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Improved synchronization analysis of competitive neural networks with time-varying delays

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Abstract. Synchronization and control are two very important aspects of any dynamical systems. Among various kinds of nonlinear systems, competitive neural network holds a very important place due to its application in diverse fields. The model is general enough to include, as subclass, the most famous neural network models such as competitive neural networks, cellular neural networks and Hopfield neural networks. In this paper, the problem of feedback controller design to guarantee synchronization for competitive neural networks with time-varying delays is investigated. The goal of this work is to derive an existent criterion of the controller for the exponential synchronization between drive and response neutral-type competitive neural networks with time-varying delays. The method used in this brief is based on feedback control gain matrix by using the Lyapunov stability theory. The synchronization conditions are given in terms of LMIs. To the best of our knowledge, the results presented here are novel and generalize some previous results. Some numerical simulations are also represented graphically to validate the effectiveness and advantages of our theoretical results.

Keywords: competitive neural networks, time-varying delays, global exponential synchronization, Lyapunov functional.

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1 Introduction

Classical concepts of the synchronization phenomenon are based on the notions of closeness of the frequencies or phases of the subsystems generating periodic oscillations. Using the traditional language of dynamical systems with continuous time, one can reveal hat synchronization of periodic oscillations that may be represented as follows. While a stable limit cycle is a geometrical image of such oscillations, an attracting two-dimensional (or n-dimensional) torus is a geometrical image of the oscillations generated by two (or n) uncoupled oscillators in a common phase space. As the parameter of coupling increases, the motions of partial subsystems are no longer independent, and a stable limit cycle is born on the torus that is still an attractor. This corresponds to the transition of the system to synchronization. The analysis of periodic systems incorporating full-time information leads to challenging control problems with a rich mathematical structure. Meanwhile, as a typical complex system, delayed neural networks have been verified to exhibit some complex and unpredictable behaviors such as periodic oscillations, bifurcation and chaotic attractors. Since synchronization of neural networks has been shown to be an important step toward both fundamental science and technological practice, much of the focus has been received and numerous research results have been reported in the literature [6, 12, 15, 26]. Many methods have been developed for synchronizing of chaos such as LMI based approach [13], adaptive control [18], passivity feedback control [24].

On the other hand, many researches have been devoted to the dynamics of various classes of neural networks (see [3, 7–10, 14, 27]). Furthermore, there is few works about the so-called competitive neural networks proposed for the first time by Meyer-Baese et al. (see [19, 21, 22, 25]), who used them to model the dynamics of cortical cognitive maps with unsupervised synaptic modifications. The model of competitive neural networks is different from the traditional neural networks with first-order interactions. In [9], based on Lyapunov functional method and Kronecker product technique, the authors proposed some sufficient conditions for global synchronization of neutral-type neural networks with constant and delayed coupling. In [10], there is proposed a simple adaptive coupling enhancement algorithm for the synchronization of two coupled identical time-varying delayed neural networks based on the invariant principle of functional differential equations.

As a continuation of their previous published results, in this paper, we consider a target model with two different state variables: the short-term memory (STM) variable describing the fast neural activity and the long-term memory (LTM) variable describing the slow unsupervised synaptic modifications. In addition, it has been reported that if the parameters and time delays are appropriately chosen, the delayed competitive neural networks can exhibit complicated behaviors even with strange chaotic attractors. Based on the aforementioned arguments, the study of delayed competitive neural networks and its analogous equations have attracted worldwide interest (see [20]).

The remainder of this paper is organized as follows. In Section 2, we present the synchronization problem for CNNs. In Section 3, we introduce preliminaries, notations and hypotheses. The controller design will be proposed in Section 4. In Section 5, we will introduce the new criteria proving the exponential synchronization of CNNs. At last, an illustrative numerical example is given.

2 Methodology and problem formulation

The competitive neural networks with time-varying delays in this brief are modeled as follows:

$$
\begin{aligned}
\text{STM:}\quad \dot{x}_i(t) &= -\alpha_i(t)x_i(t) + \sum_{j=1}^{n} D_{ij}(t)f_j\big(x_j(t)\big) + B_i(t)\sum_{j=1}^{n} m_{ij}(t)y_j \\
&\quad + \sum_{j=1}^{n} D_{ij}^{\tau}(t)f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big) + I_i(t), \\
\text{LTM:}\quad \dot{m}_{ij}(t) &= -\beta_i(t)m_{ij}(t) + y_jE_i(t)f_i\big(x_i(t)\big),
\end{aligned} \tag{1}
$$

where $i, j = 1, \dots, n$; $x_i(t)$ is the neuron current activity level; $f_j(x_j(t))$ is the output of neurons; $m_{ij}(t)$ is the synaptic efficiency; y_i is the constant external stimulus; $D_{ij}(t)$, $D_{ij}^{\tau}(t)$ represent, respectively, the connection weight and the synaptic weight of delayed feedback between the ith and jth neurons; $B_i(t)$ is the strength of the external stimulus; $E_i(t)$ denotes disposable scale; $I_i(t)$ denotes the external inputs on the ith neuron at time t; $\sigma = \max(\dot{\tau}_{ij}(t)) < 1$ for $j = 1, \dots, n$ and $t > t_0$, where σ is constant; $\alpha_i, \beta_i : \mathbb{R} \to \mathbb{R}$ are continuous functions.

By setting $S_i = \sum_{j=1}^{n} m_{ij}(t)y_i = Y^{\mathrm{T}}m_i(t)$, where $y = (y_1, y_2, \dots, y_n)^{\mathrm{T}}$, $m_i = (m_{i1}, m_{i2}, \dots, m_{in})^{\mathrm{T}}$ and, without loss of generality, the input stimulus Y is assumed to be normalized with unit magnitude $|y|^2 = 1$, summing up the LTM over j, then the above networks are simplified, and we get a state-space representation of the LTM and STM equations of the networks:

$$
\begin{aligned}
\text{STM:}\quad \dot{x}_i(t) &= -\alpha_i(t)x_i(t) + \sum_{j=1}^{n} D_{ij}(t)f_j\big(x_j(t)\big) + B_i(t)S_i(t) \\
&\quad + \sum_{j=1}^{n} D_{ij}^{\tau}(t)f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big) + I_i(t), \\
\text{LTM:}\quad \dot{S}_i(t) &= -\beta_i(t)S_i(t) + E_i(t)f_i\big(x_i(t)\big),
\end{aligned} \tag{2}
$$

$i = 1, \dots, n$. In order to observe the synchronization behavior in the class of delayed functional differential equations, we consider two delayed functional differential equations, where the drive system with state variable denoted by x_i drives the response system having identical dynamical equations denoted by state variable z_i, and S_i drives the response system having identical dynamical equations denoted by state variable W_i. However, the initial condition on the drive system is different from that of the response system. The drive system is as follows:

$$
\begin{aligned}
\text{STM:}\quad \dot{x}_i(t) &= -\alpha_i(t)x_i(t) + \sum_{j=1}^{n} D_{ij}(t)f_j\big(x_j(t)\big) + B_i(t)S_i(t) \\
&\quad + \sum_{j=1}^{n} D_{ij}^{\tau}(t)f_j\big(x_j\big(t-\tau_{ij}(t)\big)\big) + I_i(t), \\
\text{LTM:}\quad \dot{S}_i(t) &= -\beta_i(t)S_i(t) + E_i(t)f_i\big(x_i(t)\big),
\end{aligned} \tag{3}
$$

$i = 1, \dots, n$, with the initial condition

$$\big(x_1(t), \dots x_n(t), S_1(t), \dots, S_n(t)\big) = \big(\varphi_1(t), \dots, \varphi_n(t), \phi_1(t), \dots, \phi_n(t)\big) \\ \in \big(C\big([-\tau^*, 0], \mathbb{R}\big)\big)^{2n}.$$

In practice, the output signals of system (3) can be received by system (4). Therefore, the goal of control is to design and implement an appropriate controller $u_i^+(t) = (u_i(t), \tilde{u}_i(t))$ for the second system such that the controlled response system can synchronize with the drive system (3). The response system is as follows:

$$\begin{aligned} \text{STM:}\quad \dot{z}_i(t) &= -\alpha_i(t) z_i(t) + \sum_{j=1}^{n} D_{ij}(t) f_j\big(z_j(t)\big) \\ &\quad + \sum_{j=1}^{n} D_{ij}^{\tau}(t) f_j\big(z_j\big(t - \tau_{ij}(t)\big)\big) + B_i(t) W_i(t) + I_i(t) + u_i(t), \\ \text{LTM:}\quad \dot{W}_i(t) &= -\beta_i(t) W_i(t) + E_i(t) f_i\big(z_i(t)\big) + \tilde{u}_i(t), \end{aligned} \tag{4}$$

where u_i and $\tilde{u}_i$ are the control terms respectively for STM and LTM with the initial condition

$$\big(z_1(t), \dots z_n(t), W_1(t), \dots, W_n(t)\big) = \big(\widetilde{\varphi}_1(t), \dots, \widetilde{\varphi}_n(t), \widetilde{\phi}_1(t), \dots, \widetilde{\phi}_n(t)\big) \\ \in \big(C\big([-\tau^*, 0], \mathbb{R}\big)\big)^{2n},$$

where $\widetilde{\varphi}_i(\cdot)$ and $\widetilde{\phi}_i(\cdot)$ are the real-valued bounded differentiable functions defined on $[-\tau^*, 0]$, $i = 1, \dots, n$.

3 Preliminaries, notations and hypotheses

In this paper, we always consider the vectorial space $\mathbb{R}^n$ for $n \in \mathbb{N}^*$ equipped with the Euclidean norm (denoted by $\|\cdot\|$) in $\mathbb{R}^n$. Let $\mathbb{R}^n$ be n-dimensional Euclidean space. In all that follows, we denote be $I_n \in \mathbb{R}^{n\times n}$ and $O_n \in \mathbb{R}^{n\times n}$ identity matrix and zero matrix, respectively. For all $x, S, y, Z : \mathbb{R} \to \mathbb{R}$, we define the zero norm by

$$\big\|\big(x(t), S(t)\big), \big(y(t), Z(t)\big)\big\|_0 = \max\big\{\big\|x(t) - y(t)\big\|_\infty, \big\|S(t) - Z(t)\big\|_\infty\big\}.$$

For convenience, we introduce the following notations:

$$\begin{aligned} B_i^+ &= \sup_{t\in\mathbb{R}} \big|B_i(t)\big|, & E_i^+ &= \sup_{t\in\mathbb{R}} \big|E_i(t)\big|, \\ \alpha_{\min}(t) &= \min_{i=1,\dots,n} \big\{\alpha_i(t)\big\}, & \alpha_{\max}(t) &= \max_{i=1,\dots,n} \big\{\alpha_i(t)\big\}, \\ \underline{\alpha}^{\min} &= \inf_{t\in\mathbb{R}} \big\{\alpha_{\min}(t)\big\}, & \overline{\alpha}^{\max} &= \sup_{t\in\mathbb{R}} \big\{\alpha_{\max}(t)\big\}, \\ \beta_{\min}(t) &= \min_{i=1,\dots n} \big\{\beta_i(t)\big\}, & \underline{\beta}^{\min} &= \inf_{t\in\mathbb{R}} \big\{\beta_{\min}(t)\big\}, \\ D_{ij}^+ &= \sup_{t\in\mathbb{R}} \big|D_{ij}(t)\big|, & \big(D_{ij}^{\tau}\big)^+ &= \sup_{t\in\mathbb{R}} \big|D_{ij}^{\tau}(t)\big|, \end{aligned}$$

$$\tau^* = \max_{i,j}\big(\sup\big(\tau_{ij}(t)\big)\big) \quad \text{for } i, j = 1, \dots, n.$$

Let us list some assumptions, which will be used throughout the rest of this paper:

(H1) The functions $\alpha_i, \beta_i : \mathbb{R} \to \mathbb{R}_+$ are continuous and positive.

(H2) The functions $f_j(\cdot)$ are differential and satisfy the Lipschitz condition, i.e., there are constants $k_j > 0$ such that for all $x, y \in \mathbb{R}$ and for all $1 \leqslant j \leqslant n$, one has $|f_j(x) - f_j(y)| \leqslant k_j|x - y|$ and $f_j(0) = 0$.

Definition 1. Let $Z^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t), S_1^*(t), S_2^*(t), \dots, S_n^*(t))^{\mathrm{T}}$ be solution of system (3) with the initial value $\psi^*(t) = (\varphi_1^*(t), \varphi_2^*(t), \dots, \varphi_n^*(t), \phi_1^*(t), \phi_2^*(t), \dots, \phi_n^*(t))^{\mathrm{T}}$, and let $Z(t) = (x_1(t), x_2(t), \dots, x_n(t), S_1(t), S_2(t), \dots, S_n(t))^{\mathrm{T}}$ be the solution of uncontrolled system (4) with the initial value $\psi(t) = (\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t), \phi_1(t), \phi_2(t), \dots, \phi_n(t))^{\mathrm{T}}$. If there exist constants $\rho > 0$ and $M > 1$ such that for every solution $Z(t)$ of system (4) with any initial value $\psi(t)$,

$$\begin{aligned}
\|Z(t) - Z^*(t)\|_0 &= \left(\max\left\{\|x(t) - x^*(t)\|_\infty^2, \|S(t) - S^*(t)\|_\infty^2\right\}\right)^{1/2} \\
&\leqslant M \exp\left(-\rho(t - t_0)\right)\|\psi\|_1 \\
&= M \exp\left(-\rho(t - t_0)\right) \\
&\quad \times \sup_{t\in[-\tau^*,0]} \left(\max\left\{\|\varphi(t) - \varphi^*(t)\|_\infty^2, \|\phi(t) - \phi^*(t)\|_\infty^2\right\}\right)^{1/2}
\end{aligned}$$

for all $t \geqslant t_0$, then system (3) synchronizes exponentially with system (4).

Remark 1. The exponential synchronization problem considered here is to determine the control inputs $u_i(t)$ and $\tilde{u}_i(t)$ associated with the state-feedback for the purpose of exponentially synchronizing the two identical chaotic nonlinear neural networks (3) and (4) with the same system parameters except the differences in initial conditions.

4 Controller design

Let us define the synchronization error signal $e_i(t) = x_i(t) - z_i(t)$ and $\tilde{e}_i(t) = S_i(t) - W_i(t)$, where $x_i(t)$, $S_i(t)$ and $z_i(t)$, $W_i(t)$ are the ith state variables of the drive and response competitive neural networks, respectively. Therefore, the error dynamics between (3) and (4) can be expressed by

$$\begin{aligned}
\text{STM:}\quad \dot{e}_i(t) &= -\alpha_i(t)e_i(t) + \sum_{j=1}^n D_{ij}(t)f_j\left(e_j(t)\right) \\
&\quad + \sum_{j=1}^n D_{ij}^\tau(t)f_j\left(e_j\left(t - \tau_{ij}(t)\right)\right) + B_i(t)\tilde{e}_i(t) - u_i(t), \qquad (5)
\end{aligned}$$

$$\text{LTM:}\quad \dot{\tilde{e}}_i(t) = -\beta_i(t)e_i(t) + E_i(t)f_i\left(e_i(t)\right) - \tilde{u}_i(t) \qquad (6)$$

for $i = 1, \dots, n$, where

$$\begin{aligned}
f_j\left(e_j(t)\right) &= f_j\left(x_j(t)\right) - f_j\left(z_j(t)\right), \\
f_j\left(e_j\left(t - \tau_{ij}(t)\right)\right) &= f_j\left(x_j\left(t - \tau_{ij}(t)\right)\right) - f_j\left(z_j\left(t - \tau_{ij}(t)\right)\right).
\end{aligned}$$

From hypothesis (H2) we can have that $f_i(\cdot)$ satisfies

$$0 \leqslant e_i(t) f_i\big(e_i(t)\big) \leqslant k_i e_i^2(t), \quad 0 \leqslant \tilde{e}_i(t) f_i\big(\tilde{e}_i(t)\big) \leqslant k_i \tilde{e}_i^2(t).$$

If the state variables of the drive system are used to drive the response system, then the control input vector with state feedback is designed as follows:

$$\begin{aligned}
\begin{pmatrix} u_1(t) \\ \vdots \\ u_n(t) \\ \tilde{u}_1(t) \\ \vdots \\ \tilde{u}_n(t) \end{pmatrix}
&= \begin{pmatrix} \sum_{j=1}^n \omega_{1,j} \times (x_j(t) - z_j(t)) \\ \vdots \\ \sum_{j=1}^n \omega_{n,j} \times (x_j(t) - z_j(t)) \\ \sum_{j=1}^n \tilde{\omega}_{1,j} \times (S_j(t) - W_j(t)) \\ \vdots \\ \sum_{j=1}^n \tilde{\omega}_{n,j} \times (S_j(t) - W_j(t)) \end{pmatrix} \\
&= \begin{pmatrix} \omega_{1,1} & \dots & \omega_{1,n} & \omega_{1,n+1} & \dots & \omega_{1,2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega_{n,1} & \dots & \omega_{n,n} & \omega_{n,n+1} & \dots & \omega_{n,2n} \end{pmatrix}
\begin{pmatrix} x_1(t) - z_1(t) \\ \vdots \\ x_n(t) - z_n(t) \\ S_1(t) - W_1(t) \\ \vdots \\ S_n(t) - W_n(t) \end{pmatrix} \\
&= \Omega e(t),
\end{aligned}$$

where $e(t) = (e_1(t), \dots, e_n(t))^{\mathrm{T}}$, and $\Omega = (\omega_{i,j})_{n \times n} \in \mathbb{R}^{n \times 2n}$ is the gain matrix to be determined for synchronizing both a drive system and response system.

Besides, if new errors are defined by $\hat{e}_i(t)$ and $\hat{\tilde{e}}_i(t)$ is defined by $\hat{e}_i(t) = \mathrm{e}^{\rho t} e_i(t)$ and $\hat{\tilde{e}}_i(t) = \mathrm{e}^{\rho t} \tilde{e}_i(t)$, respectively, the dynamics of (5) and (6) can be transformed into the following forms:

$$\begin{aligned}
\text{STM:} \quad \dot{\hat{e}}_i(t) &= -\alpha_i(t)\hat{e}_i(t) + \rho \hat{e}_i(t) + \sum_{j=1}^n D_{ij}(t) F_j\big(\hat{e}_j(t)\big) \\
&\quad + \sum_{j=1}^n D_{ij}^{\tau}(t) F_j\big(\hat{e}_j\big(t - \tau_{ij}(t)\big)\big) + B_i(t)\hat{\tilde{e}}_i(t) - \sum_{j=1}^n \omega_{i,j} \hat{e}_j(t), \qquad (7)
\end{aligned}$$

$$\text{LTM:} \quad \dot{\hat{\tilde{e}}}_i(t) = -\beta_i(t)\hat{\tilde{e}}_i(t) + \rho \hat{\tilde{e}}_i(t) + E_i(t) F_i\big(\hat{e}_i(t)\big) - \sum_{j=1}^n \tilde{\omega}_{i,j} \hat{\tilde{e}}_j(t), \qquad (8)$$

where $F_j(\hat{e}_j(t)) = \mathrm{e}^{\rho t} f_j(e_j(t))$ and $F_j(\hat{e}_j(t - \tau_{ij}(t))) = \mathrm{e}^{\rho t} f_j(e_j(t - \tau_{ij}(t)))$.

5 Main results

The exponential synchronization problem of systems (3) and (4) can be solved if the controller gain matrix is suitably designed. The exponential synchronization condition is established in the following main results.

Theorem 1. *Let* (H1)–(H2) *hold. Assume that there exist two strictly positive constants* ρ *and* $\sigma < 1$ *such that*:

(i) $\sum_{i=1}^{n} (D_{ij}^{\tau})^{+} < \frac{e^{2\rho\tau^*}}{1-\sigma}$ *for all* $j = 1, \dots, n$,

and there exist n *strictly positive constants* $q_1, q_2, \dots, q_n$ *such that*:

(ii)
$$\frac{2\rho}{\overline{\alpha}_{\max}} + \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} \frac{D_{ij}^{+}}{\underline{\alpha}_{\min}} + \max_{1\leqslant j\leqslant n} \frac{k_j \sum_{i=1}^{n} D_{ij}^{+}}{\underline{\alpha}_{\min}} - 2 \max_{1\leqslant j\leqslant n} \sum_{i=1}^{n} \frac{\omega_{i,j}}{\underline{\alpha}_{\min}} + \max_{1\leqslant i\leqslant n} \left(\frac{B_i^{+}}{\underline{\alpha}_{\min}} + \frac{q_i}{\underline{\alpha}_{\min}} k_i E_i^{+} \right) < 2;$$

(iii)
$$\frac{1}{2} \max_{1\leqslant i\leqslant n} \left(\frac{B_i^{+}}{\underline{\alpha}_{\min}} + \frac{q_i}{\underline{\alpha}_{\min}} k_i E_i^{+} \right) \leqslant \max_{1\leqslant i\leqslant n} q_i \underline{\beta}^{\min} + \max_{1\leqslant j\leqslant n} \sum_{i=1}^{n} q_i \tilde{\omega}_{i,j}.$$

Then the drive system (3) *synchronizes exponentially with the response system* (4).

Proof. To confirm that the origin of (7) is globally exponentially convergent, a continuous Lyapunov functional $V(t)$ is defined as follows:

$$V(t) = \sum_{i=1}^{n} \frac{1}{\alpha_i(t)} \hat{e}_i^2(t) + 2 \sum_{i=1}^{n} q_i \int_{0}^{\hat{\dot{e}}_i(t)} s \, \mathrm{d}s + \frac{e^{2\rho\tau^*}}{1-\sigma} \sum_{j=1}^{n} (D_{ij}^{\tau})^{+} \int_{t-\tau_{ij}(t)}^{t} F_j^2(\hat{e}_j(s)) \, \mathrm{d}s.$$

It is easy to verify that $V(t)$ is a nonnegative function over $[-\tau^*, +\infty)$ and $\lim_{\hat{e}(t)\to+\infty} V(t) = +\infty$. By the expression of $f_j(e_j(t))$, $F_j(\hat{e}_j(t))$ and assumption (H2) we obtain

$$\left|f_j(e_j(t))\right| \leqslant k_j \left|e_j(t)\right|,$$
$$\left|F_j(\hat{e}_j(t))\right| = \left|e^{\rho t} F_j(e_j(t))\right| \leqslant k_j \left|e^{\rho t} e_j(t)\right| = k_j \left|\hat{e}_j(t)\right|.$$

Evaluating the time derivative of V along the trajectory of (7) and (8)

$$\begin{aligned}
\dot{V}(t) &= 2 \sum_{i=1}^{n} \frac{1}{\alpha_i(t)} \hat{e}_i(t) \dot{\hat{e}}_i(t) + 2 \sum_{i=1}^{n} q_i \hat{\dot{e}}_i(t) \dot{\hat{\dot{e}}}_i(t) + \frac{e^{2\rho\tau^*}}{1-\sigma} \sum_{j=1}^{n} (D_{ij}^{\tau})^{+} F_j^2(\hat{e}_j(t)) \\
&\quad - \frac{e^{2\rho\tau^*}}{1-\sigma} \sum_{j=1}^{n} (D_{ij}^{\tau})^{+} F_j^2(\hat{e}_j(t - \tau_{ij}(t)))(1 - \dot{\tau}_{ij}(t)) \\
&\leqslant -2 \sum_{i=1}^{n} \hat{e}_i^2(t) + 2\rho \sum_{i=1}^{n} \frac{1}{\alpha_i(t)} \hat{e}_i^2(t) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{D_{ij}(t)}{\alpha_i(t)} \hat{e}_i(t) F_j(\hat{e}_j(t)) \\
&\quad + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{D_{ij}^{\tau}(t)}{\alpha_i(t)} \hat{e}_i(t) F_j(\hat{e}_j(t - \tau_{ij}(t)))
\end{aligned}$$

$$
\begin{aligned}
&+ 2\sum_{i=1}^{n}\frac{1}{\alpha_i(t)}\hat{e}_i(t)B_i(t)\hat{\hat{e}}_i(t) - 2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\omega_{i,j}}{\alpha_i(t)}\hat{e}_j^2(t) \\
&- 2\sum_{i=1}^{n}q_i\hat{\hat{e}}_i^2(t) + 2\sum_{i=1}^{n}q_i\hat{\hat{e}}_i(t)E_i(t)F_i\bigl(\hat{e}_i(t)\bigr) - 2\sum_{i=1}^{n}\sum_{j=1}^{n}q_i\tilde{\omega}_{i,j}\hat{\hat{e}}_j^2(t) \\
&+ \frac{\mathrm{e}^{2\rho\tau^*}}{1-\sigma}\sum_{j=1}^{n}\bigl(D_{ij}^{\tau}\bigr)^{+}F_j^2\bigl(\hat{e}_j(t)\bigr) - \frac{\mathrm{e}^{2\rho\tau^*}}{1-\sigma}\sum_{j=1}^{n}\bigl(D_{ij}^{\tau}\bigr)^{+}F_j^2\bigl(\hat{e}_j\bigl(t-\tau_{ij}(t)\bigr)\bigr). \qquad (9)
\end{aligned}
$$

Applying the inequality $2|a|b \leqslant a^2 + b^2$, we have that

$$
\begin{aligned}
&2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{D_{ij}(t)}{\alpha_i(t)}\hat{e}_i(t)F_j\bigl(\hat{e}_j(t)\bigr) \\
&\quad\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{D_{ij}^{+}}{\underline{\alpha}^{\min}}\hat{e}_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{D_{ij}^{+}}{\underline{\alpha}^{\min}}F_j^2\bigl(\hat{e}_j(t)\bigr), \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
&2\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{D_{ij}^{\tau}(t)}{\alpha_i(t)}\hat{e}_i(t)F_j\bigl(\hat{e}_j\bigl(t-\tau_{ij}(t)\bigr)\bigr) \\
&\quad\leqslant \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(D_{ij}^{\tau})^{+}}{\underline{\alpha}^{\min}}\hat{e}_i^2(t) + \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{(D_{ij}^{\tau})^{+}}{\underline{\alpha}^{\min}}F_j^2\bigl(\hat{e}_j\bigl(t-\tau_{ij}(t)\bigr)\bigr) \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
&2\sum_{i=1}^{n}\left(\frac{B_i(t)}{\alpha_i(t)} + \frac{q_i}{\alpha_i(t)}k_iE_i(t)\right)\bigl|\hat{e}_i(t)\bigr|\hat{\hat{e}}_i(t) \\
&\quad\leqslant \sum_{i=1}^{n}\left(\frac{B_i^{+}}{\underline{\alpha}^{\min}} + \frac{q_i}{\underline{\alpha}^{\min}}k_iE_i^{+}\right)\bigl(\hat{e}_i^2(t) + \hat{\hat{e}}_i^2(t)\bigr). \qquad (12)
\end{aligned}
$$

Substituting (10), (11) and (12) in derivative (9), we obtain easily

$$
\begin{aligned}
\dot{V}(t) \leqslant{}& \Biggl\{-2 + \frac{2\rho}{\underline{\alpha}^{\min}} + \max_{1\leqslant i\leqslant n}\sum_{j=1}^{n}\frac{D_{ij}^{+}}{\underline{\alpha}^{\min}} + \max_{1\leqslant j\leqslant n}\frac{k_j\sum_{i=1}^{n}D_{ij}^{+}}{\underline{\alpha}^{\min}} \\
&- 2\max_{1\leqslant j\leqslant n}\sum_{i=1}^{n}\frac{\omega_{i,j}}{\underline{\alpha}^{\min}} + \max_{1\leqslant i\leqslant n}\left(\frac{B_i^{+}}{\underline{\alpha}^{\min}} + q_ik_iE_i^{+}\right)\Biggr\}\bigl\|\hat{e}(t)\bigr\|^2 \\
&+ \sum_{j=1}^{n}\left(\max_{1\leqslant j\leqslant n}\sum_{i=1}^{n}\bigl(D_{ij}^{\tau}\bigr)^{+} - \frac{\mathrm{e}^{2\rho\tau^*}}{1-\sigma}\right)F_j^2\bigl(\hat{e}_j\bigl(t-\tau_{ij}(t)\bigr)\bigr) \\
&+ \left\{\max_{1\leqslant i\leqslant n}\left(\frac{B_i^{+}}{\underline{\alpha}^{\min}} + q_ik_iE_i^{+}\right) - 2\max_{1\leqslant i\leqslant n}q_i\underline{\beta}^{\min} - 2\max_{1\leqslant j\leqslant n}\sum_{i=1}^{n}q_i\tilde{\omega}_{i,j}\right\} \\
&\times \bigl\|\hat{\hat{e}}(t)\bigr\|^2.
\end{aligned}
$$

In addition, we have

$$\begin{aligned}\frac{1}{\overline{\alpha}_{\max}}\|\hat{e}(t)\|^2 &\leqslant V(t)\\ &\leqslant \frac{1}{\underline{\alpha}_{\min}}\|\hat{e}(t)\|^2+\frac{q_{\max}}{\underline{\alpha}_{\min}}\|\hat{\tilde{e}}(t)\|^2\\ &\quad+\frac{\tau^*\mathrm{e}^{2\rho\tau^*}}{1-\sigma}\frac{K_{\max}^2\sum_{j=1}^n(D_{ij}^{\tau})^+}{\underline{\alpha}_{\min}}\|\hat{e}(t)\|^2.\end{aligned}$$

Besides,

$$\begin{aligned}\frac{q_{\min}}{\overline{\alpha}_{\max}}\|\hat{\tilde{e}}(t)\|^2 &\leqslant V(t)\\ &\leqslant \frac{1}{\underline{\alpha}_{\min}}\|\hat{e}(t)\|^2+\frac{q_{\max}}{\underline{\alpha}_{\min}}\|\hat{\tilde{e}}(t)\|^2\\ &\quad+\frac{\tau^*\mathrm{e}^{2\rho\tau^*}}{1-\sigma}\frac{K_{\max}^2\sum_{j=1}^n(D_{ij}^{\tau})^+}{\underline{\alpha}_{\min}}\|\hat{e}(t)\|^2.\end{aligned}$$

Therefore, we have

$$\sqrt{\frac{1+q_{\min}}{\overline{\alpha}_{\max}}}\max\{\|\hat{e}(t)\|,\hat{\tilde{e}}(t)\|\}\leqslant\sqrt{2V(t)}\leqslant\sqrt{2V(t_0)}.$$

For $t=t_0$, we have

$$\begin{aligned}\sqrt{V(t_0)}\leqslant{}&\left(\sqrt{\frac{1}{q_{\min}}}+\sqrt{\frac{q_{\max}}{\underline{\alpha}_{\min}}}+\sqrt{\frac{\tau^*\mathrm{e}^{2\rho\tau^*}}{1-\sigma}\frac{K_{\max}^2\sum_{j=1}^n(D_{ij}^{\tau})^+}{\underline{\alpha}_{\min}}}\right)\exp(\rho t_0)\\ &\times\sup_{t\in[-\tau^*,0]}\max\{\|\varphi(t)\|,\|\phi(t)\|\}.\end{aligned}$$

Then

$$\max\{\|e(t)\|,\|\tilde{e}(t)\|\}\leqslant M\exp\bigl(-\rho(t-t_0)\bigr)\sup_{t\in[-\tau^*,0]}\max\{\|\varphi(t)\|,\|\phi(t)\|\},$$

where

$$\begin{aligned}M={}&\sqrt{\frac{\overline{\alpha}_{\max}}{(1+q_{\min})\underline{\alpha}_{\min}}}+\sqrt{\frac{\overline{\alpha}_{\max}q_{\max}}{(1+q_{\min})\underline{\alpha}_{\min}}}\\ &+\sqrt{\frac{\tau^*\mathrm{e}^{2\rho\tau^*}}{1-\sigma}\frac{K_{\max}^2\overline{\alpha}_{\max}\sum_{j=1}^n(D_{ij}^{\tau})^+}{(1+q_{\min})\underline{\alpha}_{\min}}}\\ >{}&1.\end{aligned}$$

Therefore, system (3) synchronizes exponentially with system (4). □

If $q_1=q_2=\cdots=q_n=1$, we obtain the following result.

Corollary 1. *The drive system* (3) *synchronizes exponentially with the response system* (4) *if there exist two strictly positive constants* ρ *and* $\sigma < 1$ *such that*:

$$\text{(i)} \quad \sum_{i=1}^{n} \left(D_{ij}^{\tau}\right)^{+} < \frac{\mathrm{e}^{2\rho\tau^*}}{1-\sigma} \quad \textit{for all } j = 1, \dots, n;$$

$$\text{(ii)} \quad \frac{2\rho}{\overline{\alpha}_{\max}} + \max_{1\leqslant i\leqslant n} \sum_{j=1}^{n} \frac{D_{ij}^{+}}{\underline{\alpha}_{\min}} + \max_{1\leqslant j\leqslant n} \frac{k_j \sum_{i=1}^{n} D_{ij}^{+}}{\underline{\alpha}_{\min}} + \max_{1\leqslant i\leqslant n} \left(\frac{B_i^{+}}{\underline{\alpha}_{\min}} + \frac{1}{\underline{\alpha}_{\min}} k_i E_i^{+}\right) < 2 + 2 \max_{1\leqslant j\leqslant n} \sum_{i=1}^{n} \frac{\omega_{i,j}}{\underline{\alpha}_{\min}};$$

$$\text{(iii)} \quad \frac{1}{2} \max_{1\leqslant i\leqslant n} \left(\frac{B_i^{+}}{\underline{\alpha}_{\min}} + \frac{1}{\underline{\alpha}_{\min}} k_i E_i^{+}\right) \leqslant \max_{1\leqslant i\leqslant n} \underline{\beta}^{\min} + \max_{1\leqslant j\leqslant n} \sum_{i=1}^{n} \tilde{\omega}_{i,j}.$$

Remark 2. To the best of our knowledge, no paper in the literature has investigated the synchronization problem of system (3) with system (4). Furthermore, this paper improves and generalizes the outcomes in [10, 20].

6 Simulation results

Example 1. The parameters of a two-dimensional nonlinear competitive neural networks with time-varying delays (3) and (4) is given by the following system of equations:

$$\alpha_1(t) = \alpha_2(t) = \exp\left(-\frac{t^2}{2}\right) + 0.5, \qquad \beta_1(t) = \beta_2(t) = 2\exp\left(-\frac{t^2}{2}\right),$$

$$I_i(t) = 2\cos(\sqrt{2}t),$$

$$D(t) = \left(D_{ij}(t)\right)_{2\times 2} = \begin{pmatrix} 2\exp(-t^2) & -0.3\cos t \\ -2\exp(-t^2) & 8\cos(\sqrt{2}t) \end{pmatrix},$$

$$D^{\tau}(t) = \left(D_{ij}^{\tau}(t)\right)_{2\times 2} = \begin{pmatrix} 2\cos(\sqrt{2}t) & 0.1\cos t \\ \exp(-t^2) & -8\cos t \end{pmatrix},$$

$$B(t) = \begin{pmatrix} 4\exp(-t^2) \\ 4\exp(-t^2) \end{pmatrix}, \qquad E(t) = \begin{pmatrix} 2\exp(-t^2) \\ 1.5\exp(-t^2) \end{pmatrix}.$$

The delays $(\tau_{ij}(t))_{1\leqslant i,j\leqslant 2} = 0.7\exp t/(1+\exp t)$ satisfy

$$0 \leqslant \tau_{ij}(t) \leqslant 0.7 = \tau^*, \qquad 0 \leqslant \dot{\tau}_{ij}(t) \leqslant 0.7.$$

We choose the activation functions of competitive neural networks as the type of hyperbolic tangent function $f_j(x) = 0.3\tanh(x)$. Figures 1–10 show the oscillation of the delayed competitive neural networks (3) and (4) with above coefficients and initial values $x_1(\theta) = -1$, $x_2(\theta) = -0.2$, $S_1(\theta) = -2$, $S_2(\theta) = -0.6$ for $\theta \in [-0.7, 0]$.

The oscillation of solution of drive system is clearly presented in Figs. 1–4.

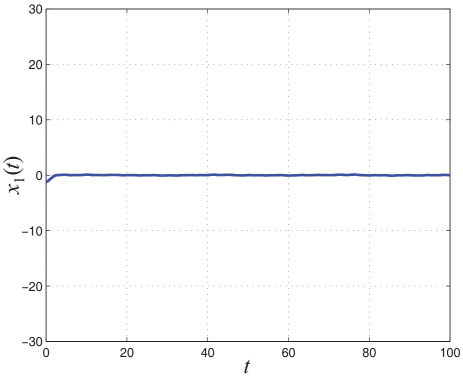

Figure 1. Trajectory of x_1 for $t \in [0, 100]$.

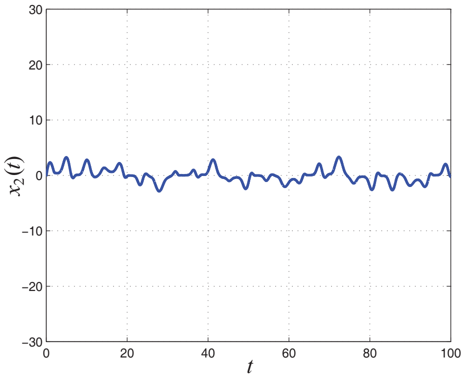

Figure 2. Trajectory of x_2 for $t \in [0, 100]$.

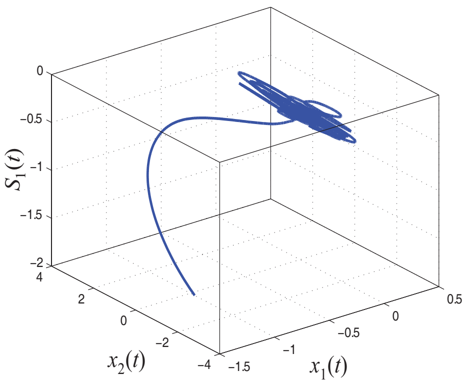

Figure 3. Phase plot of x_1, x_2, S_1 for $t \in [0, 100]$.

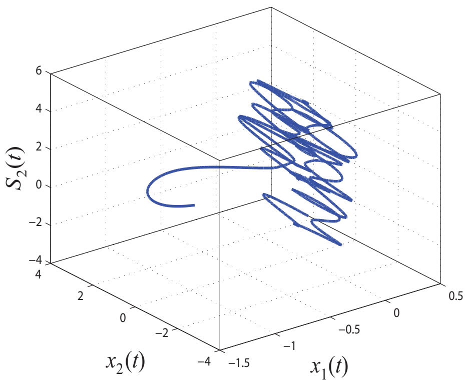

Figure 4. Phase plot of x_1, x_2, S_2 for $t \in [0, 100]$.

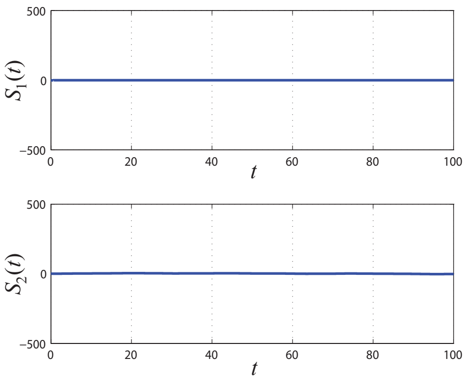

Figure 5. Trajectory of S_1 and S_2 for $t \in [0, 100]$.

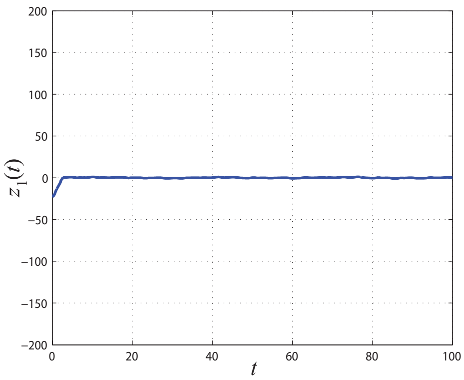

Figure 6. Trajectory of z_1 for $t \in [0, 100]$.

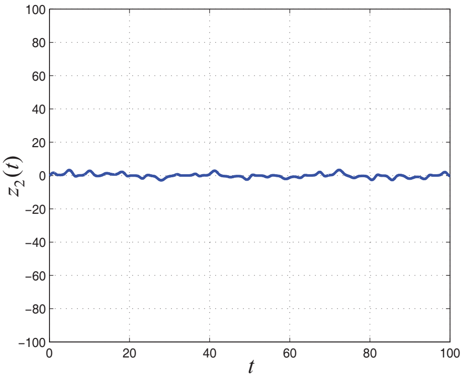

Figure 7. Trajectory of z_2 for $t \in [0, 100]$.

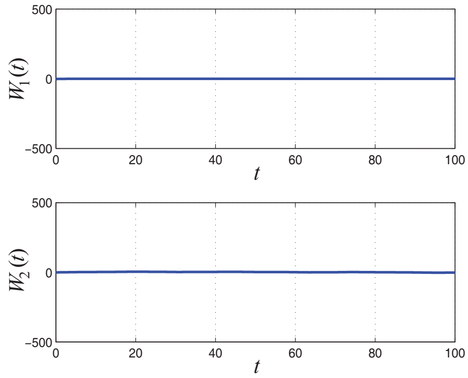

Figure 8. Trajectory of W_1 and W_2 for $t \in [0, 100]$.

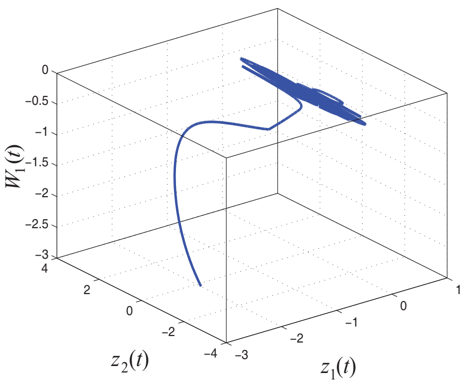

Figure 9. Phase plot of z_1, z_2, W_1 for $t \in [0, 100]$.

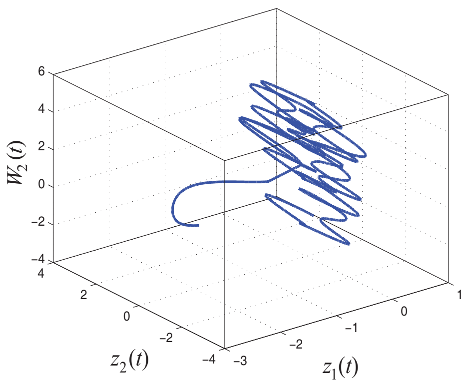

Figure 10. Phase plot of z_1, z_2, W_2 for $t \in [0, 100]$.

It follows from the main theorem that if the controls input $u_i(t)$ and $\tilde{u}_i(t)$ are chosen as

$$\begin{aligned} u_1(t) &= 9.2e_1(t), \qquad & u_2(t) &= 9.2e_2(t), \\ \tilde{u}_1(t) &= 1.2\tilde{e}_1(t), \qquad & \tilde{u}_2(t) &= 1.2\tilde{e}_2(t), \end{aligned}$$

then the matrix of control is as follows:

$$\Omega = \begin{pmatrix} 9.2 & 0 & 1.2 & 0 \\ 0 & 9.2 & 0 & 1.2 \end{pmatrix}.$$

The oscillation of solution of response system with above coefficients and initial values $x_1(\theta) = -2$, $x_2(\theta) = -0.2$, $S_1(\theta) = -3$, $S_2(\theta) = -0.5$ for $\theta \in [-0.7, 0]$ is clearly presented in Figs. 6–10.

It is evident that hypotheses (H1)–(H2) hold ($k_1 = k_2 = 0.3$). By choosing $\rho = 0.5$ and $\sigma = 0.9$ we have

$$\sum_{i=1}^{2} \left(D_{i1}^{\tau}\right)^{+} = 3 < \frac{\mathrm{e}^{2\rho\tau^*}}{1-\sigma} = 14.1, \qquad \sum_{i=1}^{2} \left(D_{i2}^{\tau}\right)^{+} = 8.1 < \frac{\mathrm{e}^{2\rho\tau^*}}{1-\sigma} = 14.1.$$

Besides,

$$\begin{aligned} &\frac{2\rho}{\overline{\alpha}_{\max}} + \max_{1\leqslant i\leqslant 2} \sum_{j=1}^{2} \frac{D_{ij}^{+}}{\underline{\alpha}_{\min}} + \max_{1\leqslant j\leqslant 2} \frac{k_j \sum_{i=1}^{n} D_{ij}^{+}}{\underline{\alpha}_{\min}} + \max_{1\leqslant i\leqslant 2} \left(\frac{B_i^{+}}{\underline{\alpha}_{\min}} + \frac{1}{\underline{\alpha}_{\min}} k_i E_i^{+} \right) \\ &\quad = \frac{1}{1.5} + \max\left\{ \frac{2.3}{0.5}, \frac{10}{0.5} \right\} + \max\left\{ 0.3\frac{4}{0.5}, 0.3\frac{8.3}{0.5} \right\} \\ &\qquad + \max\left\{ \frac{4}{0.5} + 2\frac{0.3}{0.5}, \frac{4}{0.5} + 1.5\frac{0.3}{0.5} \right\} \\ &\quad = 0.66 + 20 + 4.98 + 9.2 < 2 + 2\max_{1\leqslant j\leqslant 2} \sum_{i=1}^{2} \frac{\omega_{i,j}}{\underline{\alpha}_{\min}} = 38.8 \end{aligned}$$

and

$$\begin{aligned} \frac{1}{2} \max_{1\leqslant i\leqslant 2} \left(\frac{B_i^{+}}{\underline{\alpha}_{\min}} + \frac{1}{\underline{\alpha}_{\min}} k_i E_i^{+} \right) &= \frac{1}{2} \max\left\{ \frac{4}{0.5} + 2\frac{0.3}{0.5}, \frac{4}{0.5} + 1.5\frac{0.3}{0.5} \right\} \\ &\leqslant \max_{1\leqslant i\leqslant 2} \underline{\beta}^{\min} + \max_{1\leqslant j\leqslant n} \sum_{i=1}^{2} \tilde{\omega}_{i,j} = 9.2. \end{aligned}$$

Conditions (i), (ii) and (iii) of Corollary 1 are satisfied. Hence, by using Corollary 1, the drive system (3) can be synchronized by the corresponding response system (4). Figures 11–14 reveal the synchronization error of the state variables between the drive system and the corresponding response system.

Remark 3. The major improvement over [20, 22] is that in our approach, it is very easy to verify the criteria by simple algebraic calculus.

Remark 4. The competitive neural networks models investigated in [20] and [22] are considered with constant coefficients. However, in this work, we study the model with time-varying coefficients. Furthermore, our system include models in [20,22] and [23] as special cases when $D_{ij}(t) = D_{ij}$, $D_{ij}^{\tau}(t) = D_{ij}^{\tau}$, $B_i(t) = B_i$, $E_i(t) = E_i$, $I_i(t) = I_i$ and $J_i(t) = J_i$. Hence, our results have been shown to be the generalization and improvement of existing results reported recently in the literature.

Remark 5. For system (1), if there is no external stimulus, i.e., $y_j = 0$, then system (1) degenerates into general neural networks with time-varying delay, which contains neural models studied in [4, 11, 16].

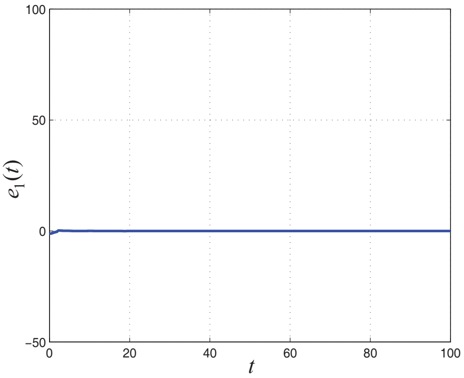

Figure 11. Trajectory of e_1 and x_2 for $t \in [0, 100]$.

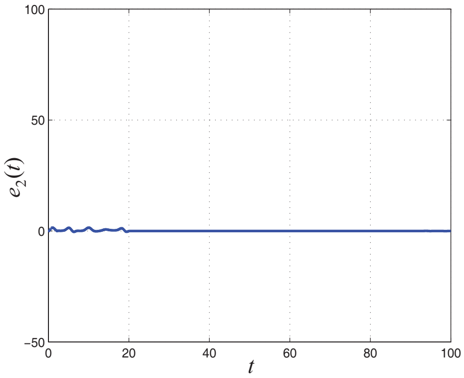

Figure 12. Trajectory of e_2 for $t \in [0, 100]$.

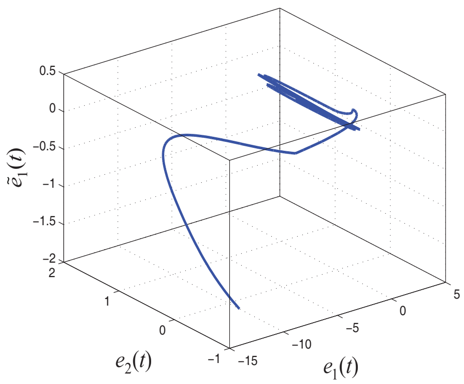

Figure 13. Phase plot of e_1, e_2, $\tilde{e}_1$ for $t \in [0, 100]$.

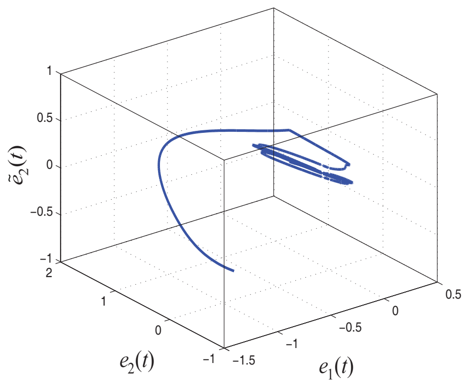

Figure 14. Phase plot of e_1, e_2, $\tilde{e}_2$ for $t \in [0, 100]$.

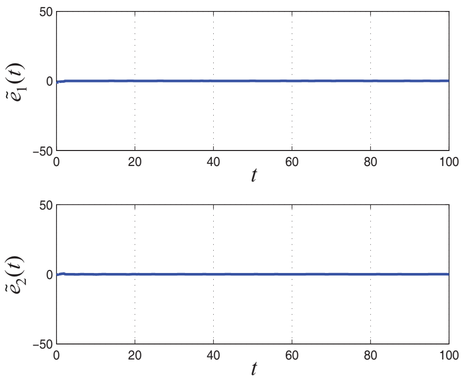

Figure 15. Trajectory of $\tilde{e}_1$ and $\tilde{e}_2$ for $t \in [0, 100]$.

Example 2. The parameters of a two-dimensional nonlinear competitive neural networks with time-varying delays (3) and (4) are almost some parameters of Example 1. The only difference compared to Example 1 is about the activation functions. Furthermore, in this example, we choose the activation functions as follows: $f_j(x) = 0.3\sin(0.5x)$, $j = 1, \dots, n$, and

$$D(t) = \big(D_{ij}(t)\big)_{2\times 2} = \begin{pmatrix} 2\exp(-t^2) & -0.3\cos t \\ -2\exp(-t^2) & 5\cos(\sqrt{2}t) \end{pmatrix},$$

$$D^\tau(t) = \big(D^\tau_{ij}(t)\big)_{2\times 2} = \begin{pmatrix} 2\cos(\sqrt{2}t) & 0.1\cos t \\ \exp(-t^2) & -5\cos t \end{pmatrix}.$$

The delays $(\tau_{ij}(t))_{1\leqslant i,j\leqslant 2} = 0.7\exp t/(1+\exp t)$ are time-varying and satisfy $0 \leqslant \tau_{ij}(t) \leqslant 0.7 = \tau^*$, $0 \leqslant \dot{\tau}_{ij}(t) \leqslant 0.7$.

Figures 16–19 show the oscillation of the delayed competitive neural networks (3) and (4) with above coefficients and initial values $x_1(\theta) = -1$, $x_2(\theta) = -0.2$, $S_1(\theta) = -2$, $S_2(\theta) = 0.6$ for $\theta \in [-0.7, 0]$. The driver system's oscillating solution is clearly presented in Figs. 21–24.

It follows from the main theorem that if the controls input $u_i(t)$, $\tilde{u}_i(t)$ are chosen as

$$u_1(t) = 9.2e_1(t) \qquad u_2(t) = 9.2e_2(t), \qquad \tilde{u}_1(t) = 1.2\tilde{e}_1(t), \qquad \tilde{u}_2(t) = 1.2\tilde{e}_2(t),$$

then the matrix of control is as follows:

$$\Omega = \begin{pmatrix} 9.2 & 0 & 1.2 & 0 \\ 0 & 9.2 & 0 & 1.2 \end{pmatrix}.$$

The responser system's oscillating solution with above coefficients and initial values $x_1(\theta) = -2$, $x_2(\theta) = -0.2$, $S_1(\theta) = -3$, $S_2(\theta) = -0.5$ for $\theta \in [-0.7, 0]$ is clearly presented in Figs. 26–29.

It is evident that hypotheses (H1)–(H2) hold ($k_1 = k_2 = 0.3$). By choosing $\rho = 0.5$ and $\sigma = 0.9$, we have

$$\sum_{i=1}^{2} (D^\tau_{i1})^+ = 3 < \frac{\mathrm{e}^{2\rho\tau^*}}{1-\sigma} = 14.1, \qquad \sum_{i=1}^{2} (D^\tau_{i2})^+ = 5.1 < \frac{\mathrm{e}^{2\rho\tau^*}}{1-\sigma} = 14.1.$$

Besides,

$$\begin{aligned}
&\frac{2\rho}{\overline{\alpha}_{\max}} + \max_{1\leqslant i\leqslant 2} \sum_{j=1}^{2} \frac{D^+_{ij}}{\underline{\alpha}_{\min}} + \max_{1\leqslant j\leqslant 2} \frac{k_j \sum_{i=1}^{n} D^+_{ij}}{\underline{\alpha}_{\min}} + \max_{1\leqslant i\leqslant 2}\left(\frac{B^+_i}{\underline{\alpha}_{\min}} + \frac{1}{\underline{\alpha}_{\min}} k_i E^+_i\right) \\
&\quad = \frac{1}{1.5} + \max\left\{\frac{2.3}{0.5}, \frac{7}{0.5}\right\} + \max\left\{0.3\frac{4}{0.5}, 0.3\frac{5.3}{0.5}\right\} \\
&\qquad + \max\left\{\frac{4}{0.5} + 2\frac{0.3}{0.5}, \frac{4}{0.5} + 1.5\frac{0.3}{0.5}\right\} \\
&\quad = 0.66 + 14 + 3.18 + 9.2 < 2 + 2\max_{1\leqslant j\leqslant 2} \sum_{i=1}^{2} \frac{\omega_{i,j}}{\underline{\alpha}_{\min}} = 38.8
\end{aligned}$$

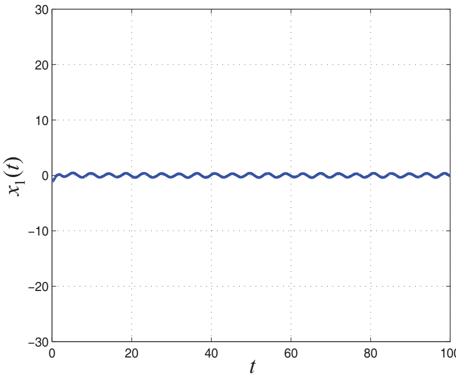

Figure 16. Trajectory of x_1 for $t \in [0, 100]$.

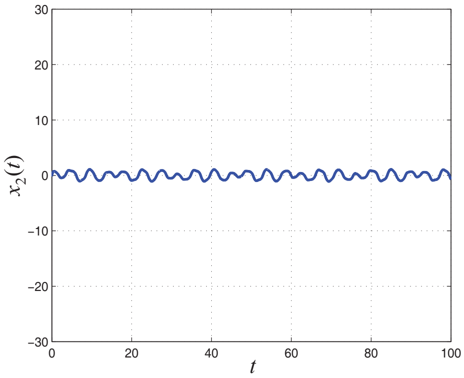

Figure 17. Trajectory of x_2 for $t \in [0, 100]$.

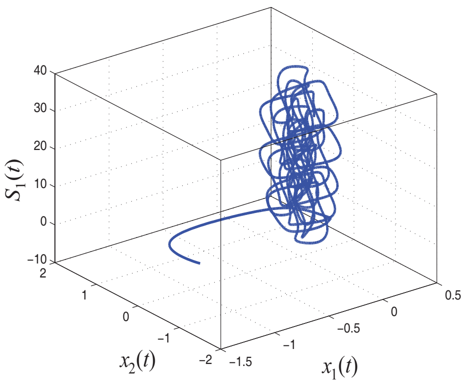

Figure 18. Phase plot of x_1, x_2, S_1 for $t \in [0, 100]$.

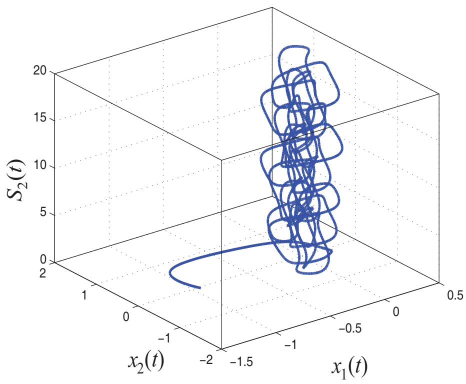

Figure 19. Phase plot of x_1, x_2, S_2 for $t \in [0, 100]$.

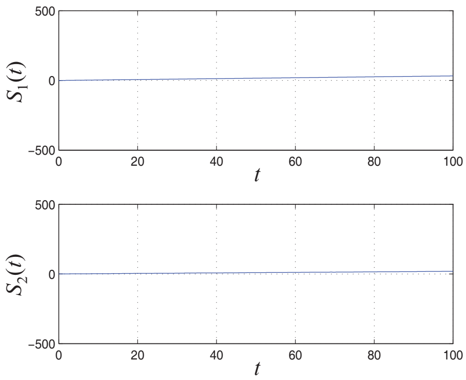

Figure 20. Trajectory of S_1 and S_2 for $t \in [0, 100]$.

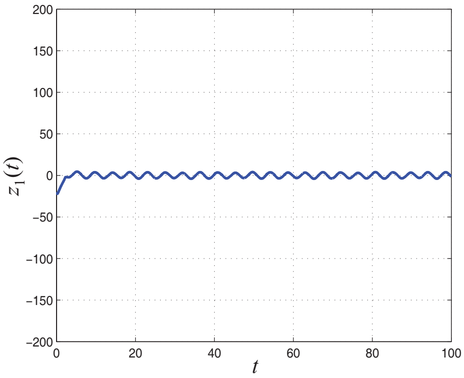

Figure 21. Trajectory of z_1 for $t \in [0, 100]$.

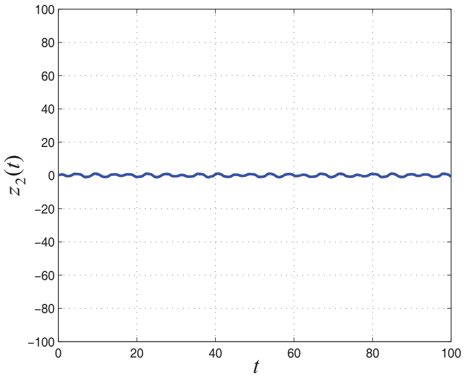

Figure 22. Trajectory of z_2 for $t \in [0, 100]$.

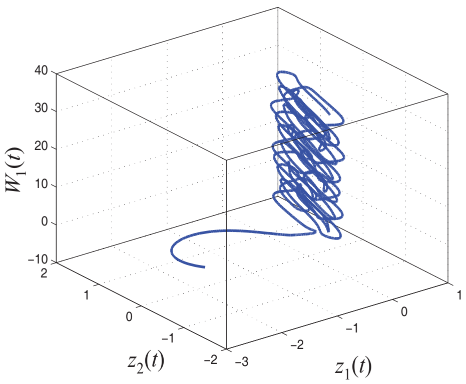

Figure 23. Phase plot of z_1, z_2, W_1 for $t \in [0, 100]$.

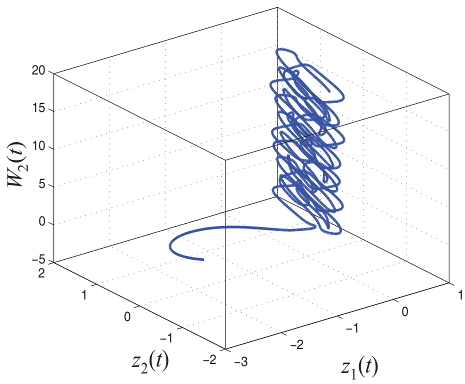

Figure 24. Phase plot of z_1, z_2, W_2 for $t \in [0, 100]$.

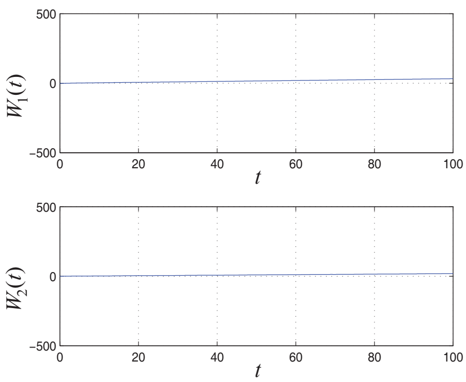

Figure 25. Trajectory of W_1 and W_2 for $t \in [0, 100]$.

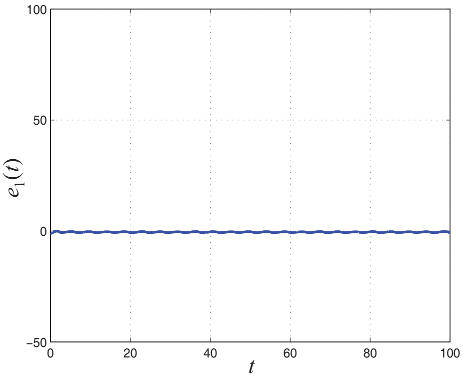

Figure 26. Trajectory of e_1 and x_2 for $t \in [0, 100]$.

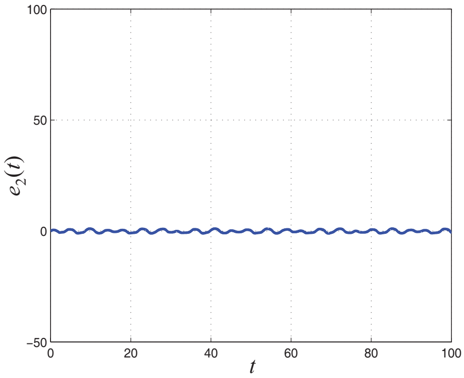

Figure 27. Trajectory of e_2 for $t \in [0, 100]$.

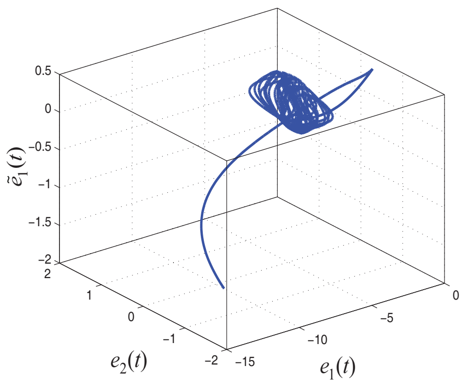

Figure 28. Phase plot of e_1, e_2, $\tilde{e}_1$ for $t \in [0, 100]$.

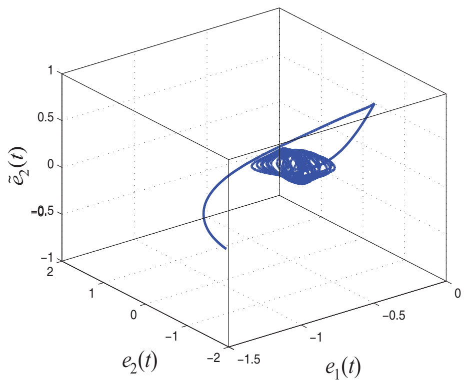

Figure 29. Phase plot of e_1, e_2, $\tilde{e}_2$ for $t \in [0, 100]$.

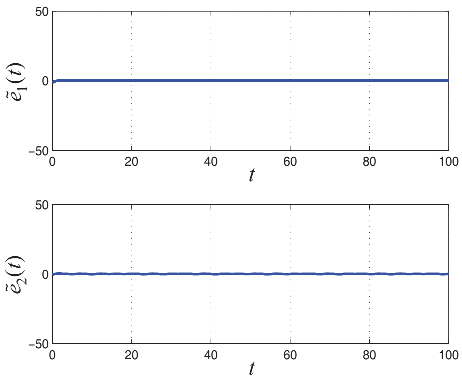

Figure 30. Trajectory of $\tilde{e}_1$ and $\tilde{e}_2$ for $t \in [0, 100]$.

and

$$\frac{1}{2}\max_{1\leqslant i\leqslant 2}\left(\frac{B_i^+}{\underline{\alpha}_{\min}}+\frac{1}{\underline{\alpha}_{\min}}k_iE_i^+\right)=\frac{1}{2}\max\left\{\frac{4}{0.5}+2\frac{0.3}{0.5},\frac{4}{0.5}+1.5\frac{0.3}{0.5}\right\}$$
$$\leqslant \max_{1\leqslant i\leqslant 2}\underline{\beta}^{\min}+\max_{1\leqslant j\leqslant n}\sum_{i=1}^{2}\tilde{\omega}_{i,j}=9.2.$$

Hence, using Corollary 1, the drive system (3) can be synchronized by the corresponding response system (4). Figures 24–29 reveal the synchronization error of the state variables between the drive system and the corresponding response system.

Remark 6. However, in most papers, the activation functions are assumed to be monotonically nondecreasing. In this paper, from Theorem 1 the restriction is removed, and thus, the results obtained here extend and improve those in [11, 16, 17]. However the activation functions here are not monotonous in Example 2, the results in [11, 16, 17] are not applicable.

Remark 7. The above examples show that the result of the proposed control law ensures exponential synchronization of the competitive neural networks constituting of two or multi-neurons with/without time delays.

7 Conclusion and future works

In the present work, we demonstrate that two different chaotic nonlinear competitive neural networks with time-varying delays can be synchronized using active control. More precisely, this paper has presented sufficient conditions to guarantee the exponential synchronization of a class of chaotic nonlinear competitive neural networks with time-varying delays. Moreover, the proposed criteria were dependent of the delay parameter, which may possess important significance in the design of chaos of delayed competitive neural networks. A numerical example and its simulation have been given to demonstrate the effectiveness and advantage of the theory results. Furthermore, the synchronization degree can be easily estimated. Finally, an illustrative example has been given to verify the theoretical results. However, to the best of our knowledge, there are few results concerning the exponential synchronization for competitive neural networks. This technique is applicable for the high-order Hopfield neural networks [2]. Furthermore, there are no studies investigating the problem of exponential synchronization of competitive neural networks with mixed time-varying delays in the leakage terms [5] and the high-order competitive neural networks with mixed time-varying delays in the leakage terms [1]. This is some interesting problems and will become our future investigative direction. Besides, more methods and tools should be explored and developed in this direction. Along, the future work spawning from this paper would be to train this model that can be applied in various areas including pattern recognition, associate memory, cryptography etc.

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