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New uniqueness results for boundary value problem of fractional differential equation*

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Abstract. In this paper, uniqueness results for boundary value problem of fractional differential equation are obtained. Both the Banach's contraction mapping principle and the theory of linear operator are used, and a comparison between the obtained results is provided.

Keywords: fractional differential equation, uniqueness results, Banach's contraction mapping principle.

1 Introduction

In this paper, we consider the uniqueness of solutions of the following boundary value problems for nonlinear fractional differential equation:

$$\begin{gathered} D_{0+}^{\alpha} u(t) + f\big(t, u(t)\big) = 0, \quad t \in (0,1),\ 1 < \alpha \leqslant 2, \\ u(0) = u(1) = 0. \end{gathered} \tag{1}$$

Boundary value problems for nonlinear fractional differential equation have been investigated extensively. The motivation for those works arises from both the development of the theory of fractional calculus itself and the study of models of viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [5, 7, 8, 16]). In applications, one

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is interested in showing the existence and multiplicity of solution (or positive solution). Consequently, there has been a significant development in the study of boundary value problems of fractional differential equation, see [1, 3, 6, 7, 9, 11, 12, 15]. In [3], the authors considered the existence and multiplicity of positive solutions of BVP (1) by means of the Krasnosel'skii fixed-point theorem and the Leggett–Williams fixed-point theorem. However, there are few works on the uniqueness for boundary value problems of fractional differential equations [2, 4, 10, 13, 14]. In [13], the authors studied the following multi-point boundary value problems of fractional order:

$$\begin{aligned} &D_t^{\alpha} y(t) = f\bigl(t, y(t), D_t^{\beta} y(t)\bigr), \quad t \in (0,1), \\ &y(0) = 0, \qquad D_t^{\beta} y(1) - \sum_{i=1}^{m-2} \varsigma_i D_t^{\beta} y(\xi_i) = y_0, \end{aligned} \tag{2}$$

where $1 < \alpha \leqslant 2$, $0 < \beta < 1$, $0 < \xi_i < 1$ $(i = 1, 2, \dots, m-2)$, $\varsigma_i \geqslant 0$ with $\sum_{i=1}^{m-2} \varsigma_i \xi_i^{\alpha-\beta-1} < 1$, and D_t^{α} denotes the standard Riemann–Liouville fractional derivative. They proved the uniqueness of solutions to BVP (2) by means of the Banach's contraction mapping principle. Recently, the following nonlinear fractional differential equations with two point boundary conditions was also studied by Cui [4]:

$$\begin{aligned} &D^p x(t) + p(t) f\bigl(t, x(t)\bigr) + q(t) = 0, \quad t \in (0,1), \\ &x(0) = x'(0) = 0, \qquad x(1) = 0, \end{aligned}$$

where $2 < p \leqslant 3$ is a real number. By use of u_0-positive operator, a uniqueness result is proved, provided that f is a Lipschitz continuous function. The novelty of [4] is that the Lipschitz constant is related to the first eigenvalue corresponding to the relevant operator.

Motivated by the results mentioned above, in this paper, we study the uniqueness of solutions for BVP (1) based on the Banach's contraction mapping principle and the theory of linear operator. It should be remarked that the method used in [4] is not suitable for BVP (1).

2 Preliminaries and lemmas

For convenience, we present here the necessary definitions from fractional calculus theory. These definitions can be found in the literature.

Definition 1. (See [12].) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, \mathrm{d}s,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2. (See [12]) The Riemann–Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}}\,\mathrm{d}s,$$

where $n-1 \leqslant \alpha < n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Let $C[0,1]$ denote the Banach space of real-valued continuous function with norm $\|u\| = \max_{t\in[0,1]} |u(t)|$.

Lemma 1. (*See* [3].) *Given* $h \in C[0,1]$ *and* $1 < \alpha \leqslant 2$*, the unique solution of*

$$D_{0+}^{\alpha} u(t) + h(t) = 0, \quad t \in (0,1),$$
$$u(0) = u(1) = 0$$

is

$$u(t) = \int_0^1 G(t,s)h(s)\,\mathrm{d}s,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (t(1-s))^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ (t(1-s))^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

By Lemma 1, BVP (1) can be converted into a fixed-point problem $x = Ax$, where $A : C[0,1] \to C[0,1]$ is presented by

$$(Au)(t) = \int_0^1 G(t,s) f\bigl(s, u(s)\bigr)\,\mathrm{d}s.$$

Clearly, BVP (1) has a solution if and only if the associated fixed-point problem $x = Ax$ has a fixed point.

Lemma 2. *Let* $E = \{a \in \mathbb{R}\colon\ a(1-t) \geqslant 1 + t^{\alpha+1} - 2t^{\alpha},\ t \in [0,1]\}$*. Then* $E \neq \emptyset$ *and* $M = \inf E \in [1,\alpha]$.

Proof. By the Lagrange mean value theorem, we have

$$1 + t^{\alpha+1} - 2t^{\alpha} = 1 - t^{\alpha} - t^{\alpha}(1-t) \leqslant 1 - t^{\alpha} = \alpha\xi^{\alpha-1}(1-t)$$
$$\leqslant \alpha(1-t), \quad \xi \in (t,1).$$

This implies that $E \neq \emptyset$ and $M \leqslant \alpha$.

On the other hand, substituting the value $t = 0$ to the inequality $a(1-t) \geqslant 1 + t^{\alpha+1} - 2t^{\alpha}$, we get $a \geqslant 1$ and $M \geqslant 1$. This completes the proof. □

3 Main results

We first prove a uniqueness result based on the Banach's contraction mapping principle.

Theorem 1. *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a constant $L > 0$ such that*

$$\left|f(t,x) - f(t,y)\right| \leqslant L|x-y|, \quad t \in [0,1],\ x, y \in \mathbb{R}.$$

If $L < \alpha^2\Gamma(\alpha)((\alpha-1)/\alpha)^{1-\alpha}$, then BVP (1) *has a unique solution in $C[0,1]$.*

Proof. We now show that A is a contraction on $C[0,1]$. To see this, let $u, v \in C[0,1]$ and notice

$$\begin{aligned}
\Gamma(\alpha)\int_0^1 G(t,s)\,\mathrm{d}s &= \int_0^1 \bigl(t(1-s)\bigr)^{\alpha-1}\,\mathrm{d}s - \int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s \\
&= t^{\alpha-1}B(\alpha,1) - t^{\alpha}\int_0^1 (1-s)^{\alpha-1}\,\mathrm{d}s \\
&= t^{\alpha-1}B(\alpha,1) - t^{\alpha}B(\alpha,1) = \frac{1}{\alpha}\bigl(t^{\alpha-1} - t^{\alpha}\bigr) \\
&\leqslant \frac{1}{\alpha}\bigl(t^{\alpha-1} - t^{\alpha}\bigr)\Big|_{t=(\alpha-1)/\alpha} = \frac{1}{\alpha^2}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}.
\end{aligned} \tag{3}$$

Thus, for $t \in [0,1]$,

$$\begin{aligned}
\bigl|(Au)(t) - (Av)(t)\bigr| &= \left|\int_0^1 G(t,s)\bigl(f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr)\,\mathrm{d}s\right| \\
&\leqslant \int_0^1 G(t,s)\bigl|f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr|\,\mathrm{d}s \\
&\leqslant L\int_0^1 G(t,s)\bigl|u(s) - v(s)\bigr|\,\mathrm{d}s \leqslant L\int_0^1 G(t,s)\,\mathrm{d}s\|u-v\| \\
&\leqslant \frac{L}{\Gamma(\alpha)\alpha^2}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\|u-v\|
\end{aligned}$$

and therefore

$$\|Au - Av\| \leqslant \frac{L}{\Gamma(\alpha)\alpha^2}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\|u-v\|.$$

Since $L < \Gamma(\alpha)\alpha^2((\alpha-1)/\alpha)^{1-\alpha}$, the Banach's contraction mapping principle implies that there is a unique u with $u = Au$, equivalently, BVP (1) has a unique solution $u \in C[0,1]$. The proof is completed. □

Next, we prove a uniqueness result by means of the theory of linear operator.

Theorem 2. *Suppose that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and there exists a constant $L > 0$ such that*

$$\bigl|f(t,x) - f(t,y)\bigr| \leqslant L|x-y|, \quad t \in [0,1],\ x,y \in \mathbb{R}.$$

If $L < 2\Gamma(2\alpha)/(M\Gamma(\alpha))$, then BVP (1) *has a unique solution in $C[0,1]$.*

Proof. We define an operator T on $C[0,1]$ by

$$(Tu)(t) = \int_0^1 G(t,s)u(s)\,\mathrm{d}s, \quad u \in C[0,1].$$

In the following, we separate the proof into the following four steps.

Step 1. For any given $u \in C[0,1]$ with $u(t) \geqslant 0$ ($t \in [0,1]$), there is a constant $N = N(u)$ such that

$$(Tu)(t) \leqslant Nu_0(t), \quad t \in [0,1],$$

where $u_0(t) = t^{\alpha-1}(1-t)$. In fact, it follows from (3) that

$$\begin{aligned}\Gamma(\alpha)\int_0^1 G(t,s)u(s)\,\mathrm{d}s &\leqslant \left(\int_0^1 \bigl(t(1-s)\bigr)^{\alpha-1}\,\mathrm{d}s - \int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s\right)\|u\| \\ &\leqslant \frac{\|u\|}{\alpha}\bigl(t^{\alpha-1} - t^{\alpha}\bigr) = \frac{\|u\|}{\alpha}u_0(t), \quad t \in [0,1].\end{aligned}$$

Step 2. $(Tu_0)(t) \leqslant \beta u_0(t)$, where $\beta = M\Gamma(\alpha)/(2\Gamma(2\alpha))$. This can be obtained by the following inequality:

$$\begin{aligned}&\Gamma(\alpha)(Tu_0)(t) \\ &\quad= \Gamma(\alpha)\int_0^1 G(t,s)s^{\alpha-1}(1-s)\,\mathrm{d}s \\ &\quad= \int_0^1 \bigl(t(1-s)\bigr)^{\alpha-1}s^{\alpha-1}(1-s)\,\mathrm{d}s - \int_0^t (t-s)^{\alpha-1}s^{\alpha-1}(1-s)\,\mathrm{d}s \\ &\quad= t^{\alpha-1}B(\alpha+1,\,\alpha) - \int_0^t (t-s)^{\alpha-1}s^{\alpha-1}\,\mathrm{d}s + \int_0^t (t-s)^{\alpha-1}s^{\alpha}\,\mathrm{d}s \\ &\quad= t^{\alpha-1}B(\alpha+1,\,\alpha) - t^{2\alpha-1}\int_0^1 (1-s)^{\alpha-1}s^{\alpha-1}\,\mathrm{d}s + t^{2\alpha}\int_0^1 (1-s)^{\alpha-1}s^{\alpha}\,\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= t^{\alpha-1}B(\alpha+1,\,\alpha) - t^{2\alpha-1}B(\alpha,\alpha) + t^{2\alpha}B(\alpha,\,\alpha+1)\\
&= t^{\alpha-1}\frac{\Gamma(\alpha+1)\Gamma(\alpha)}{\Gamma(2\alpha+1)} - t^{2\alpha-1}\frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)} + t^{2\alpha}\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\\
&= t^{\alpha-1}\frac{\alpha\Gamma(\alpha)\Gamma(\alpha)}{2\alpha\Gamma(2\alpha)} - t^{2\alpha-1}\frac{\Gamma(\alpha)\Gamma(\alpha)}{\Gamma(2\alpha)} + t^{2\alpha}\frac{\alpha\Gamma(\alpha)\Gamma(\alpha)}{2\alpha\Gamma(2\alpha)}\\
&= t^{\alpha-1}\frac{B(\alpha,\alpha)}{2}\bigl(1+t^{\alpha+1}-2t^{\alpha}\bigr)\\
&\leqslant \frac{MB(\alpha,\alpha)}{2}(1-t)t^{\alpha-1} = \frac{M\Gamma(\alpha)\Gamma(\alpha)}{2\Gamma(2\alpha)}u_0(t), \quad t\in[0,1].
\end{aligned}
$$

Step 3. To show the existence of the solution, select $v_1 \in C[0,1]$. Let

$$
v_{n+1} = Av_n, \quad w_n(t) = \bigl|v_{n+1}(t) - v_n(t)\bigr|, \quad n = 1,2,\dots.
$$

Notice for $n \in \{1,2,\dots\}$ that

$$
\begin{aligned}
w_n(t) &= \bigl|v_{n+1}(t) - v_n(t)\bigr| = \bigl|(Av_n)(t) - (Av_{n-1})(t)\bigr|\\
&\leqslant \int_0^1 G(t,s)\bigl|f\bigl(s,v_n(s)\bigr) - f\bigl(s,v_{n-1}(s)\bigr)\bigr|\,\mathrm{d}s\\
&\leqslant L\int_0^1 G(t,s)\bigl|v_n(s) - v_{n-1}(s)\bigr|\,\mathrm{d}s\\
&= L(Tw_{n-1})(t), \quad t\in[0,1].
\end{aligned}
$$

Then by Steps 1 and 2, we have

$$
\begin{aligned}
w_n(t) &\leqslant L(Tw_{n-1})(t) \leqslant \cdots \leqslant L^{n-1}\bigl(T^{n-1}w_1\bigr)(t) \leqslant N(w_1)L^{n-1}\bigl(T^{n-2}u_0\bigr)(t)\\
&\leqslant N(w_1)L^{n-1}\beta^{n-2}u_0(t), \quad t\in[0,1].
\end{aligned}
$$

Thus, for $m > n$ where $n \in \{1,2,\dots\}$,

$$
\begin{aligned}
\bigl|v_m(t) - v_n(t)\bigr| &\leqslant \bigl|v_m(t) - v_{m-1}(t)\bigr| + \cdots + \bigl|v_{n+1}(t) - v_n(t)\bigr|\\
&= w_{m-1}(t) + \cdots + w_n(t)\\
&\leqslant N(w_1)L^{m-2}\beta^{m-3}u_0(t) + \cdots + N(w_1)L^{n-1}\beta^{n-2}u_0(t)\\
&\leqslant N(w_1)L^{n-1}\beta^{n-2}u_0(t)\bigl(1 + L\beta + (L\beta)^2 + \cdots\bigr)\\
&= N(w_1)L^{n-1}\beta^{n-2}u_0(t)\frac{1}{1-L\beta}, \quad t\in[0,1].
\end{aligned}
$$

That is, for $m > n$ where $n \in \{1,2,\dots\}$,

$$
\bigl|v_m(t) - v_n(t)\bigr| \leqslant N(w_1)L^{n-1}\beta^{n-2}\frac{1}{1-L\beta}. \tag{4}
$$

This shows that $\{v_n\}$ is a Cauchy sequence, and since $C[0,1]$ is complete, there exists $v^* \in C[0,1]$ with $\lim_{n\to\infty} v_n = v^*$. Moreover, the continuity of A yields

$$v^* = \lim_{n\to\infty} v_n = \lim_{n\to\infty} A(v_{n-1}) = Av^*,$$

therefore, v^* is a fixed point of A. Finally, letting $m \to \infty$ in (4) implies

$$\|v^* - v_n\| \leqslant N(w_1)L^{n-1}\beta^{n-2}\frac{1}{1-L\beta}.$$

Step 4. We show the uniqueness of the solution. Suppose that there exist $u, v \in C[0,1]$ with $u = Au$ and $v = Av$. Similarly to the proof of Step 3, we obtain

$$\begin{aligned}
\bigl|u(t) - v(t)\bigr| &\leqslant \bigl|\bigl(A^n u\bigr)(t) - \bigl(A^n v\bigr)(t)\bigr| \\
&\leqslant \int_0^1 G(t,s)\bigl|f\bigl(s, \bigl(A^{n-1}u\bigr)(s)\bigr) - f\bigl(s, \bigl(A^{n-1}v\bigr)(s)\bigr)\bigr| \, \mathrm{d}s \\
&\leqslant L\int_0^1 G(t,s)\bigl|\bigl(A^{n-1}u\bigr)(s) - \bigl(A^{n-1}v\bigr)(s)\bigr| \, \mathrm{d}s \\
&= LT\bigl(\bigl|A^{n-1}u - A^{n-1}v\bigr|\bigr)(t) \\
&\leqslant \cdots \leqslant L^n T^n\bigl(|u-v|\bigr)(t) \leqslant N\bigl(|u-v|\bigr)L^n\bigl(T^{n-1}u_0\bigr)(t) \\
&\leqslant N\bigl(|u-v|\bigr)L^n\beta^{n-1}u_0(t) \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Therefore, $u = v$. This completes the proof. □

From the proof of Theorems 1 and 2 we have

$$\begin{aligned}
&t^{\alpha-1}\frac{B(\alpha,\alpha)}{2}\bigl(1 + t^{\alpha+1} - 2t^\alpha\bigr) \\
&\quad = \Gamma(\alpha)(Tu_0)(t) \leqslant \Gamma(\alpha)\int_0^1 G(t,s)\, \mathrm{d}s \|u_0\| = \frac{1}{\alpha}\bigl(t^{\alpha-1} - t^\alpha\bigr)\|u_0\| \\
&\quad = \frac{1}{\alpha}\bigl(t^{\alpha-1} - t^\alpha\bigr)\frac{1}{\alpha}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1} = \frac{1}{\alpha^2}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1}\bigl(t^{\alpha-1} - t^\alpha\bigr).
\end{aligned}$$

Using this and Lemma 2, we get

$$\frac{\Gamma(\alpha)M}{2\Gamma(2\alpha)} \leqslant \frac{1}{\alpha^2\Gamma(\alpha)}\left(\frac{\alpha-1}{\alpha}\right)^{\alpha-1},$$

that is,

$$\alpha^2\Gamma(\alpha)\left(\frac{\alpha-1}{\alpha}\right)^{1-\alpha} \leqslant \frac{2\Gamma(2\alpha)}{\Gamma(\alpha)M}.$$

This implies that Theorem 2 provides the same results with weaker conditions. In order to apply the above results, one needs to compute some of the following three values:

$$M, \quad \alpha^2\Gamma(\alpha)\left(\frac{\alpha-1}{\alpha}\right)^{1-\alpha}, \quad \frac{2\Gamma(2\alpha)}{\Gamma(\alpha)M}.$$

When $\alpha = 2$, three values are $5/4$, 8 and $48/5$, respectively; if $\alpha = 3/2$, three values are $M \approx 1.0507$, 3.454 and 4.296, respectively.

Similarly to the proof of Theorem 2, we can obtain the uniqueness result of nonnegative solution for BVP (1).

Theorem 3. *Suppose that $f : [0,1] \times \mathbb{R} \to [0,+\infty)$ is a continuous function and there exists a constant $L > 0$ such that*

$$\left|f(t,x) - f(t,y)\right| \leqslant L|x-y|, \quad t \in [0,1],\ x,y \in [0,+\infty).$$

If $L < 2\Gamma(2\alpha)/(M\Gamma(\alpha))$, then BVP (1) has a unique nonnegative solution in $C[0,1]$.

References

1. R.P. Agarwal, M. Benchohra, S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions, *Acta Appl. Math.*, **109**(3):973–1033, 2010.

2. B. Ahmad, A. Alsaedi, Existence and uniqueness of solutions for coupled systems of higher-order nonlinear fractional differential equations, *Fixed Point Theory Appl.*, **2010**:364560, 2010.

3. Z. Bai, H. Lü, Positive solutions for boundary value problems of nonlinear fractional differential equation, *J. Math. Anal. Appl.*, **311**(2):495–505, 2005.

4. Y. Cui, Uniqueness of solution for boundary value problems for fractional differential equations, *Appl. Math. Lett.*, **51**:48–54, 2016.

5. K. Diethelm, A.D. Freed, On the solution of nonlinear fractional order differential equation used in the modeling of viscoplasticity, in F. Keil, W. Maskens, H. Voss, J. Werther (Eds.), *Scientific Computing in Chemical Engineering II. Computational Fluid Dynamics and Molecular Properties*, Springer, Berlin, Heideberg, 1999, pp. 217–224.

6. L. Guo, L. Liu, Y. Wu, Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions, *Nonlinear Anal. Model. Control*, **21**(5):635–650, 2016.

7. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud., Vol. 204, Elsevier, Amsterdam, 2006.

8. W. Lin, Global existence theory and chaos control of fractional differential equation, *J. Math. Anal. Appl.*, **332**(1):709–726, 2007.

9. L. Liu, F. Sun, X. Zhang, Y. Wu, Bifurcation analysis for a singular differential system with two parameters via to degree theory, *Nonlinear Anal. Model. Control*, **22**(1):31–50, 2017.

10. L. Liu, X. Zhang, J. Jiang, Y. Wu, The unique solution of a class of sum mixed monotone operator equations and its application to fractional boundary value problems, *J. Nonlinear Sci. Appl.*, **9**(5):2943–2958, 2016.

11. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equation*, Wiley, New York, 1993.

12. I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*, Academic Press, New York, 1999.

13. M.U. Rehman, R.A. Khan, Existence and uniqueness of solutions for multi-point boundary value problems for fractional differential equations, *Appl. Math. Lett.*, **23**(9):1038–1044, 2010.

14. X. Zhang, L. Liu, Y. Wu, The uniqueness of positive solution for a singular fractional differential system involving derivatives, *Commun. Nonlinear Sci.*, **18**(6):1400–1409, 2013.

15. X. Zhang, L. Liu, Y. Wu, B. Wiwatanapataphee, The spectral analysis for a singular fractional differential equation with a signed measure, *Appl. Math. Comput.*, **257**:252–263, 2015.

16. Y. Zhou, F. Jiao, Nonlocal cauchy problem for fractional evolution equations, *Nonlinear Anal., Theory Methods Appl.*, **11**(5):4465–4475, 2010.