin{aligned} &\left(u_{nt} + a\Delta^2 u_n + \mu\nabla\cdot\left(\frac{\nabla u_n}{1+|\nabla u_n|^2}\right), y_j\right) = (Bw, y_j), \\ &\big(u_n(\cdot,0), y_j\big) = \big(u_{n0}(\cdot), y_j\big), \quad j = 1,2,\dots,N. \end{aligned} \tag{6}$$

Obviously, the equation of (6) is an ordinary differential equation, and according to ODE theory, there exists a unique solution to the equation of (6) in the interval $[0, t_n)$. What we should do is to show that the solution is uniformly bounded when $t_n \to T$. We also need to show that the times t_n are not decaying to 0 as $n \to \infty$. Therefore, we shall prove the existence of solution in the following steps.

Step 1. Multiplying the equation of (6) by u_n, integrating with respect to x over Ω, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2 + a\|\Delta u_n\|^2 = \mu\int_\Omega \frac{|\nabla u_n|^2}{1+|\nabla u_n|^2}\,\mathrm{d}x + \int_\Omega u_n Bw\,\mathrm{d}x.$$

Noticing that

$$\mu\int_\Omega \frac{\nabla u_n^2}{1+|\nabla u_n|^2}\,\mathrm{d}x \leqslant \mu|\Omega|,$$

$$\int_\Omega u_n Bw\,\mathrm{d}x \leqslant \|u_n\|\|Bw\| \leqslant \frac{1}{2}\|u_n\|^2 + \frac{1}{2}\|Bw\|^2.$$

Summing up, we immediately get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2 + 2a\|\Delta u_n\|^2 \leqslant \|u_n\|^2 + \|Bw\|^2 + 2\mu|\Omega|. \tag{7}$$

Since $Bw \in L^2(0,T;H)$ is the control item, we can assume that $\|Bw\| \leqslant M$, where M is a positive constant. Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u_n\|^2 + 2a\|\Delta u_n\|^2 \leqslant \|u_n\|^2 + M^2 + 2\mu|\Omega|.$$

Using Gronwall's inequality, we obtain

$$\begin{aligned}\|u_n\|^2 &\leqslant \mathrm{e}^t\|u_{n,0}\|^2 + M^2 + 2\mu|\Omega| \\ &\leqslant \mathrm{e}^T\|u_{n,0}\|^2 + M^2 + 2\mu|\Omega| = c_1^2 \quad \forall t \in [0,T].\end{aligned} \tag{8}$$

Step 2. Multiplying the equation of (6) by Δu_n, integrating with respect to x over Ω, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n\|^2 + a\|\nabla\Delta u_n\|^2 + \mu\int\limits_\Omega \frac{\nabla u_n}{1+|\nabla u_n|^2}\nabla\Delta u_n\,\mathrm{d}x = -\int\limits_\Omega \Delta u_n Bw\,\mathrm{d}x. \tag{9}$$

Noticing that

$$\|\Delta u_n\|^2 \leqslant \frac{1}{a}\|\nabla u_n\|^2 + \frac{a}{4}\|\nabla\Delta u_n\|^2$$

and

$$\int\limits_\Omega \left(\frac{\nabla u_n}{1+|\nabla u_n|^2}\right)^2 \mathrm{d}x \leqslant \int\limits_\Omega |\nabla u_n|^2\,\mathrm{d}x = \|\nabla u_n\|^2.$$

It then follows from (9) that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n\|^2 + a\|\nabla\Delta u_n\|^2 \\ &\quad\leqslant \frac{a}{4}\|\nabla\Delta u_n\|^2 + \frac{\mu^2}{a}\int\limits_\Omega \left(\frac{\nabla u_n}{1+|\nabla u_n|^2}\right)^2 \mathrm{d}x + \|\Delta u_n\|^2 + \frac{1}{4}\|Bw\|^2 \\ &\quad\leqslant \frac{a}{4}\|\nabla\Delta u_n\|^2 + \frac{\mu^2}{a}\|\nabla u_n\|^2 + \frac{a}{4}\|\nabla\Delta u_n\|^2 + \frac{1}{a}\|\nabla u_n\|^2 + \frac{1}{4}\|Bw\|^2,\end{aligned}$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u_n\|^2 + a\|\nabla\Delta u_n\|^2 \leqslant \frac{2(\mu^2+1)}{a}\|\nabla u_n\|^2 + \frac{1}{2}\|Bw\|^2. \tag{10}$$

Since $\|Bw\| \leqslant M$, using Gronwall's inequality, we obtain

$$\|\nabla u_n\|^2 \leqslant \mathrm{e}^{2(\mu^2+1)t/a}\big\|\nabla u_n(0)\big\|^2 + \frac{aM^2}{4\mu^2+4} \leqslant c_2^2 \quad \forall t \in [0,T]. \tag{11}$$

Step 3. Multiplying the equation of (5) by $\Delta^2 u_n$, integrating with respect to x over Ω, we deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_n\|^2 + a\|\Delta^2 u_n\|^2 + \mu\int_\Omega \nabla\cdot\left(\frac{\nabla u_n}{1+|\nabla u_n|^2}\right)\Delta^2 u_n\,\mathrm{d}x = \int_\Omega \Delta^2 u_n Bw\,\mathrm{d}x.$$

Noticing that

$$\int_\Omega \nabla\cdot\left(\frac{\nabla u_n}{1+|\nabla u_n|^2}\right)\Delta^2 u_n\,\mathrm{d}x = \int_\Omega \frac{\Delta u_n\Delta^2 u_n}{1+|\nabla u_n|^2}\,\mathrm{d}x - \int_\Omega \frac{2|\nabla u_n|^2\Delta u_n\Delta^2 u_n}{(1+|\nabla u_n|^2)^2}\,\mathrm{d}x.$$

Hence,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_n\|^2 + a\|\Delta^2 u_n\|^2 \\
&\quad = \mu\int_\Omega \frac{2|\nabla u_n|^2\Delta u_n\Delta^2 u_n}{(1+|\nabla u_n|^2)^2}\,\mathrm{d}x - \mu\int_\Omega \frac{\Delta u_n\Delta^2 u_n}{1+|\nabla u_n|^2}\,\mathrm{d}x + \int_\Omega \Delta^2 u_n Bw\,\mathrm{d}x \\
&\quad \leqslant \mu\sup_{x\in\bar{\Omega}} \frac{2|\nabla u_n|^2}{(1+|\nabla u_n|^2)^2}\cdot\|\Delta u_n\|\|\Delta^2 u_n\| \\
&\qquad + \mu\sup_{x\in\bar{\Omega}} \frac{1}{1+|\nabla u_n|^2}\cdot\|\Delta u_n\|\|\Delta^2 u_n\| + \|Bw\|\|\Delta^2 u_n\| \\
&\quad \leqslant \frac{3\mu}{2}\|\Delta u_n\|\|\Delta^2 u_n\| + \|Bw\|\|\Delta^2 u_n\| \\
&\quad \leqslant \frac{a}{2}\|\Delta^2 u_n\|^2 + \frac{9\mu^2}{4a}\|\Delta u_n\|^2 + \frac{1}{a}\|Bw\|^2.
\end{aligned}$$

Therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u_n\|^2 + a\|\Delta^2 u_n\|^2 \leqslant \frac{9\mu^2}{2a}\|\Delta u_n\|^2 + \frac{2}{a}\|Bw\|^2. \tag{12}$$

Since $\|Bw\| \leqslant M$, using Gronwall's inequality, we obtain

$$\begin{aligned}
\|\Delta u_n\|^2 &\leqslant \mathrm{e}^{9\mu^2 t/(2a)}\|\Delta u_n(0)\|^2 + \frac{4}{9\mu^2}M^2 \\
&\leqslant \mathrm{e}^{9\mu^2 T/(2a)}\|\Delta u_n(0)\|^2 + \frac{4}{9\mu^2}M^2 = c_3^2,
\end{aligned}$$

where $t \in [0, T]$. Adding (8) and (11) together gives

$$\|u_n\|^2_{L^2(0,T;V)} = \int_0^T \left(\|u_n\|^2 + \|\nabla u_n\|^2 + \|\Delta u_n\|^2\right)\mathrm{d}t \leqslant c_4. \tag{13}$$

Then the uniform $L^2(0, T; V)$ bounded on a sequence $\{u_n\}$ is proved.

Step 4. We prove a uniform $L^2(0,T;V^*)$ bound on a sequence $\{u_{n,t}\}$. Noticing that

$$-\big(\Delta^2 u_n, \eta\big) \leqslant \big|(\Delta u_n, \Delta\eta)\big| \leqslant \|\Delta u_n\|\|\Delta\eta\| \leqslant \|\Delta u_n\|\|\eta\|_V,$$

$$\left(\nabla\cdot\left[\frac{\nabla u_n}{1+|\nabla u_n|^2}\right], \eta\right) = -\left(\frac{\nabla u_n}{1+|\nabla u_n|^2}, \nabla\eta\right) \leqslant \|\nabla u_n\|\|\nabla\eta\| \leqslant \|\nabla u_n\|\|\eta\|_V,$$

$$(Bw, \eta) \leqslant \|Bw\|\|\eta\| \leqslant \|Bw\|\|\eta\|_V.$$

Therefore, by (11), we have

$$\begin{aligned}
&\|u_{n,t}\|_{V^*} \\
&\leqslant a\big\|\Delta^2 u_n\big\|_{V^*} + \mu\left\|\nabla\cdot\left(\frac{\nabla u_n}{1+|\nabla u_n|^2}\right)\right\|_{V^*} + \|Bw\|_{V^*} \\
&\leqslant c\left(\sup\frac{|(\Delta^2 u_n, \eta)_{V^*,V}|}{\|\eta\|_V} + \sup\frac{|(\nabla\cdot[\frac{\nabla u_n}{1+|\nabla u_n|^2}]_x, \eta)_{V^*,V}|}{\|\eta\|_V} + \sup\frac{|(Bw,\eta)_{V^*,V}|}{\|\eta\|_V}\right) \\
&\leqslant c\big(\|\Delta u_n\| + \|\nabla u_n\| + \|Bw\|\big) \leqslant c(c_2 + c_3 + M).
\end{aligned}$$

Hence, we get

$$\|u_{n,t}\|^2_{L^2(0,T;V^*)} = \int_0^T \|u_{n,t}\|^2_{V^*}\,\mathrm{d}t \leqslant \big[c(c_2+c_3+M)\big]^2 T = c_5.$$

Step 5. Integrating (10) and (12) with respect to $[0,T]$, we derive that

$$\|\nabla\Delta u_n\|^2_{L^2(0,T;H)} + \big\|\Delta^2 u_n\big\|^2_{L^2(0,T;H)} \leqslant c_6, \tag{14}$$

combining (13) and (14) together, we deduce that

$$\|u_n\|_{L^2(0,T;H^4)} \leqslant c_7.$$

It then follows from Aubin–Lions lemma that

$$\begin{aligned}
W(0,T;V,V^*) &\hookrightarrow C\big(0,T;H^1\big), \\
W(0,T;V,V^*) &\hookrightarrow C(0,T;H),
\end{aligned}$$

and

$$\overline{W}\big(0,T;H^4,V^*\big) \hookrightarrow L^2(0,T;V)$$

are compact. Therefore, there exist $u \in C(0,T;H^1)$ and $u \in L^2(0,T;H^2)$ such that, up to a subsequence,

$$\begin{aligned}
&u_n \to u \text{ strongly in } C\big(0,T;H^1\big), \\
&u_n \to u \text{ strongly in } L^2\big(0,T;H^2\big).
\end{aligned} \tag{15}$$

By Sobolev's embedding theorem, we get

$$H^2(\Omega) \hookrightarrow L^\infty(\Omega), \qquad H^2(\Omega) \hookrightarrow W^{1,4}(\Omega).$$

It then follows from (15) that

$$\|u_n - u\|_{C(0,T;H^1)} \to 0, \qquad \|\Delta u_n - \Delta u\|_{L^2(0,T;H^2)} \to 0.$$

According to the previous subsequences $\{u_n\}$, we conclude that $\nabla\cdot[\nabla u_n/(1+|\nabla u_n|^2)]$ weakly converges to $\nabla\cdot[\nabla u/(1+|\nabla u|^2)]$ in $L^2(0,T;H)$. In fact, setting $\varphi(s) = s/(1+s^2)$, we have

$$\varphi'(s) = \frac{1-s^2}{(1+s^2)^2} \leqslant \frac{1}{(1+s^2)^2} \leqslant 1$$

and

$$\begin{aligned}\varphi''(s) &= \frac{-2s}{(1+s^2)^2} - \frac{4-4s^4}{(1+s^2)^4} \leqslant \frac{1}{1+s^2} + \frac{4}{(1+s^2)^4} + \frac{4s^4}{(1+s^2)^4}\\ &\leqslant 1+4+\frac{1}{4} = \frac{21}{4}.\end{aligned}$$

Hence, for any $w \in L^2(0,T;H)$, by differential mean value theorems, we have

$$\begin{aligned}&\left|\int_0^T \big(\nabla\varphi(\nabla u_n) - \nabla\varphi(\nabla u), w\big)\,\mathrm{d}t\right|\\
&\quad= \left|\int_0^T \big(\varphi'(\nabla u_n)\Delta u_n - \varphi'(\nabla u)\Delta u, w\big)\,\mathrm{d}t\right|\\
&\quad\leqslant \left|\int_0^T \big(\varphi'(\nabla u_n)\Delta u_n - \varphi'(\nabla u)\Delta u_n, w\big)\,\mathrm{d}t\right| + \left|\int_0^T \big(\varphi'(\nabla u)\Delta u_n - \varphi'(\nabla u)\Delta u, w\big)\,\mathrm{d}t\right|\\
&\quad= \left|\int_0^T \big(\varphi''\big(\theta\nabla u_n + (1-\theta)\nabla u\big)(\nabla u_n - \nabla u)\Delta u_n, w\big)\,\mathrm{d}t\right|\\
&\qquad+ \left|\int_0^T \big(\varphi'(\nabla u)\Delta u_n - \varphi'(\nabla u)\Delta u, w\big)\,\mathrm{d}t\right|\\
&\quad\leqslant \int_0^T \big\|\varphi''\big(\theta\nabla u_n + (1-\theta)\nabla u\big)\big\|_\infty \|\nabla u_n - \nabla u\|\|\Delta u_n\|_\infty \|w\|\,\mathrm{d}t\\
&\qquad+ \int_0^T \big\|\varphi'(\nabla u)\big\|_\infty \|\Delta u_n - \Delta u\|\|w\|\,\mathrm{d}t\\
&\quad\leqslant \frac{21}{4}\|\nabla u_n - \nabla u\|_{C(0,T;H)}\|\Delta u_n\|_{L^2(0,T;H^2)}\|w\|_{L^2(0,T;H)}\\
&\qquad+ \|\Delta u_n - \Delta u\|_{L^2(0,T;H)}\|w\|_{L^2(0,T;H)}\\
&\quad\leqslant C\big(\|u_n - u\|_{C(0,T;H^1)} + \|\Delta u_n - \Delta u\|_{L^2(0,T;H)}\big)\|w\|_{L^2(0,T;H)},\end{aligned} \tag{16}$$

where $\theta \in (0,1)$. By (16), we know that there exists a subsequence $\{u_n(x,t)\}$ such that $\nabla\cdot[\nabla u_n/(1+|\nabla u_n|^2)]$ weakly converges to $\nabla\cdot[\nabla u/(1+|\nabla u|^2)]$. On the other hand, the subsequence $\{u_{n,t}\}$ weakly converges to $\{u_t\}$ in $L^2(0,T;V^*)$.

Based on the above discussion, we conclude that there exists a function $u(x,t) \in W(0,T;V)$, which satisfies (7).

Now, we prove the uniqueness of the solutions for problem (5). Suppose that there is another solution $\tilde{u} \in W(0,T;V)$. Let $v = u - \tilde{u}$. Then v satisfies

$$v_t + a\Delta^2 v + \mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2} - \frac{\nabla\tilde{u}}{1+|\nabla\tilde{u}|^2}\right) = 0, \quad (x,t) \in \Omega\times(0,T),$$
$$\frac{\partial v}{\partial n} = \frac{\partial\Delta v}{\partial n} = 0, \quad x \in \partial\Omega, \qquad v(x,0) = v_0(x) = 0, \quad x \in \Omega.$$

Taking the scalar product with v under the duality between V and V^*, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 + a\|\Delta v\|^2 - 2k\|\nabla v\|^2 = \mu\left(\frac{\nabla u}{1+|\nabla u|^2} - \frac{\nabla\tilde{u}}{1+|\nabla\tilde{u}|^2}, \nabla v\right).$$

Note that

$$\begin{aligned}
&\mu\left(\frac{\nabla u}{1+|\nabla u|^2} - \frac{\nabla\tilde{u}}{1+|\nabla\tilde{u}|^2}, \nabla v\right)\\
&\quad= \mu\left(\frac{(1-\nabla u\nabla\tilde{u})\nabla v}{(1+|\nabla u|^2)(1+|\nabla\tilde{u}|^2)}\right)\\
&\quad= \mu\left(\frac{1}{(1+|\nabla u|^2)(1+|\nabla\tilde{u}|^2)}, |\nabla v|^2\right) - \mu\left(\frac{\nabla u\nabla\tilde{u}}{(1+|\nabla u|^2)(1+|\nabla\tilde{u}|^2)}, |\nabla v|^2\right)\\
&\quad\leqslant \mu\|\nabla v\|^2 + \mu\left\|\frac{\nabla u}{1+|\nabla u|^2}\right\|_\infty\left\|\frac{\nabla\tilde{u}}{1+|\nabla\tilde{u}|^2}\right\|_\infty\|\nabla v\|^2\\
&\quad\leqslant \frac{5}{4}\mu\|\nabla v\|^2 \leqslant a\|\Delta v\|^2 + \frac{5\mu^2}{64a}\|v\|^2.
\end{aligned}$$

Summing up, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|^2 \leqslant \frac{5\mu^2}{32a}\|v\|^2.$$

Using Gronwall's inequality, we get

$$\|v\|^2 \leqslant \mathrm{e}^{5\mu^2 t/(32a)}\|v_0\|^2.$$

Therefore, $v = 0$. The uniqueness of solutions is proved. We complete the proof of Theorem 1. □

For the relation among the norm of weak solution, initial value and control item, basing on the above discussion, we obtain the following theorem immediately.

Corollary 1. *Assume that $Bw \in L^2(0,T;H)$, $u_0 \in V$, then there exists positive constants C' and C'' such that*

$$\|u\|^2_{W(0,T;V)} \leqslant C'\bigl(\|u_0\|^2_V + \|w\|^2_{L^2(Q_0)}\bigr) + C''.$$

3 Optimal control problem

In this section, we consider the optimal control problem associated with problem (5) and prove the existence of optimal solution. By virtue of Theorem 1, we can define the solution map $w \to u(w)$ of $L^2(Q_0)$ into $W(0,T;V)$. The solution u is called the state of the control system (5). The observation of the state is assumed to be given by Cu.

Let $X = W(0,T;V) \times L^2(Q_0)$ and $Y = L^2(0,T;V) \times H$. We define an operator $e = e(e_1, e_2) : X \to Y$, where

$$\begin{aligned} e_1 &= G = \left(\Delta^2\right)^{-1}\left[\frac{\partial u}{\partial t} + a\Delta^2 u + \mu\nabla\cdot\left(\frac{\nabla u}{1+|\nabla u|^2}\right) - Bw\right],\\ e_2 &= u(x,0) - u_0. \end{aligned}$$

Here Δ^2 is an operator from V to V^*. Then we write (4) in the following form:

$$\min J(u,w) \quad \text{subject to } e(u,w) = 0.$$

Theorem 2. *Assume that $Bw \in L^2(0,T;H)$, $u_0 \in V$, then there exists an optimal control solution (u^*, w^*) to problem* (5).

Proof. Suppose (u,w) satisfies the equation $e(u,w) = 0$. In view of (4), we get

$$J(u,w) \geqslant \frac{\delta}{2}\|w\|^2_{L^2(Q_0)}.$$

By Corollary 1, we obtain

$$\|u\|_{W(0,T;V)} \to \infty \quad \text{yields} \quad \|w\|_{L^2(Q_0)} \to \infty.$$

Therefore,

$$J(u,w) \to \infty \quad \text{when } \|(u,w)\|_X \to \infty. \tag{17}$$

As the norm is weakly lower semi-continuous, we achieve that J is weakly lower semi-continuous. Since for all $(u,w) \in X$, $J(u,w) \geqslant 0$, there exists $\lambda \geqslant 0$ defined by

$$\lambda = \inf\bigl\{J(u,w) \bigm| (u,w) \in X,\ e(u,w) = 0\bigr\},$$

which means the existence of a minimizing sequence $\{(u^n, w^n)\}_{n\in N}$ in X such that

$$\lambda = \lim_{n\to\infty} J(u^n, w^n) \quad \text{and} \quad e\bigl(u^n, w^n\bigr) = 0 \quad \forall n \in N.$$

From (17) there exists an element $(u^*, w^*) \in X$ such that when $n \to \infty$,

$$\begin{aligned} u^n &\to u^* \text{ weakly}, \quad u \in W(0,T;V),\\ w^n &\to w^* \text{ weakly}, \quad w \in L^2(Q_0). \end{aligned} \tag{18}$$

Using (18), we get

$$\lim_{n\to\infty} \int_0^T \bigl(u_t^n(x,t) - u_t^*,\ \psi(t)\bigr)_{V^*,V}\,\mathrm{d}t = 0 \quad \forall \psi \in L^2(0,T;V).$$

Since $W(0,T;V)$ is continuously embedded into $L^2(0,T;L^\infty)$, we have $u^n \to u^*$ strongly in $L^2(0,T;L^\infty)$. On the other hand, we know that $u_n \in L^\infty(0,T;V)$ and

$u_{n,t} \in L^2(0,T;V^*)$. Hence, by [16, Lemma 4] we have $u^n \to u^*$ strongly in $C(0,T;H)$ as $n \to \infty$.

As the sequence $\{u^n\}_{n\in N}$ converges weakly, then $\|u^n\|_{W(0,T;V)}$ is bounded, and $\|u^n\|_{L^2(0,T;L^\infty)}$ is also bounded based on the embedding theorem. Because $u^n \to u^*$ in $L^2(0,T;L^\infty)$ as $n \to \infty$, we know that $\|u^*\|_{L^2(0,T;L^\infty)}$ is bounded too.

Using (18), we deduce that

$$\begin{aligned}
&\left|\int_0^T\int_\Omega \left[\nabla\cdot\left(\frac{\nabla u^n}{1+|\nabla u^n|^2}\right) - \nabla\cdot\left(\frac{\nabla u^*}{1+|\nabla u^*|^2}\right)\right]\eta\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad = \left|\int_0^T\int_\Omega \left(\frac{\nabla u^n}{1+|\nabla u^n|^2} - \frac{\nabla u^*}{1+|\nabla u^*|^2}\right)\nabla\eta\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad = \left|\int_0^T\int_\Omega \frac{(1-\nabla u^n\nabla u^*)(\nabla u^n-\nabla u^*)}{(1+|\nabla u^n|^2)(1+|\nabla u^*|^2)}\nabla\eta\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \leqslant \left|\int_0^T\int_\Omega \frac{\nabla u^n-\nabla u^*}{(1+|\nabla u^n|^2)(1+|\nabla u^*|^2)}\nabla\eta\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad + \left|\int_0^T\int_\Omega \frac{\nabla u^n}{1+|\nabla u^n|^2}\frac{\nabla u^*}{1+|\nabla u^*|^2}\big(\nabla u^n-\nabla u^*\big)\nabla\eta\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \leqslant c\int_0^T \|\nabla u^n-\nabla u^*\|_H\|\nabla\eta\|_H\,\mathrm{d}t \leqslant c\|\nabla u^n-\nabla u^*\|_{L^2(0,T;H)}\|\nabla\eta\|_{L^2(0,T;H)} \\
&\quad \to 0, \quad n\to\infty,\ \forall\eta\in L^2(0,T;V).
\end{aligned}$$

Hence, we have $u = u(w)$ and therefore

$$J(u,w) \leqslant \lim_{n\to\infty} J\big(u^n,w^n\big) = \lambda.$$

In view of the above discussion, we get

$$e_1(u^*,w^*) = 0 \quad \forall n \in N.$$

Noticing that $u^* \in W(0,T;V)$, we derive that $u^*(0) \in H$. Since $u^n \to u^*$ weakly in $W(0,T;V)$, we can infer that $u^n(0) \to u^*(0)$ weakly when $n \to \infty$. Thus, we obtain

$$\big(u^n(0) - u^*(0),\, \eta\big) \to 0, \quad n\to\infty,\ \forall\eta\in H,$$

which means $e_2(u^*,w^*) = 0$. Therefore, we obtain

$$e(u^*,w^*) = 0 \quad \text{in } Y.$$

So, there exists an optimal solution (u^*,w^*) to problem (5). Then the proof of Theorem 2 is completed. □

4 Optimality conditions

It is well known that the optimality conditions for w is given by the variational inequality

$$J'(u,w)(v-w) \geqslant 0 \quad \forall v \in L^2(Q_0), \tag{19}$$

where $J'(u,w)$ denotes the Gateaux derivative of $J(u,v)$ at $v=w$.

The following lemma is essential in deriving necessary optimality conditions.

Lemma 1. *The map $v \to u(v)$ of $L^2(Q_0)$ into $W(0,T;V)$ is weakly Gateaux differentiable at $v=w$ and such the Gateaux derivative of $u(v)$ at $v=w$ in the direction $v-w \in L^2(Q_0)$, say $z = \mathcal{D}u(w)(v-w)$, is a unique weak solution of the following problem*:

$$\begin{aligned} & z_t + a\Delta^2 z + \mu\nabla\cdot\left(\frac{(1-|\nabla u|^2)}{(1+|\nabla u|^2)^2}\nabla z\right) = B(v-w), \\ & 0<t\leqslant T,\ x\in\Omega, \\ & \frac{\partial z}{\partial n}\bigg|_{\partial\Omega} = \frac{\partial \Delta z}{\partial n}\bigg|_{\partial\Omega} = 0, \qquad z(0)=0, \quad x\in\Omega. \end{aligned} \tag{20}$$

Proof. Let $0\leqslant h\leqslant 1$, u_h and u be the solutions of (5) corresponding to $w+h(v-w)$ and w, respectively. We prove the lemma in the following two steps:

Step 1. We prove $u_h \to u$ strongly in $C(0,T;H^1(\Omega))$ as $h\to 0$. Let $q = u_h - u$, then

$$\begin{aligned} & \frac{\mathrm{d}q}{\mathrm{d}t} + a\Delta q + \mu\nabla\cdot\left(\frac{\nabla u_h}{1+|\nabla u_h|^2} - \frac{\nabla u}{1+|\nabla u|^2}\right) = hB(v-w), \\ & 0<t\leqslant T,\ x\in\Omega, \\ & \frac{\partial q}{\partial n}\bigg|_{\partial\Omega} = \frac{\partial \Delta q}{\partial n}\bigg|_{\partial\Omega} = 0, \qquad q(0)=0, \quad x\in\Omega. \end{aligned} \tag{21}$$

Taking the scalar product of (21) with q, we have

$$\begin{aligned} & \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|^2 + a\|\Delta q\|^2 \\ & \quad = \mu\left(\frac{\nabla u_h}{1+|\nabla u_h|^2} - \frac{\nabla u}{1+|\nabla u|^2},\ \nabla q\right) + \big(hB(v-w),\ q\big) \\ & \quad = \mu\left(\frac{1-\nabla u\nabla u_h}{(1+|\nabla u_h|^2)(1+|\nabla u|^2)}\nabla q,\ \nabla q\right) + \big(hB(v-w),\ q\big) \\ & \quad \leqslant \mu\|\nabla q\|^2 + \mu\left\|\frac{\nabla u}{1+|\nabla u|^2}\frac{\nabla u_h}{1+|\nabla u_h|^2}\right\|_\infty \|\nabla q\|^2 + h\big\|B(v-w)\big\|\|q\| \\ & \quad \leqslant c_0\|\nabla q\|^2 + h\big\|B(v-w)\big\|\|q\| \\ & \quad \leqslant \frac{a}{2}\|\Delta q\|^2 + \left(\frac{c_0^2}{2a} + \frac{1}{4}\right)\|q\|^2 + h^2\big\|B(v-w)\big\|^2. \end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|q\|^2 + a\|\Delta q\|^2 \leqslant \left(\frac{c_0^2}{a} + \frac{1}{2}\right)\|q\|^2 + 2h^2\big\|B(v-w)\big\|^2.$$

Using Gronwall's inequality, it is easy to see that $\|q\|^2 \to 0$ as $h \to 0$. Then $u_h \to u$ strongly in $C(0,T;H)$ as $h \to 0$.

Taking the scalar product of (21) with $-\Delta q$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q\|^2 + a\|\nabla\Delta q\|^2 \\
&\quad = -\mu\left(\frac{\nabla u_h}{1+|\nabla u_h|^2} - \frac{\nabla u}{1+|\nabla u|^2},\ \nabla\Delta q\right) - \big(hB(v-w),\ \Delta q\big) \\
&\quad \leqslant c'\|\nabla q\|\|\nabla\Delta q\| + h\big\|B(v-w)\big\|\|\Delta q\| \\
&\quad \leqslant \frac{a}{2}\|\nabla\Delta q\|^2 + c''\|\nabla q\|^2 + h^2\big\|B(v-w)\big\|^2.
\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla q\|^2 + a\|\nabla\Delta q\|^2 \leqslant 2c''\|\nabla q\|^2 + 2h^2\big\|B(v-w)\big\|^2.$$

Using Gronwall's inequality, it is easy to see that $\|\nabla q\|^2 \to 0$ as $h \to 0$. Then $\nabla u_h \to \nabla u$ strongly in $C(0,T;H)$ as $h \to 0$.

Hence, $u_h \to u$ strongly in $C(0,T;H^1(\Omega))$ as $h \to 0$.

Step 2. We prove that $(u_h - u)/h \to z$ strongly in $W(0,T;V)$. Rewrite (21) in the following form:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u_h-u}{h}\right) + a\Delta^2\left(\frac{u_h-u}{h}\right) + \frac{\mu}{h}\nabla\cdot\left(\frac{\nabla u_h}{1+|\nabla u_h|^2} - \frac{\nabla u}{1+|\nabla u|^2}\right) \\
&\quad = B(v-w), \quad 0 < t \leqslant T, \\
&\frac{\partial}{\partial n}\left(\frac{u_h-u}{h}\right)_{\partial\Omega} = \frac{\partial}{\partial n}\left(\Delta\frac{u_h-u}{h}\right)_{\partial\Omega} = 0, \qquad \frac{u_h-u}{h}(0) = 0, \quad x \in \Omega.
\end{aligned} \tag{22}$$

We can easily verify that the above problem possesses a unique weak solution in $W(0,T;V)$. On the other hand, it is easy to check that the linear problem (20) possesses a unique weak solution $z \in W(0,T;V)$. Let $r = (u_h - u)/h - z$, thus r satisfies

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}r + a\Delta^2 r + \mu\left[\frac{1}{h}\nabla\cdot\left(\frac{\nabla u_h}{1+|\nabla u_h|^2} - \frac{\nabla u}{1+|\nabla - u|^2}\right)\right. \\
&\quad \left. - \nabla\cdot\left(\frac{(1-|\nabla u|^2)\nabla z}{(1+|\nabla u|^2)^2}\right)\right] = 0, \quad 0 < t \leqslant T, \\
&\left.\frac{\partial r}{\partial n}\right|_{\partial\Omega} = \left.\frac{\partial\Delta r}{\partial n}\right|_{\partial\Omega} = 0, \qquad r(0) = 0, \quad x \in \Omega.
\end{aligned} \tag{23}$$

Taking the scalar product of (23) with r, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|r\|^2 + a\|\Delta r\|^2 \\
&\quad = \mu\left(\frac{1}{h}\left(\frac{\nabla u_h}{1+|\nabla u_h|^2} - \frac{\nabla u}{1+|\nabla u|^2}\right) - \frac{(1-|\nabla u|^2)\nabla z}{(1+|\nabla u|^2)^2},\, \nabla r\right) \\
&\quad = \mu\left(\left[\frac{1}{1+|\nabla u+\theta(\nabla u_h-\nabla u)|^2} - \frac{2|\nabla u+\theta(\nabla u_h-\nabla u)|^2}{(1+|\nabla u+\theta(\nabla u_h-\nabla u)|^2)^2}\right]\frac{\nabla u_h-\nabla u}{h}\right. \\
&\qquad \left. - \frac{(1-|\nabla u|^2)\nabla z}{(1+|\nabla u|^2)^2},\, \nabla r\right) \\
&\quad \leqslant \mu\left\|\left[\frac{1}{1+|\nabla u+\theta(\nabla u_h-\nabla u)|^2} - \frac{2|\nabla u+\theta(\nabla u_h-\nabla u)|^2}{(1+|\nabla u+\theta(\nabla u_h-\nabla u)|^2)^2}\right]\frac{\nabla u_h-\nabla u}{h}\right. \\
&\qquad \left. - \frac{(1-|\nabla u|^2)\nabla z}{(1+|\nabla u|^2)^2}\right\|\|\nabla r\|,
\end{aligned}$$

where $\theta \in (0,1)$. Noticing that $u_h \to u$ strongly in $C(0,T;H^1)$ as $h \to 0$, then

$$\begin{aligned}
&\left\|\left[\frac{1}{1+|\nabla u+\theta(\nabla u_h-\nabla u)|^2} - \frac{2|\nabla u+\theta(\nabla u_h-\nabla u)|^2}{(1+|\nabla u+\theta(\nabla u_h-\nabla u)|^2)^2}\right]\frac{\nabla u_h-\nabla u}{h}\right. \\
&\qquad \left. - \frac{(1-|\nabla u|^2)\nabla z}{(1+|\nabla u|^2)^2}\right\|\|\nabla r\| \to \left\|\frac{(1-|\nabla u|^2)}{(1+|\nabla u|^2)^2}\left(\frac{\nabla u_h-\nabla u}{h} - \nabla z\right)\right\|\|\nabla r\| \\
&\quad \leqslant c_0\|\nabla r\|^2 \leqslant \frac{a}{2}\|\Delta r\|^2 + \frac{c_0^2}{2a}\|r\|^2 \quad \text{as } h \to 0.
\end{aligned}$$

Therefore,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|r\|^2 + a\|\Delta r\|^2 \leqslant \frac{c_0^2}{a}\|r\|^2.$$

Using Gronwall's inequality, it is easy to check that $(u_h - u)/h$ is strongly convergent to z in $W(0,T;V)$.

Then Lemma 1 is proved. □

As in [8], we denote $\Lambda =$ canonical isomorphism of S onto S^*, where S^* is the dual space of S. By calculating the Gateaux derivative of (17) via Lemma 1, we see that the cost $J(v)$ is weakly Gateaux differentiable at w in the direction $v - w$. Then (19) can be rewritten as

$$\begin{aligned}
&\big(C^*\Lambda\big(Cu(w) - z_d\big), z\big)_{W(V)^*,W(V)} \\
&\qquad + \frac{\delta}{2}(w,\, v-w)_{L^2(Q_0)} \geqslant 0 \quad \forall v \in L^2(Q_0),
\end{aligned} \tag{24}$$

where z is the solution of (20).

Now we study the necessary conditions of optimality. To avoid the complexity of observation states, we consider the two types of distributive and terminal value observations.

Case 1: $C \in \mathcal{L}(L^2(0,T;V);S)$.

In this case, $C^* \in \mathcal{L}(S^*;L^2(0,T;V^*))$ and (24) may be written as

$$\int_0^T \big(C^* \Lambda\big(Cu(w) - z_d\big),\, z\big)_{V^*,\,V}\,\mathrm{d}t + \frac{\delta}{2}(w,\, v-w)_{L^2(Q_0)} \geqslant 0 \quad \forall v \in L^2(Q_0). \tag{25}$$

We introduce the adjoint state $p(v)$ by

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}t}p(v) + a\Delta^2 p(v) + \mu\nabla\cdot\left(\frac{(1-|\nabla u|^2)\nabla p}{(1+|\nabla u|^2)^2}\right) \\ &\quad = C^*\Lambda\big(Cu(v) - z_d\big) \quad \text{in } (0,T), \\ &\frac{\partial p}{\partial n}\bigg|_{\partial\Omega} = \frac{\partial \Delta p}{\partial n}\bigg|_{\partial\Omega} = 0, \qquad p(x,T;v) = 0. \end{aligned} \tag{26}$$

According to Theorem 1, the above problem admits a unique solution (after changing t into $T-t$).

Multiplying both sides of (26) (with $v = w$) by z, using Lemma 1, we get

$$\begin{gathered} \int_0^T \left(-\frac{\mathrm{d}}{\mathrm{d}t}p(w), z\right)_{V^*,\,V}\mathrm{d}t = \int_0^T \left(p(w), \frac{\mathrm{d}}{\mathrm{d}t}z\right)\mathrm{d}t, \\ \int_0^T \big(\Delta^2 p(w), z\big)_{V^*,\,V}\,\mathrm{d}t = \int_0^T \big(p(w), \Delta^2 z\big)\,\mathrm{d}t \end{gathered}$$

and

$$\int_0^T \left(\nabla\cdot\frac{(1-|\nabla u|^2)\nabla p(w)}{(1+|\nabla u|^2)^2},\, z\right)_{V^*,\,V}\mathrm{d}t = \int_0^T \left(p(w),\, \nabla\cdot\left(\frac{(1-|\nabla u|^2)\nabla z}{(1+|\nabla u|^2)^2}\right)\right)\mathrm{d}t.$$

Then we obtain

$$\begin{aligned} &\int_0^T \big(C^*\Lambda\big(Cu(w) - z_d\big), z\big)_{V^*,\,V}\,\mathrm{d}t \\ &\quad = \int_0^T \left(p(w),\, z_t + a\Delta^2 z + \mu\nabla\cdot\left(\frac{(1-|\nabla u|^2)\nabla z}{(1+|\nabla u|^2)^2}\right)\right)\mathrm{d}t \\ &\quad = \int_0^T \big(p(w),\, Bv - Bw\big)\,\mathrm{d}t = \big(B^*p(w),\, v-w\big). \end{aligned}$$

Hence, (25) may be written as

$$\int_0^T \int_0^1 B^* p(w)(v-w)\,\mathrm{d}x\,\mathrm{d}t + \frac{\delta}{2}(w,\, v-w)_{L^2(Q_0)} \geqslant 0 \quad \forall v \in L^2(Q_0). \tag{27}$$

Therefore, we have proved the following theorem:

Theorem 3. *We assume that all conditions of Theorem* 2 *hold. Let us suppose that* $C \in \mathcal{L}(L^2(0,T;V);S)$. *The optimal control* w *is characterized by the system of two PDEs and an inequality:* (5), (26) *and* (27).

Case 2: $C \in \mathcal{L}(H;S)$.

In this case, we observe $Cu(v) = Du(T;v)$, $D \in \mathcal{L}(H;H)$. The associated cost function is expressed as

$$J(u,v) = \big\|Du(T;v) - z\big\|_S^2 + \frac{\delta}{2}\|v\|_{L^2(Q_0)}^2 \quad \forall v \in L^2(Q_0). \tag{28}$$

Then, for all $v \in L^2(Q_0)$, the optimal control w for (28) is characterized by

$$\big(Du(T;w) - z,\, Du(T;v) - Du(T;w)\big)_{V^*,\,V} + \frac{\delta}{2}(w,\, v-w)_{L^2(Q_0)} \geqslant 0. \tag{29}$$

We introduce the adjoint state $p(v)$ by

$$\begin{aligned} &-\frac{\mathrm{d}}{\mathrm{d}t}p(v) + a\Delta^2 p(v) + \mu\nabla\cdot\left(\frac{(1-|\nabla u|^2)\nabla p}{(1+|\nabla u|^2)^2}\right) = 0 \quad \text{in } (0,T),\\ &\frac{\partial p}{\partial n}\bigg|_{\partial\Omega} = \frac{\partial \Delta p}{\partial n}\bigg|_{\partial\Omega} = 0, \qquad p(T;v) = D^*\big(Du(T;v) - z_d\big). \end{aligned} \tag{30}$$

According to Theorem 1, the above problem admits a unique solution (after changing t into $T-t$).

Let us set $v = w$ in the above equations and scalar multiply both side of the first equation of (30) by $u(v) - u(w)$ and integrate from 0 to T. A simple calculation shows that (29) is equivalent to

$$\int_0^T \int_0^1 B^* p(w)(v-w)\,\mathrm{d}x\,\mathrm{d}t + \frac{\delta}{2}(w,\, v-w)_{L^2(Q_0)} \geqslant 0 \quad \forall v \in L^2(Q_0). \tag{31}$$

Then we have the following theorem:

Theorem 4. *We assume that all conditions of Theorem* 2 *hold. Let us suppose that* $D \in \mathcal{L}(H;H)$. *The optimal control* w *is characterized by the system of two PDEs and an inequality*: (5), (30) *and* (31).

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