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Impulsive mean square exponential synchronization of stochastic dynamical networks with hybrid time-varying delays*

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Abstract. This paper investigates the mean square exponential synchronization problem for complex dynamical networks with stochastic disturbances and hybrid time-varying delays, both internal delay and coupling delay are considered in the model. At the same time, the coupled time-delay is also probabilistic in two time interval. Impulsive control method is applied to force all nodes synchronize to a chaotic orbit, and impulsive input delay is also taken into account. Based on the theory of stochastic differential equation, an impulsive differential inequality, and some analysis techniques, several simple and useful criteria are derived to ensure mean square exponential synchronization of the stochastic dynamical networks. Furthermore, pinning impulsive strategy is studied. An effective method is introduced to select the controlled nodes at each impulsive constants. Numerical simulations are exploited to demonstrate the effectiveness of the theory results in this paper.

Keywords: stochastic dynamical networks, hybrid time-delay, impulsive input delay, probabilistic time-delay.

1 Introduction

A complex network consists of some nodes and edges, which provides complex systems in real world with great convenience. Complex networks, such as neural networks, genetic regulatory networks, traffic networks, social networks, etc., have been infiltrated into science, social, biology, and so on [5, 17, 20, 36]. In past decades, complex networks have attracted many people from various fields, among which, the synchronization behavior has lots of applications in communication system, distributed real-time systems, pattern recognition, and so on [2, 11, 13, 18, 19, 32]. Consequently, synchronization of complex networks aroused people's great interest.

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Note that some networks cannot be synchronized through time and space evolution by their intrinsic structure. Synchronization control problem has been a hot research topic in recently years. One can divide control strategies into two variants: continuous control and discontinuous control. Compared with continuous control method, discontinuous control can reduce transmit data in the network. As an important discontinuous control method, impulsive control just add controlled quantity at some discrete moments, which got a wide application. Some results have also been obtained for impulsive synchronization. In [35], the function projective synchronization for a class of time-delay chaotic system via impulsive control method has been studied. The problem of impulsive synchronization for discrete-time delayed neural networks has been investigated in [7]. Synchronization of complex networks via impulsive pinning control has been described in [25]. More results can be found in [10, 21, 28, 34].

However, input delay has not been considered in above impulsive control results. Actually, for a complex network under impulsive control, it is significant to take time delay into impulsive controllers when modeling the transmission of signals, especially in a networked environment. There were just few results about impulsive models, which considered impulsive delay. In [31], the stochastic synchronization problem has been studied for a class of delayed dynamical networks under delayed impulsive control. Global exponential synchronization of nonlinear time-delay Lur'e systems via delayed impulsive control has been investigated in [6]. A delayed impulsive control strategy is introduced for synchronization of discrete-time complex networks with distributed delays in [14]. More recently, exponential synchronization of coupled stochastic memristor-based neural networks with impulsive delay has been discussed in [1].

For many networks in real world, all nodes' dynamics would be effected by time-delay. Furthermore, communication delays among a network are also needed to be considered. There are some results, which have take transmission delay and inner delay into the dynamical networks models, one can see [3, 9, 16, 26]. Most of which have identical transmission delay for all nodes. In a real-world signal transmission process, however, the delay may affect both the nodes own state and neighbors state, and self delay may be different from neighboring delay. Heterogeneous constant delays have been considered in some previous results [22, 27, 30]. Just a few results have been reported on the synchronization of dynamical networks with different transmission time-varying delays and internal time-varying delays, despite their importance in modeling realistic dynamical networks [4]. Time delays in networks are often stochastic, and their probabilistic characteristics can be obtained easily by statistical methods [1, 12, 15, 24]. In real networks, the probability of large delay is often very small. Under these circumstances, it is significant to consider probabilistic time-varying delay.

On the other hand, stochastic disturbances cannot be neglected in real systems [8, 23, 33]. Motivated by the above discussions, this paper is concerned with mean square exponential synchronization of complex dynamical networks with hybrid time-delays and stochastic disturbances. Probabilistic heterogeneous time-varying delays have been studied. Impulsive control has been applied, impulsive input delay also has been considered. Some synchronization criterions have been derived. Then pinning impulsive control strategy has been used, and a specific pinning method has been given. Numerical examples are finally given to demonstrate the effectiveness of the proposed impulsive strategy.

The rest of the paper is organized as follows: In Section 2, we introduce some definitions and some lemmas, which are necessary for presenting our results in the following. The main results about synchronization control problem will be presented in Section 3. Then some examples are given to demonstrate the effectiveness of our results in Section 4. Conclusions are finally drawn in Section 5.

Notations. Let $\mathbb{R}$ be the set of real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n_1\times n_2}$ refer to the n-dimensional real vector and $n_1 \times n_2$ real matrices. The superscript "T" denotes matrix transposition. I_n denotes the n-dimensional identify matrix. For a vector $x \in \mathbb{R}^n$, $\|x\|$ is defined as $\|x\| = \sqrt{x^{\mathrm{T}}x}$. For $P \in \mathbb{R}^{n\times n}$, $\lambda_{\max}(P)$ represents the maximum eigenvalue of P. $\otimes$ denotes Kronecker product. $\mathbf{P}\{\cdot\}$ is the probability of the event $\{\cdot\}$, and the expected value operator is $\mathbf{E}\{\cdot\}$. $\mathbb{C}([-\hat{\tau}, 0]; \mathbb{R}^n)$ denotes the family of piecewise continuous functions from $[-\hat{\tau}, 0]$ to $\mathbb{R}^n$. Moreover, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$ be complete probability space with filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e., the filtration contains all $\mathbf{P}$-null sets and is right continuous).

2 Preliminaries and problem formulation

Consider a network consisting of N nodes, and let $x_i(t)$ be the state of the ith node. The dynamics of each node is defined as follows:

$$\begin{aligned} \mathrm{d}x_i(t) = \Bigg(& f\big(t, x_i(t), x_i\big(t - \tau_0(t)\big)\big) + c_1 \sum_{j=1,\, j\neq i}^{N} a_{ij}\Gamma_1\big(x_j(t) - x_i(t)\big) \\ & + c_2 \sum_{j=1, j\neq i}^{N} b_{ij}\Gamma_2\big(x_j\big(t - \tau_1(t)\big) - x_i\big(t - \tau_2(t)\big)\big)\Bigg)\,\mathrm{d}t \\ & + g\big(t, x_i(t), x_i\big(t - \tau_0(t)\big)\big)\,\mathrm{d}\omega(t). \end{aligned} \tag{1}$$

$x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^{\mathrm{T}} \in \mathbb{R}^n$ $(i = 1, 2, \dots, N)$ is the state variables of the ith node. $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuously vector value function. $\tau_0(t) \in [0, \tau_0]$ represents the internal delay occurring inside the node; $\tau_1(t) \in [0, \tau_1]$, $\tau_2(t) \in [0, \tau_2]$ denote the transmission delay for signal sent from jth node to ith node; here τ_0, τ_1, τ_2 are known constants. $\omega(t)$ is an n-dimensional Wiener process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbf{P})$, and $g : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \times\mathbb{R}^{n\times n}$ is noise function matrix. $\Gamma_1, \Gamma_2 \in \mathbb{R}^{n\times n}$ are diagonal matrices with positive diagonal elements. $A = (a_{ij}) \in \mathbb{R}^{N\times N}$ and $B = (b_{ij}) \in \mathbb{R}^{N\times N}$ are the weight configuration matrices with $a_{ij} \geqslant 0$ and $b_{ij} \geqslant 0$ when $i \neq j$. The diagonal elements of matrix A and B are defined by

$$a_{ii} = -\sum_{j=1,j\neq i}^{N} a_{ij}, \qquad b_{ii} = -\sum_{j=1,j\neq i}^{N} b_{ij}. \tag{2}$$

The initial values associated with system (1) are $x_i(t) = \phi_i(t) \in \mathbb{C}([-\hat{\tau}, 0]; \mathbb{R}^n)$ $(i = 1, 2, \dots, N)$, where $\hat{\tau} = \max\{\tau_0, \tau_1, \tau_2\}$.

The transmission time-varying delays $\tau_k(t)$ $(k = 1, 2)$ satisfy the following assumptions:

Assumption 1. The $\tau_k(t)$ $(k = 1, 2)$ obey the following probability distribution: $\mathbf{P}\{0 \leqslant \tau_k(t) \leqslant \tau_k^l(t)\} = \beta_k$, $\mathbf{P}\{\tau_k^l(t) < \tau_k(t) \leqslant \tau_k^u(t)\} = 1 - \beta_k$, where $0 \leqslant \beta_k \leqslant 1$ are constants.

Then define a random variable $\beta_k(t)$ as follows:

$$\beta_k(t) = \begin{cases} 1, & 0 \leqslant \tau_k(t) \leqslant \tau_k^l(t), \\ 0, & \tau_k^l(t) < \tau_k(t) \leqslant \tau_k^u(t). \end{cases}$$

According above assumption, we have

$$\begin{aligned} &\mathbf{P}\{\beta_k(t) = 1\} = \mathbf{P}\{0 \leqslant \tau_k(t) \leqslant \tau_k^l(t)\} = \mathbf{E}\{\beta_k(t)\} = \beta_k, \\ &\mathbf{P}\{\beta_k(t) = 0\} = \mathbf{P}\{\tau_k^l(t) < \tau_k(t) \leqslant \tau_k^u(t)\} = \mathbf{E}\{1 - \beta_k(t)\} = 1 - \beta_k. \end{aligned}$$

Based on the stochastic variables $\beta_k(t)$, system (1) can be rewritten as

$$\begin{aligned} \mathrm{d}x_i(t) = \Bigg(& f\big(t, x_i(t), x_i\big(t - \tau_0(t)\big)\big) + c_1 \sum_{j=1, j\neq i}^{N} a_{ij} \Gamma_1 \big(x_j(t) - x_i(t)\big) \\ & + c_2 \sum_{j=1, j\neq i}^{N} b_{ij} \Gamma_2 \big(\beta_1(t) x_j\big(t - \tau_1^l(t)\big) - \beta_2(t) x_i\big(t - \tau_2^l(t)\big)\big) \\ & + c_2 \sum_{j=1, j\neq i}^{N} b_{ij} \Gamma_2 \big(\big(1 - \beta_1(t)\big) x_j\big(t - \tau_1^u(t)\big) \\ & - \big(1 - \beta_2(t)\big) x_i\big(t - \tau_2^u(t)\big)\big) \Bigg) \mathrm{d}t + g\big(t, x_i(t), x_i\big(t - \tau_0(t)\big)\big) \,\mathrm{d}\omega(t), \end{aligned}$$

where $\tau_k^l(t)$ and $\tau_k^u(t)$ $(k = 1, 2)$ are defined as

$$\tau_k(t) = \begin{cases} \tau_k^l(t), & 0 \leqslant \tau_k(t) \leqslant \tau_k^l(t), \\ \tau_k^u(t), & \tau_k^l(t) < \tau_k(t) \leqslant \tau_k^u(t). \end{cases}$$

After some simple calculation based on (2), one has

$$\begin{aligned} \mathrm{d}x_i(t) = \Bigg(& f\big(t, x_i(t), x_i\big(t - \tau_0(t)\big)\big) + c_1 \sum_{j=1}^{N} a_{ij} \Gamma_1 x_j(t) \\ & + c_2 \beta_1(t) \sum_{j=1}^{N} b_{ij} \Gamma_2 x_j\big(t - \tau_1^l(t)\big) + c_2 \big(1 - \beta_1(t)\big) \sum_{j=1}^{N} b_{ij} \Gamma_2 x_j\big(t - \tau_1^u(t)\big) \\ & - c_2 b_{ii} \Gamma_2 \big(\beta_1(t) x_i\big(t - \tau_1^l(t)\big) - \beta_2(t) x_i\big(t - \tau_2^l(t)\big)\big) \end{aligned}$$

$$- c_2 b_{ii} \Gamma_2 \big(\big(1 - \beta_1(t)\big) x_i\big(t - \tau_1^u(t)\big) - \big(1 - \beta_2(t)\big) x_i\big(t - \tau_2^u(t)\big) \big) \Bigg) \, \mathrm{d}t$$
$$+ g\big(t, x_i(t), x_i\big(t - \tau_0(t)\big)\big) \, \mathrm{d}\omega(t).$$

Remark 1. The model of dynamical networks described by (1) is a generalization of most of exists results. Without the disturbances and probabilistic time-varying delays, the model would degradation to the model in [4]. Let $\tau_1(t) = \tau_2(t) = \tau(t)$, the model would change to the model in [1]. Noting that the transmission delay in network (1) are all time-varying and different from each other, which may lead a different synchronized state, one can see the following analysis.

Definition 1. Synchronization manifold $\mathcal{S} = \{(x_1^{\mathrm{T}}(t), x_2^{\mathrm{T}}(t), \dots, x_N^{\mathrm{T}}(t)) \in \mathbb{R}^{nN} \mid x_i(t) = x_j(t)\}$ for $i, j = 1, 2, \dots, N$, where $x_i(t) = (x_{i1}(t), x_{i2}(t), \dots, x_{in}(t))^{\mathrm{T}} \in \mathbb{R}^n$ ($i = 1, 2, \dots, N$) are the state variables of the ith node.

Once network (1) reaches complete synchronization, i.e., $x_i(t) = s(t)$, $i = 1, 2, \dots, N$, the following synchronized state equation can be derived:

$$\begin{aligned} \mathrm{d}s(t) = {} & \big(f\big(t, s(t), s\big(t - \tau_0(t)\big)\big) - c_2 b_{ii} \Gamma_2 \big(\beta_1(t) s\big(t - \tau_1^l(t)\big) - \beta_2(t) s\big(t - \tau_2^l(t)\big)\big) \\ & - c_2 b_{ii} \Gamma_2 \big(\big(1 - \beta_1(t)\big) s\big(t - \tau_1^u(t)\big) - \big(1 - \beta_2(t)\big) s\big(t - \tau_2^u(t)\big)\big)\big) \, \mathrm{d}t \\ & + g\big(t, s(t), s\big(t - \tau_0(t)\big)\big) \, \mathrm{d}\omega(t). \end{aligned} \tag{3}$$

Note that the synchronized goal is system (3), instead of the isolated node's state. However, system (3) may be nonidentical due to different b_{ii}, $i = 1, 2, \dots, N$. We should assume that $b_{11} = b_{22} = \dots = b_{NN}$, which is difficult to a square matrix B. Fortunately, b_{ii} cannot effect the dynamic of system (1), which just need satisfied (2). In this paper, the method to design matrix B proposed in [4] would be adopted, in which: let $\tilde{b}_{ij} = G_{ij} / \sum_{j=1, j \neq i}^{N} G_{ij}$ when $i \neq j$, where $G = (G_{ij})_{N \times N}$ is any weighted matrix for the network with $G_{ij} > 0$ when there is a connection from node i to node j, otherwise, $G_{ij} = 0$. Then let $b_{ij} = \theta \tilde{b}_{ij}$ when $i \neq j$. It is obviously that $b_{ii} = -\theta$ for $i = 1, 2, \dots, N$. Different θ will cause different synchronized trajectories. Consequently, the synchronized state could be identical and formed as follows:

$$\begin{aligned} \mathrm{d}s(t) = {} & \big(f\big(t, s(t), s\big(t - \tau_0(t)\big)\big) + c_2 \theta \Gamma_2 \big(\beta_1(t) s\big(t - \tau_1^l(t)\big) - \beta_2(t) s\big(t - \tau_2^l(t)\big)\big) \\ & + c_2 \theta \Gamma_2 \big(\big(1 - \beta_1(t)\big) s\big(t - \tau_1^u(t)\big) - \big(1 - \beta_2(t)\big) s\big(t - \tau_2^u(t)\big)\big)\big) \, \mathrm{d}t \\ & + g\big(t, s(t), s\big(t - \tau_0(t)\big)\big) \, \mathrm{d}\omega(t). \end{aligned}$$

Remark 2. When all nodes are synchronized, the nodes cannot force to the isolated orbit determined by $\mathrm{d}s(t) = (f(t, s(t), s(t - \tau_0(t))) + g(t, s(t), s(t - \tau_0(t))) \, \mathrm{d}\omega(t)$ due to the different transmission delay $\tau_1(t)$ and $\tau_2(t)$. A similar situation has been studied in [1], in which the coupled matrix B is assumed to satisfied $b_{11} = b_{22} = \dots = b_{NN}$. It is obviously that B in this paper became less conservative than in [1].

Let $e_i(t) = x_i(t) - s(t)$ be synchronization errors, then, based on (2), the error dynamical system is described by:

$$\begin{aligned}
\mathrm{d}e_i(t) = \Bigg(&\bar{f}\big(t, e_i(t), e_i\big(t-\tau_0(t)\big)\big) + c_1\sum_{j=1}^{N} a_{ij}\Gamma_1 e_j(t)\\
&+ c_2\beta_1(t)\sum_{j=1}^{N} b_{ij}\Gamma_2 e_j\big(t-\tau_1^l(t)\big) + c_2\big(1-\beta_1(t)\big)\sum_{j=1}^{N} b_{ij}\Gamma_2 e_j\big(t-\tau_1^u(t)\big)\\
&+ c_2\theta\Gamma_2\big(\beta_1(t)e_i\big(t-\tau_1^l(t)\big) - \beta_2(t)e_i\big(t-\tau_2^l(t)\big)\big)\\
&+ c_2\theta\Gamma_2\big(\big(1-\beta_1(t)\big)e_i\big(t-\tau_1^u(t)\big) - \big(1-\beta_2(t)\big)e_i\big(t-\tau_2^u(t)\big)\big)\Bigg)\,\mathrm{d}t\\
&+ \bar{g}\big(t, e_i(t), e_i\big(t-\tau_0(t)\big)\big)\,\mathrm{d}\omega(t),
\end{aligned}$$

where $\bar{f}(t, e_i(t), e_i(t-\tau_0(t))) = f(t, x_i(t), x_i(t-\tau_0(t))) - f(t, s(t), s(t-\tau_0(t)))$, $\bar{g}(t, e_i(t), e_i(t-\tau_0(t))) = g(t, x_i(t), x_i(t-\tau_0(t))) - g(t, s(t), s(t-\tau_0(t)))$.

In order to achieve the control objective, the impulsive controllers are designed, then the following errors systems can be derived:

$$\begin{aligned}
\mathrm{d}e_i(t) = \Bigg(&\bar{f}\big(t, e_i(t), e_i\big(t-\tau_0(t)\big)\big) + c_1\sum_{j=1}^{N} a_{ij}\Gamma_1 e_j(t)\\
&+ c_2\beta_1(t)\sum_{j=1}^{N} b_{ij}\Gamma_2 e_j\big(t-\tau_1^l(t)\big)\\
&+ c_2\big(1-\beta_1(t)\big)\sum_{j=1}^{N} b_{ij}\Gamma_2 e_j\big(t-\tau_1^u(t)\big)\\
&+ c_2\theta\Gamma_2\big(\beta_1(t)e_i\big(t-\tau_1^l(t)\big) - \beta_2(t)e_i\big(t-\tau_2^l(t)\big)\big)\\
&+ c_2\theta\Gamma_2\big(\big(1-\beta_1(t)\big)e_i\big(t-\tau_1^u(t)\big) - \big(1-\beta_2(t)\big)e_i\big(t-\tau_2^u(t)\big)\big)\Bigg)\,\mathrm{d}t\\
&+ \bar{g}\big(t, e_i(t), e_i\big(t-\tau_0(t)\big)\big)\,\mathrm{d}\omega(t),\\
\Delta\big(e_i\big(t_k^+\big)\big) = {}& e_i\big(t_k^+\big) - e_i\big(t_k^-\big) = \eta e_i\big(t_k^-\big) + \mu e_i\big(t_k^- - \tau_3\big(t_k^-\big)\big).
\end{aligned} \tag{4}$$

where $\eta \in (-2, 0]$ and $\mu \in (-\infty, 0]$ are constants, which denote the impulsive strength for ith node; the time series $\{t_1, t_2, \dots\}$ is a strictly increasing impulsive instants satisfying $\lim_{k\to\infty} t_k = +\infty$. Without loss of generality, in this paper, we assume that $x_i(t_k^+) = x_i(t_k)$. $\delta(\cdot)$ is the Dirac impulsive function, and $\tau_3(t_k^-)$ is impulsive input time-varying delay at impulsive instant t_k. We assume that $0 \leqslant \tau_3(t_k^-) \leqslant \tau_3$, where τ_3 is a positive scalar.

Remark 3. Impulsive control method has got lots of results, but just few results have considered impulsive input delay. Indeed, the input delay $\tau_3(t_k^-)$ is very meaningful for two reasons in a network environment: Firstly, data from controllers to actuators must have a time delay due to a finite speed of transmission; Secondly, as impulsive controllers, the state $x_i(t_k^+)$ also effected by both $x_i(t_k^-)$ and some state before. Thus, the $\tau_3(t_k^-)$ could effect synchronization performance, at the same time, it may also steer synchronization.

Assumption 2. For the vector-valued function $f(t, x(t), x(t-\tau_1(t)))$, suppose that there exists positive constants $f_1 > 0$ and $f_2 > 0$ such that, for any $x(t), y(t) \in \mathbb{R}^n$,

$$\begin{aligned}&\big(x(t) - y(t)\big)^{\mathrm{T}} \big(f\big(t, x(t), x\big(t - \tau_1(t)\big)\big) - f\big(t, y(t), y\big(t - \tau_1(t)\big)\big)\big)\\ &\quad \leqslant f_1 \big\|x(t) - y(t)\big\|^2 + f_2 \big\|x\big(t - \tau_1(t)\big) - y\big(t - \tau_1(t)\big)\big\|^2.\end{aligned}$$

Assumption 3. Assume that for the noise intensity function matrix g, there exist nonnegative constants g_1 and g_2 such that

$$\begin{aligned}&\operatorname{trace}\big[\big(g(t, x_1, y_1) - g(t, x_1, y_1)\big)^{\mathrm{T}} \big(g(t, x_1, y_1) - g(t, x_1, y_1)\big)\big]\\ &\quad \leqslant g_1 \|x_1 - x_2\|^2 + g_2 \|y_1 - y_2\|^2,\end{aligned}$$

where $x_1, x_2, y_1, y_2 \in \mathbb{R}^n$ and $t \in \mathbb{R}$.

Lemma 1. *If x and y are real matrices with appropriate dimensions, then there exists a positive constant $\varepsilon > 0$ such that*

$$x^{\mathrm{T}} y = y^{\mathrm{T}} x = \frac{1}{2}\big(x^{\mathrm{T}} y + y^{\mathrm{T}} x\big) \leqslant \frac{\varepsilon}{2} x^{\mathrm{T}} x + \frac{1}{2\varepsilon} y^{\mathrm{T}} y.$$

Lemma 2. *For matrices A, B, C and D with appropriate dimensions, the Kronecker product $\otimes$ satisfies*

$$\begin{gathered}(A + B) \otimes C = A \otimes C + B \otimes C; \qquad (A \otimes B)(C \otimes D) = (AC) \otimes (BD);\\ \big(A^{\mathrm{T}} \otimes B^{\mathrm{T}}\big) = (A \otimes B)^{\mathrm{T}}; \qquad \lambda_{\max}(A \otimes B) = \lambda_{\max}(A)\lambda_{\max}(B).\end{gathered}$$

Lemma 3. (*See* [29].) *Consider the following impulsive differential inequalities*:

$$\begin{aligned}&D^+ v(t) \leqslant a v(t) + b_1 \big[v(t)\big]_{\tau_1} + b_2 \big[v(t)\big]_{\tau_2} + \cdots + b_m \big[v(t)\big]_{\tau_m},\\ &\quad t \neq t_k,\ t \geqslant t_0,\\ &v\big(t_k^+\big) \leqslant p_k v\big(t_k^-\big) + q_k^1 \big[v\big(t_k^-\big)\big]_{\tau_1} + q_k^2 \big[v(t_k^-)\big]_{\tau_2} + \cdots + q_k^m \big[v\big(t_k^-\big)\big]_{\tau_m},\\ &\quad k \in \mathbb{N}^+,\\ &v(t) = \varphi(t), \quad t \in [t - \tau, t_0],\end{aligned}$$

where a, b, p_k, q_k^i and τ_i are constants, $i = 1, 2, \dots, m$, and $v(t) \geqslant 0$. $[v(t)]_{\tau_i} = \sup_{t-\tau_i \leqslant s \leqslant t} v(s)$, $[v(t_k)]_{\tau_i} = \sup_{t_k - \tau_i(t_k) \leqslant s \leqslant t_k} v(s)$, $\varphi(t)$ is continous on $[t - \tau, t_0]$, and $v(t)$ is continuous except at t_k, $k \in \mathbb{N}^+$, where there are jump discontinuities. The time series $\{t_1, t_2, \dots\}$ is a strictly increasing impulsive instants satisfying $\lim_{k\to\infty} t_k = +\infty$. Suppose that $p_k + \sum_{i=1}^m q_k^i < 1$ and $a + \sum_{i=1}^m b_i/(p_k + \sum_{j=1}^m q_k^j) + \ln(p_k + \sum_{j=1}^m q_k^j)/(t_{k+1} - t_k) < 0$, then there exist constants $\beta > 1$ and $\lambda > 0$ such that $v(t) \leqslant \|\varphi\|_\tau \beta \mathrm{e}^{\lambda(t-t_0)}$, where $\|\varphi\|_\tau = \sup_{t_0-\tau \leqslant s \leqslant t} \|\varphi(s)\|$ and $\tau = \max_{i=1,2,\dots,m}\{t_i\}$.

3 Main results

In this section, the impulsive control synchronization criterions are derived at first, then, based a effective pinned method, some pinning impulsive criterions are also derived to ensure the mean square exponential synchronization of stochastic dynamical networks (1).

3.1 Impulsive control synchronization

Theorem 1. *Under assumptions mentioned in last section, network* (1) *can be exponentially synchronized to* $s(t)$ *with convergence rate* ρ_2 *if there exist positive constants* $\varepsilon_1 - \varepsilon_6$ *such that*

$$\text{(i)}\ \rho_1+\rho_2<1, \qquad \text{(ii)}\ \zeta_1+\frac{\sum_{k=2}^{6}\zeta_k}{\rho_1+\rho_2}+\frac{\ln(\rho_1+\rho_2)}{t_{k+1}-t_k}<1, \tag{5}$$

where $\rho_1=(1+\eta)^2+\mu(1+\eta)$, $\rho_2=\mu^2+\mu(1+\eta)$, $\zeta_1-\zeta_6$ *are defined as follows*:

$$\begin{aligned}
\zeta_1 &= f_1+\frac{g_1}{2}+\frac{c_2\theta\beta_1\varepsilon_1}{2}+\frac{c_2\theta\beta_2\varepsilon_2}{2}+\frac{c_2\theta(1-\beta_1)\varepsilon_3}{2}+\frac{c_2\theta(1-\beta_2)\varepsilon_4}{2}\\
&\quad+\lambda_{\max}(A\otimes\Gamma_1)+\frac{c_1c_2\beta_1\varepsilon_5}{2}\lambda_{\max}\big(BB^{\mathrm T}\big)\lambda_{\max}\big(\Gamma_2\Gamma_2^{\mathrm T}\big)\\
&\quad+\frac{c_1c_2(1-\beta_1)\varepsilon_6}{2}\lambda_{\max}\big(BB^{\mathrm T}\big)\lambda_{\max}\big(\Gamma_2\Gamma_2^{\mathrm T}\big),\\
\zeta_2 &= f_2+\frac{g_2}{2}, \qquad \zeta_3=\frac{c_2\theta\beta_1\lambda_{\max}(\Gamma_2^{\mathrm T}\Gamma_2)}{2\varepsilon_1}+\frac{c_1c_2\beta_1}{\varepsilon_5},\\
\zeta_4 &= \frac{c_2\theta(1-\beta_1)\lambda_{\max}(\Gamma_2^{\mathrm T}\Gamma_2)}{2\varepsilon_3}+\frac{c_1c_2(1-\beta_1)}{\varepsilon_6},\\
\zeta_5 &= \frac{c_2\theta\beta_2\lambda_{\max}(\Gamma_2^{\mathrm T}\Gamma_2)}{2\varepsilon_2}, \qquad \zeta_6=\frac{c_2\theta(1-\beta_2)\lambda_{\max}(\Gamma_2^{\mathrm T}\Gamma_2)}{2\varepsilon_4}.
\end{aligned}$$

Proof. Consider the following Lyapunov function:

$$V(t)=\frac{1}{2}\sum_{i=1}^{N}e_i^{\mathrm T}(t)e_i(t).$$

Using the weak infinitesimal operator $\mathcal{L}$ on the function $V(t)$ along solution (4) for $t\neq t_k$, one has:

$$\begin{aligned}
\mathcal{L}V(t) &= \sum_{i=1}^{N}e_i^{\mathrm T}(t)\Bigg(\bar f\big(t,e_i(t),e_i\big(t-\tau_0(t)\big)\big)+c_1\sum_{j=1}^{N}a_{ij}\Gamma_1e_j(t)\\
&\quad+c_2\beta_1(t)\sum_{j=1}^{N}b_{ij}\Gamma_2e_j\big(t-\tau_1^l(t)\big)+c_2\big(1-\beta_1(t)\big)\sum_{j=1}^{N}b_{ij}\Gamma_2e_j\big(t-\tau_1^u(t)\big)\\
&\quad+c_2\theta\Gamma_2\big(\beta_1(t)e_i\big(t-\tau_1^l(t)\big)-\beta_2(t)e_i\big(t-\tau_2^l(t)\big)\big)
\end{aligned}$$

$$+ c_2\theta\Gamma_2\big(\big(1-\beta_1(t)\big)e_i\big(t-\tau_1^u(t)\big) - \big(1-\beta_2(t)\big)e_i\big(t-\tau_2^u(t)\big)\big)\bigg)$$
$$+ \frac{1}{2}\sum_{i=1}^{N}\operatorname{trace}\big[\bar g^{\mathrm T}\big(t, e_i(t), e_i\big(t-\tau_0(t)\big)\big)\bar g\big(t, e_i(t), e_i\big(t-\tau_0(t)\big)\big)\big]. \tag{6}$$

According to Assumptions 2 and 3, the following inequalities can be derived:

$$\begin{aligned}
&\sum_{i=1}^{N} e_i^{\mathrm T}(t)\bar f\big(t, e_i(t), e_i\big(t-\tau_0(t)\big)\big)\\
&\quad\leqslant f_1\sum_{i=1}^{N} e_i^{\mathrm T}(t)e_i(t) + f_2\sum_{i=1}^{N} e_i^{\mathrm T}\big(t-\tau_0(t)\big)e_i\big(t-\tau_0(t)\big),\\
&\frac{1}{2}\sum_{i=1}^{N}\operatorname{trace}\big[\bar g^{\mathrm T}\big(t, e_i(t), e_i\big(t-\tau_0(t)\big)\big)\bar g\big(t, e_i(t), e_i\big(t-\tau_0(t)\big)\big)\big]\\
&\quad\leqslant \frac{1}{2}g_1\sum_{i=1}^{N} e_i^{\mathrm T}(t)e_i(t) + \frac{1}{2}g_2\sum_{i=1}^{N} e_i^{\mathrm T}\big(t-\tau_0(t)\big)e_i\big(t-\tau_0(t)\big).
\end{aligned} \tag{7}$$

Based on Lemma 1, for any positive constants ε_1, ε_2, ε_3, ε_4, the following inequalities can be obtained:

$$\begin{aligned}
&c_2\theta\beta_1(t)\sum_{i=1}^{N} e_i^{\mathrm T}(t)\Gamma_2 e_i\big(t-\tau_1^l(t)\big)\\
&\quad\leqslant \frac{c_2\theta\beta_1(t)}{2}\left\{\varepsilon_1\sum_{i=1}^{N} e_i^{\mathrm T}(t)e_i(t) + \frac{1}{\varepsilon_1}\sum_{i=1}^{N} e_i^{\mathrm T}\big(t-\tau_1^l(t)\big)\Gamma_2^{\mathrm T}\Gamma_2 e_i\big(t-\tau_1^l(t)\big)\right\}\\
&\quad\leqslant \frac{c_2\theta\beta_1(t)\varepsilon_1}{2}\sum_{i=1}^{N} e_i^{\mathrm T}(t)e_i(t)\\
&\qquad + \frac{c_2\theta\beta_1(t)\lambda_{\max}(\Gamma_2^{\mathrm T}\Gamma_2)}{2\varepsilon_1}\sum_{i=1}^{N} e_i^{\mathrm T}\big(t-\tau_1^l(t)\big)e_i\big(t-\tau_1^l(t)\big),
\end{aligned} \tag{8}$$

$$\begin{aligned}
&-c_2\theta\beta_2(t)\sum_{i=1}^{N} e_i^{\mathrm T}(t)\Gamma_2 e_i\big(t-\tau_2^l(t)\big)\\
&\quad\leqslant \frac{c_2\theta\beta_2(t)}{2}\left\{\varepsilon_2\sum_{i=1}^{N} e_i^{\mathrm T}(t)e_i(t) + \frac{1}{\varepsilon_2}\sum_{i=1}^{N} e_i^{\mathrm T}\big(t-\tau_2^l(t)\big)\Gamma_2^{\mathrm T}\Gamma_2 e_i\big(t-\tau_2^l(t)\big)\right\}\\
&\quad\leqslant \frac{c_2\theta\beta_2(t)\varepsilon_2}{2}\sum_{i=1}^{N} e_i^{\mathrm T}(t)e_i(t)\\
&\qquad + \frac{c_2\theta\beta_2(t)\lambda_{\max}(\Gamma_2^{\mathrm T}\Gamma_2)}{2\varepsilon_2}\sum_{i=1}^{N} e_i^{\mathrm T}\big(t-\tau_2^l(t)\big)e_i\big(t-\tau_2^l(t)\big),
\end{aligned} \tag{9}$$

$$
\begin{aligned}
&c_2\theta\bigl(1-\beta_1(t)\bigr)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\Gamma_2 e_i\bigl(t-\tau_1^u(t)\bigr)\\
&\quad\leqslant \frac{c_2\theta(1-\beta_1(t))}{2}\left\{\varepsilon_3\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t)+\frac{1}{\varepsilon_3}\sum_{i=1}^{N} e_i^{\mathrm{T}}\bigl(t-\tau_1^u(t)\bigr)\Gamma_2^{\mathrm{T}}\Gamma_2 e_i\bigl(t-\tau_1^u(t)\bigr)\right\}\\
&\quad\leqslant \frac{c_2\theta(1-\beta_1(t))\varepsilon_3}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t)\\
&\qquad+\frac{c_2\theta(1-\beta_1(t))\lambda_{\max}(\Gamma_2^{\mathrm{T}}\Gamma_2)}{2\varepsilon_3}\sum_{i=1}^{N} e_i^{\mathrm{T}}\bigl(t-\tau_1^u(t)\bigr)e_i\bigl(t-\tau_1^u(t)\bigr),
\end{aligned}\tag{10}
$$

$$
\begin{aligned}
&c_2\theta\bigl(1-\beta_2(t)\bigr)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\Gamma_2 e_i\bigl(t-\tau_2^u(t)\bigr)\\
&\quad\leqslant \frac{c_2\theta(1-\beta_2(t))}{2}\left\{\varepsilon_4\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t)+\frac{1}{\varepsilon_4}\sum_{i=1}^{N} e_i^{\mathrm{T}}\bigl(t-\tau_2^u(t)\bigr)\Gamma_2^{\mathrm{T}}\Gamma_2 e_i\bigl(t-\tau_2^u(t)\bigr)\right\}\\
&\quad\leqslant \frac{c_2\theta(1-\beta_2(t))\varepsilon_4}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t)\\
&\qquad+\frac{c_2\theta(1-\beta_2(t))\lambda_{\max}(\Gamma_2^{\mathrm{T}}\Gamma_2)}{2\varepsilon_4}\sum_{i=1}^{N} e_i^{\mathrm{T}}\bigl(t-\tau_2^u(t)\bigr)e_i\bigl(t-\tau_2^u(t)\bigr).
\end{aligned}\tag{11}
$$

According to properties of the Kronecker product, which are listed in Lemma 2, and Lemma 1, for any positive constants ε_5, ε_6, one has:

$$
\begin{aligned}
&c_1\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\sum_{j=1}^{N} a_{ij}\Gamma_1 e_j(t)\\
&\quad= c_1 e^{\mathrm{T}}(t)(A\otimes\Gamma_1)e(t)\leqslant \lambda_{\max}(A\otimes\Gamma_1)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t),
\end{aligned}\tag{12}
$$

$$
\begin{aligned}
&c_1c_2\beta_1(t)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)\sum_{j=1}^{N} b_{ij}\Gamma_2 e_j\bigl(t-\tau_1^l(t)\bigr)\\
&\quad= c_1c_2\beta_1(t)e^{\mathrm{T}}(t)(B\otimes\Gamma_2)e\bigl(t-\tau_1^l(t)\bigr)\\
&\quad\leqslant \frac{c_1c_2\beta_1(t)}{2}\left\{\varepsilon_5 e^{\mathrm{T}}(t)\bigl(BB^{\mathrm{T}}\otimes\Gamma_2\Gamma_2^{\mathrm{T}}\bigr)e(t)+\frac{e^{\mathrm{T}}(t-\tau_1^l(t))(I_N\otimes I_n)e(t-\tau_1^l(t))}{\varepsilon_5}\right\}\\
&\quad\leqslant \frac{c_1c_2\beta_1(t)\varepsilon_5}{2}\lambda_{\max}\bigl(BB^{\mathrm{T}}\bigr)\lambda_{\max}\bigl(\Gamma_2\Gamma_2^{\mathrm{T}}\bigr)\sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t)\\
&\qquad+\frac{c_1c_2\beta_1(t)}{\varepsilon_5}\sum_{i=1}^{N} e_i^{\mathrm{T}}\bigl(t-\tau_1^l(t)\bigr)e_i\bigl(t-\tau_1^l(t)\bigr),
\end{aligned}\tag{13}
$$

$$
\begin{aligned}
&c_1c_2\big(1-\beta_1(t)\big)\sum_{i=1}^{N}e_i^{\mathrm T}(t)\sum_{j=1}^{N}b_{ij}\Gamma_2e_j\big(t-\tau_1^u(t)\big)\\
&\quad = c_1c_2\beta_1(t)e^{\mathrm T}(t)(B\otimes\Gamma_2)e\big(t-\tau_1^u(t)\big)\\
&\quad \leqslant \frac{c_1c_2(1-\beta_1(t))}{2}\bigg\{\varepsilon_6e^{\mathrm T}(t)\big(BB^{\mathrm T}\otimes\Gamma_2\Gamma_2^{\mathrm T}\big)e(t)\\
&\qquad + \frac{e^{\mathrm T}(t-\tau_1^u(t))(I_N\otimes I_n)e(t-\tau_1^u(t))}{\varepsilon_6}\bigg\}\\
&\quad \leqslant \frac{c_1c_2(1-\beta_1(t))\varepsilon_6}{2}\lambda_{\max}\big(BB^{\mathrm T}\big)\lambda_{\max}\big(\Gamma_2\Gamma_2^{\mathrm T}\big)\sum_{i=1}^{N}e_i^{\mathrm T}(t)e_i(t)\\
&\qquad + \frac{c_1c_2(1-\beta_1(t))}{\varepsilon_6}\sum_{i=1}^{N}e_i^{\mathrm T}\big(t-\tau_1^u(t)\big)e_i\big(t-\tau_1^u(t)\big).
\end{aligned}
\tag{14}
$$

Then, considering (6)–(14), one can derive

$$
\begin{aligned}
\mathcal{L}V(t) = {}&\bigg\{f_1+\frac{g_1}{2}+\frac{c_2\theta\beta_1(t)\varepsilon_1}{2}+\frac{c_2\theta\beta_2(t)\varepsilon_2}{2}+\frac{c_2\theta(1-\beta_1(t))\varepsilon_3}{2}\\
&+\frac{c_2\theta(1-\beta_2(t))\varepsilon_4}{2}+\lambda_{\max}(A)+\frac{c_1c_2\beta_1(t)\varepsilon_5}{2}\lambda_{\max}\big(BB^{\mathrm T}\big)\lambda_{\max}\big(\Gamma_2\Gamma_2^{\mathrm T}\big)\\
&+\frac{c_1c_2(1-\beta_1(t))\varepsilon_6}{2}\lambda_{\max}\big(BB^{\mathrm T}\big)\lambda_{\max}\big(\Gamma_2\Gamma_2^{\mathrm T}\big)\bigg\}\sum_{i=1}^{N}e_i^{\mathrm T}(t)e_i(t)\\
&+\bigg\{f_2+\frac{g_2}{2}\bigg\}\sum_{i=1}^{N}e_i^{\mathrm T}\big(t-\tau_0(t)\big)e_i\big(t-\tau_0(t)\big)\\
&+\bigg\{\frac{c_2\theta\beta_1(t)\lambda_{\max}(\Gamma_2^{\mathrm T}\Gamma_2)}{2\varepsilon_1}+\frac{c_1c_2\beta_1(t)}{\varepsilon_5}\bigg\}\sum_{i=1}^{N}e_i^{\mathrm T}\big(t-\tau_1^l(t)\big)e_i\big(t-\tau_1^l(t)\big)\\
&+\bigg\{\frac{c_2\theta(1-\beta_1(t))\lambda_{\max}(\Gamma_2^{\mathrm T}\Gamma_2)}{2\varepsilon_3}+\frac{c_1c_2(1-\beta_1(t))}{\varepsilon_6}\bigg\}\\
&\times\sum_{i=1}^{N}e_i^{\mathrm T}\big(t-\tau_1^u(t)\big)e_i\big(t-\tau_1^u(t)\big)\\
&+\frac{c_2\theta\beta_2(t)\lambda_{\max}(\Gamma_2^{\mathrm T}\Gamma_2)}{2\varepsilon_2}\sum_{i=1}^{N}\mathrm{e}_i^{\mathrm T}\big(t-\tau_2^l(t)\big)e_i\big(t-\tau_2^l(t)\big)\\
&+\frac{c_2\theta(1-\beta_2(t))\lambda_{\max}(\Gamma_2^{\mathrm T}\Gamma_2)}{2\varepsilon_4}\sum_{i=1}^{N}e_i^{\mathrm T}\big(t-\tau_2^u(t)\big)e_i\big(t-\tau_2^u(t)\big).
\end{aligned}
\tag{15}
$$

Based on Ito's formulation, one has

$$
\mathrm{d}V(t)=\mathcal{L}V(t)\,\mathrm{d}t+\sum_{i=1}^{N}e_i^{\mathrm T}(t)\bar g\big(t,e_i(t),e_i\big(t-\tau_0(t)\big)\big)\,\mathrm{d}\omega(t).
$$

Taking the mathematical expectation of both sides of the above inequality combined with (15), one gets that

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{E}(V(t))}{\mathrm{d}t} \leqslant{} & \biggl\{ f_1 + \frac{g_1}{2} + \frac{c_2\theta\beta_1\varepsilon_1}{2} + \frac{c_2\theta\beta_2(t)\varepsilon_2}{2} + \frac{c_2\theta(1-\beta_1)\varepsilon_3}{2} \\
& + \frac{c_2\theta(1-\beta_2)\varepsilon_4}{2} + \lambda_{\max}(A) + \frac{c_1c_2\beta_1\varepsilon_5}{2}\lambda_{\max}\bigl(BB^{\mathrm{T}}\bigr)\lambda_{\max}\bigl(\varGamma_2\varGamma_2^{\mathrm{T}}\bigr) \\
& + \frac{c_1c_2(1-\beta_1)\varepsilon_6}{2}\lambda_{\max}\bigl(BB^{\mathrm{T}}\bigr)\lambda_{\max}\bigl(\varGamma_2\varGamma_2^{\mathrm{T}}\bigr)\biggr\} \sum_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t) \\
& + \biggl\{ f_2 + \frac{g_2}{2} \biggr\} \sum_{i=1}^{N} e_i^{\mathrm{T}}\bigl(t-\tau_0(t)\bigr)e_i\bigl(t-\tau_0(t)\bigr) \\
& + \biggl\{ \frac{c_2\theta\beta_1\lambda_{\max}(\varGamma_2^{\mathrm{T}}\varGamma_2)}{2\varepsilon_1} + \frac{c_1c_2\beta_1}{\varepsilon_5} \biggr\} \sum_{i=1}^{N} e_i^{\mathrm{T}}\bigl(t-\tau_1^l(t)\bigr)e_i\bigl(t-\tau_1^l(t)\bigr) \\
& + \biggl\{ \frac{c_2\theta(1-\beta_1)\lambda_{\max}(\varGamma_2^{\mathrm{T}}\varGamma_2)}{2\varepsilon_3} + \frac{c_1c_2(1-\beta_1)}{\varepsilon_6} \biggr\} \\
& \times \sum_{i=1}^{N} e_i^{\mathrm{T}}\bigl(t-\tau_1^u(t)\bigr)e_i\bigl(t-\tau_1^u(t)\bigr) \\
& + \frac{c_2\theta\beta_2\lambda_{\max}(\varGamma_2^{\mathrm{T}}\varGamma_2)}{2\varepsilon_2} \sum_{i=1}^{N} e_i^{\mathrm{T}}\bigl(t-\tau_2^l(t)\bigr)e_i\bigl(t-\tau_2^l(t)\bigr) \\
& + \frac{c_2\theta(1-\beta_2)\lambda_{\max}(\varGamma_2^{\mathrm{T}}\varGamma_2)}{2\varepsilon_4} \sum_{i=1}^{N} e_i^{\mathrm{T}}\bigl(t-\tau_2^u(t)\bigr)e_i\bigl(t-\tau_2^u(t)\bigr) \\
\leqslant{} & \zeta_1\mathbf{E}\bigl(V(t)\bigr) + \zeta_2\mathbf{E}\bigl(V\bigl(t-\tau_0(t)\bigr)\bigr) + \zeta_3\mathbf{E}\bigl(V\bigl(t-\tau_1^l(t)\bigr)\bigr) \\
& + \zeta_4\mathbf{E}\bigl(V\bigl(t-\tau_1^u(t)\bigr)\bigr) + \zeta_5\mathbf{E}\bigl(V\bigl(t-\tau_2^l(t)\bigr)\bigr) + \zeta_6\mathbf{E}\bigl(V\bigl(t-\tau_2^u(t)\bigr)\bigr).
\end{aligned}$$

For any $k \in \mathbb{N}$, noting that $\eta \in (-2,0)$ and $\mu \in (-1,0)$, we yield

$$\begin{aligned}
\mathbf{E}\bigl(V\bigl(t_k^+\bigr)\bigr) &= \frac{1}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}}\bigl(t_k^+\bigr)e_i\bigl(t_k^+\bigr) \\
&= \frac{1}{2}\sum_{i=1}^{N} \bigl((1+\eta)e_i\bigl(t_k^-\bigr) + \mu e_i\bigl(t_k^- - \tau_3\bigl(t_k^-\bigr)\bigr)\bigr)^{\mathrm{T}} \\
&\qquad \times \bigl((1+\eta)e_i\bigl(t_k^-\bigr) + \mu e_i\bigl(t_k^- - \tau_3\bigl(t_k^-\bigr)\bigr)\bigr) \\
&\leqslant \bigl((1+\eta)^2 + \mu(1+\eta)\bigr)\mathbf{E}\bigl(V\bigl(t_k^-\bigr)\bigr) \\
&\quad + \bigl(\mu^2 + \mu(1+\eta)\bigr)\mathbf{E}\bigl(V\bigl(t_k^- - \tau_3\bigl(t_k^-\bigr)\bigr)\bigr).
\end{aligned}$$

According to Lemma 3, if (5) is satisfied, there exist constants $\rho_1 > 1$ and $\rho_2 > 0$ such that $\mathbf{E}(V(t)) \leqslant \|\varphi\|_\tau \rho_1 \mathrm{e}^{-\rho_2(t-t_0)}$ for $t \geqslant 0$, where $\tau = \max\{\tau_0, \tau_1^l, \tau_1^u, \tau_2^l, \tau_2^u, \tau_3\}$.

Hence, the complex dynamical network (1) can synchronize exponentially to $s(t)$, this completes our proof. $\square$

3.2 Pinning synchronization strategy

All nodes are controlled in the above result, which needs a high cost. Now, let us select a fraction of the whole nodes to add impulsive controller. Without controlling, there must be different orbits among nodes. Naturally, the farthest nodes from the goal orbit should be controlled first. Therefore, those nodes with a big synchronized errors will be selected in the following result. At impulsive moment t_k, the index set of pinning nodes $\mathcal{P}(t_k)$ is defined as follow: for the vectors $e_1(t_k), e_2(t_k), \ldots, e_N(t_k)$, one can reorder the states of the nodes such that $\|e_{p_1}(t_k)\| \geqslant \|e_{p_2}(t_k)\| \geqslant \cdots \geqslant \|e_{p_N}(t_k)\|$. Suppose that we choose l ($l < N$) nodes of network (1) for controlling, then the index set of l controlled nodes $\mathcal{P}(t_k)$ is defined as $\mathcal{P}(t_k) = \{p_1, p_2, \ldots, p_l\}$. Let $\gimel(t_k) = \min\{\|e_i(t_k)\|^2\colon i \in \mathcal{P}(t_k)\}$ and $\daleth(t_k) = \max\{\|e_i(t_k)\|^2\colon i \notin \mathcal{P}(t_k)\}$. It is easy to find that $\daleth(t_k) \leqslant \gimel(t_k)$ according to our pinning strategy. Then the following pinning synchronization criteria can be derived:

Theorem 2. *Under assumptions mentioned in last section, network* (1) *can be exponentially synchronized to $s(t)$ with convergence rate ρ_2 if there exist positive constants $\varepsilon_1, \ldots, \varepsilon_6$ such that*:

$$\text{(i)}\ \rho_0 + \rho_2 < 1, \qquad \text{(ii)}\ \zeta_1 + \frac{\sum_{k=2}^{6} \zeta_k}{\rho_0 + \rho_2} + \frac{\ln(\rho_0 + \rho_2)}{t_{k+1} - t_k} < 1,$$

where $\rho_0 = 1 + [(1+\eta)(1+\eta+\mu) - 1]/N$, $\rho_2 = \mu^2 + \mu(1+\eta)$, $\zeta_1 - \zeta_6$ are same as them in Theorem 1.

Proof. It is similar to the proof for Theorem *1* when $t \neq t_k$. At impulsive moment, one has the following results:

$$\begin{aligned}
\mathbf{E}\big(V(t_k^+)\big) &= \frac{1}{2}\sum_{i=1}^{N} e_i^{\mathrm{T}}\big(t_k^+\big)e_i\big(t_k^+\big) = \frac{1}{2}\sum_{i\in\mathcal{P}(t_k)} e_i^{\mathrm{T}}\big(t_k^+\big)e_i\big(t_k^+\big) + \frac{1}{2}\sum_{i\notin\mathcal{P}(t_k)} e_i^{\mathrm{T}}\big(t_k^-\big)e_i\big(t_k^-\big) \\
&= \frac{1}{2}\sum_{i\in\mathcal{P}(t_k)} \Big((1+\eta)e_i\big(t_k^-\big) + \mu e_i\big(t_k^- - \tau_3\big(t_k^-\big)\big)\Big)^{\mathrm{T}}\Big((1+\eta)e_i\big(t_k^-\big) \\
&\quad + \mu e_i\big(t_k^- - \tau_3\big(t_k^-\big)\big)\Big) + \frac{1}{2}\sum_{i\notin\mathcal{P}(t_k)} e_i^{\mathrm{T}}\big(t_k^-\big)e_i\big(t_k^-\big) \\
&\leqslant \frac{1}{2}\big((1+\eta)^2 + \mu(1+\eta)\big)\sum_{i\in\mathcal{P}(t_k)} e_i^{\mathrm{T}}\big(t_k^-\big)e_i\big(t_k^-\big) + \frac{1}{2}\sum_{i\notin\mathcal{P}(t_k)} e_i^{\mathrm{T}}\big(t_k^-\big)e_i\big(t_k^-\big) \\
&\quad + \frac{1}{2}\big(\mu^2 + \mu(1+\eta)\big)\sum_{i\in\mathcal{P}(t_k)} e_i^{\mathrm{T}}\big(t_k^- - \tau_3\big(t_k^-\big)\big)e_i\big(t_k^- - \tau_3\big(t_k^-\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{1}{2}\big((1+\eta)^2+\mu(1+\eta)\big)\sum_{i\in\mathcal{P}(t_k)} e_i^{\mathrm{T}}\big(t_k^-\big)e_i\big(t_k^-\big)\\
&\quad+\frac{1}{2}\sum_{i\notin\mathcal{P}(t_k)} e_i^{\mathrm{T}}\big(t_k^-\big)e_i\big(t_k^-\big)+\big(\mu^2+\mu(1+\eta)\big)\mathbf{E}\big(V\big(t_k^- -\tau_3\big(t_k^-\big)\big)\big).
\end{aligned}
$$

Note that $\rho_0 = 1 + [(1+\eta)(1+\eta+\mu)-1]/N$, thus

$$
\begin{aligned}
&(1-\rho_0)\sum_{i\notin\mathcal{P}(t_k)} e_i^{\mathrm{T}}\big(t_k^-\big)e_i\big(t_k^-\big)\\
&\quad\leqslant (1-\rho_0)(N-l)\daleth(t_k) = l\big(\rho_0-(1+\eta)(1+\eta+\mu)\big)\daleth(t_k)\\
&\quad\leqslant l\big(\rho_0-(1+\eta)(1+\eta+\mu)\big)\gimel(t_k)\\
&\quad\leqslant \big(\rho_0-(1+\eta)(1+\eta+\mu)\big)\sum_{i\in\mathcal{P}(t_k)} e_i^{\mathrm{T}}\big(t_k^-\big)e_i\big(t_k^-\big).
\end{aligned}
$$

Then one has

$$
\mathbf{E}\big(V\big(t_k^+\big)\big) \leqslant \rho_0\mathbf{E}\big(V\big(t_k^-\big)\big)+\big(\mu^2+\mu(1+\eta)\big)\mathbf{E}\big(V\big(t_k^- -\tau_3\big(t_k^-\big)\big)\big).
$$

Then, it is easy to obtain the result based on the conditions in Theorem 2 and the proof of Theorem 1. This completes our proof. □

Remark 4. According to above results, some corollaries could be obtained easily. For example, let $g(t, x_i(t), x_i(t-\tau_0(t))) = 0I_n$, then system (1) is no disturbance, $g_1 = g_2 = 0$, one can get exponential synchronization for the dynamical network (1). Similarly, let $\beta_1(t) = \beta_2(t) \equiv 0$, which means that transmission time-delays are not assumed to probabilistic, corresponding results also can be derived by some simple derivations. Those trivial results are not listed here.

Remark 5. There were some results concerns about impulsive input delay [23, 29, 31], compared with them, this paper has studied a coupled network with different transmission delays among nodes. On the other hand, the pinning impulsive method has also been investigated in this paper. Furthermore, the coupled time-delays are also probabilistic in two time interval. In general, the model in this paper contains lots of exists results.

4 Numerical simulations

In this section, some examples will be given to check our theoretical result. A isolated dynamic behaviors described by a chaotic delayed neural networks at first. Then, synchronization of the network under impulsive controllers will be shown.

4.1 The synchronized state $s(t)$

Suppose that the isolated dynamic behaviors can be described by the following delayed neural network:

$$
\dot{x}_i(t) = f\big(t, x_i(t), x_i\big(t-\tau_1(t)\big)\big) = Cx_i(t) + B_1 g_1\big(x_i(t)\big) + B_2 g_1\big(x_i\big(t-\tau_1(t)\big)\big),
$$

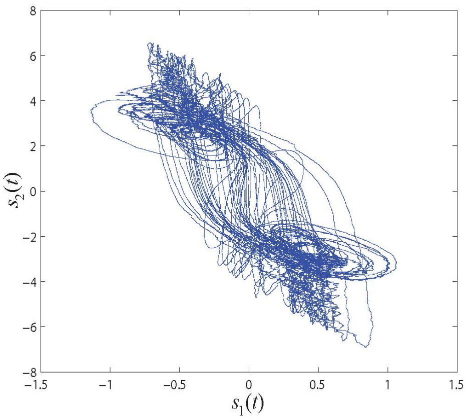

Figure 1. Chaotic attractor of $s(t)$ with $\theta = 2$.

where $x_i(t) = (x_{i1}(t), x_{i2}(t))^{\mathrm{T}} \in \mathbb{R}^2$, $\tau_1 = 1$, $g_1(x_i(t)) = g_2(x_i(t)) = (\tanh(x_{i1}(t)),$ $\tanh(x_{i2}(t)))^{\mathrm{T}}$, and

$$C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 2 & -0.1 \\ -5 & 4.5 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -1.5 & -0.1 \\ -0.2 & -4 \end{pmatrix}.$$

Consider the following parameters in system (3):

$$c_2 = 0.1, \qquad \theta = 2, \qquad \beta_1 = 0.6, \qquad \beta_2 = 0.5,$$
$$\tau_1^l = 0.25 + 0.25\sin t, \qquad \tau_1^u = 0.75 + 0.25\sin t,$$
$$\tau_2^l = 0.2 + 0.2\sin t, \qquad \tau_2^u = 0.7 + 0.3\sin t, \qquad \Gamma_2 = \operatorname{diag}\{1, 1.1\},$$
$$g\big(t, s(t), s\big(t - \tau_0(t)\big)\big) = 0.1\big(\big\|s(t)\big\| + \big\|s\big(t - \tau_0(t)\big)\big\|\big)I_2.$$

Then $s(t)$ has a chaotic attractor shown in Fig. 1 with initial condition $\phi^s(t) = [0.2, 0.5]^{\mathrm{T}}$ for $t \in [-1, 0]$.

Let us proof that $f(t, x(t), x(t - \tau_0(t)))$ conforms with Assumption 1. In fact, for any $x(t), y(t) \in \mathbb{R}^n$,

$$\begin{aligned}
&\big(x(t) - y(t)\big)^{\mathrm{T}}\big(f\big(t, x(t), x\big(t - \tau_0(t)\big)\big) - f\big(t, y(t), y\big(t - \tau_0(t)\big)\big)\big) \\
&\quad = \big(x(t) - y(t)\big)^{\mathrm{T}}\big(C\big(x(t) - y(t)\big) + B_1\big(g_1\big(x(t)\big) - g_1\big(y(t)\big)\big) \\
&\qquad + B_2\big(g_2\big(x(t - \tau_0)\big) - g_2\big(y(t - \tau_0)\big)\big)\big) \\
&\quad = \big(x(t) - y(t)\big)^{\mathrm{T}}\left(\frac{C + C^{\mathrm{T}}}{2}\right)\big(x(t) - y(t)\big) \\
&\qquad + \big(x(t) - y(t)\big)^{\mathrm{T}} B_1\big(g_1\big(x(t)\big) - g_1\big(y(t)\big)\big) \\
&\qquad + \big(x(t) - y(t)\big)^{\mathrm{T}} B_2\big(g_2\big(x(t - \tau_0)\big) - g_2\big(y(t - \tau_0)\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \left(\frac{C+C^{\mathrm{T}}}{2}+\left(\|B_1\|+\frac{\epsilon\|B_2\|}{2}\right)I_2\right)\big(x(t)-y(t)\big)^{\mathrm{T}}\big(x(t)-y(t)\big)\\
&\quad+\frac{\|B_1\|}{2}\big(g_1\big(x(t)\big)-g_1\big(y(t)\big)\big)^{\mathrm{T}}\big(g_1\big(x(t)\big)-g_1\big(y(t)\big)\big)\\
&\quad+\frac{\|B_2\|}{2\epsilon}\big(g_2\big(x(t-\tau_0)\big)-g_2\big(y(t-\tau_1)\big)\big)^{\mathrm{T}}\big(g_2\big(x(t)\big)-g_2\big(y(t)\big)\big)\\
&\leqslant \lambda_{\max}\left(\frac{C+C^{\mathrm{T}}}{2}+\left(\|B_1\|+\frac{\epsilon\|B_2\|}{2}\right)I_2\right)\big(x(t)-y(t)\big)^{\mathrm{T}}\big(x(t)-y(t)\big)\\
&\quad+\frac{\|B_2\|}{2\epsilon}\big(x(t-\tau_0)-y(t-\tau_0)\big)^{\mathrm{T}}\big(x(t-\tau_0)-y(t-\tau_0)\big).
\end{aligned}
$$

Let ϵ=2.005, then $f_1 = \lambda_{\max}((C + C^{\mathrm{T}})/2 + (\|B_1\| + \epsilon\|B_2\|/2)I_2) = 9.9288$ and $f_2 = \|B_2\|/(2\epsilon) = 1$, then Assumption 2 can be guaranteed.

Remark 6. The chaotic attractor has been shown in Fig. 1 is under $\theta = 2$, indeed, the different values θ may lead different chaotic behavior. However, the synchronized goal of lots related results are determined. The main reason is the different transmission delays $\tau_1(t)$ and $\tau_2(t)$. It is obviously that when $\tau_1(t) = \tau_2(t)$, the $\theta(-b_{ii})$ would not affect the chaotic behaviors.

4.2 Synchronization behavior among a small-world network under impulsive control with impulsive input delay

In this subsection, let us consider a small-world network, which is generated by taking initial neighboring nodes $k = 8$ and the edge adding probability $\mathbf{P} = 0.1$. The coupled matrix G is determined by the network. $a_{ij} = G_{ij}$ and $b_{ij} = \theta G_{ij}/\sum_{j=1,\, j\neq i}^{N} G_{ij}$ when $i \neq j$. a_{ii} and b_{ii} can be calculated by (2). Other parameters are given as $c_1 = 0.1$, $\mathit{\Gamma}_1 = I_2$. Without control, the nodes' state have been shown in Fig. 2, it is obviously that they cannot synchronized to $s(t)$.

Consider the impulsive control with $t_{k+1} - t_k = 0.02$ and $\eta = -0.9$. The impulsive input delay $\tau_3(t) = 0.5\cos t$ and $\mu = -0.14$. According Theorem 2, we selected 29 nodes to control. The other parameters in Theorems 1 and 2 are given as follow:

$$
\begin{aligned}
&\varepsilon_1 = 3, &&\varepsilon_2 = 1.2, &&\varepsilon_3 = 3, &&\varepsilon_4 = 6,\\
&\varepsilon_5 = 0.5, &&\varepsilon_6 = 0.6, &&g_1 = 0.1, &&g_2 = 0.1.
\end{aligned}
$$

By some simple calculation, one has

$$
\begin{aligned}
&\zeta_1 = 16.1221, \qquad \sum_{i=2}^{6}\zeta_i = 1.1796, \qquad \rho_0 = 0.7088, \qquad \rho_2 = 0.0056,\\
&\rho_0 + \rho_2 = 0.7144 < 1, \qquad \zeta_1 + \frac{\sum_{k=2}^{6}\zeta_k}{\rho_0+\rho_2} + \frac{\ln(\rho_0+\rho_2)}{t_{k+1}-t_k} = 0.9603 < 1.
\end{aligned}
$$

Then all conditions can be satisfied in Theorem 2. The results can be seen in Figs. 3 and 4.

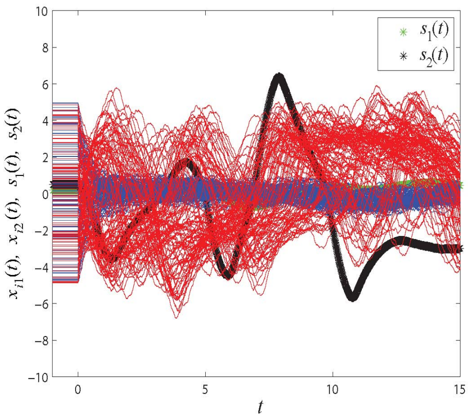

Figure 2. Time evolution of nodes' states $x_{i1}(t)$, $x_{i2}(t)$, $i = 1, 2, \dots, 100$, and $s(t)$ without control.

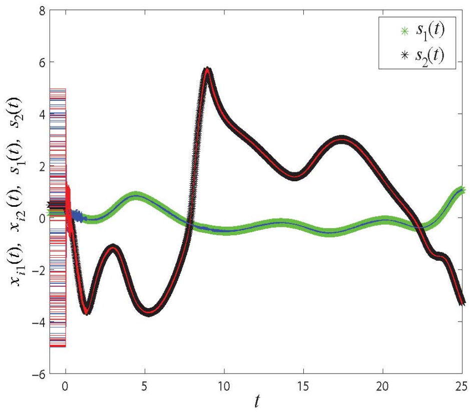

Figure 3. Time evolution of nodes' states $x_{i1}(t)$, $x_{i2}(t)$, $i = 1, 2, \dots, 100$, and $s(t)$ under pinning impulsive control.

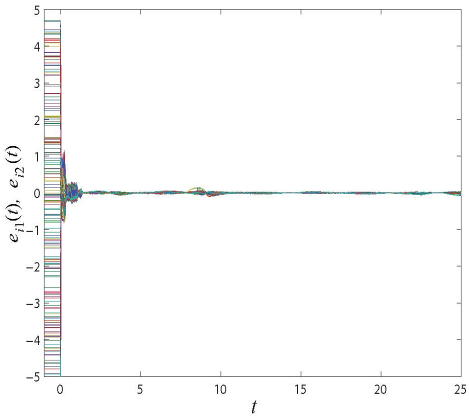

Figure 4. Time evolution of errors' states $e_{i1}(t)$, $e_{i2}(t)$, $i = 1, 2, \dots, 100$, under pinning impulsive control.

5 Conclusion

In this paper, a general hybrid-coupled network model with both the internal delay and coupling delay is investigated, stochastic disturbances also have been taken into consideration, and the impulsive synchronization of such delayed dynamical network is intensively studied. The delayed coupling term considered here includes the transmission delay and self-feedback delay, transmission time-varying delay is assumed probabilistic. Impulsive control input delays have been considered, furthermore, pinning impulsive strategy has been studied. Numerical examples are also given to demonstrate the effectiveness of our proposed control strategy.

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