Nonlinear Analysis: Modelling and Control, Vol. 22, No. 6, 739–752 ISSN 1392-5113
https://doi.org/10.15388/NA.2017.6.1

C-class functions on fixed and common fixed point results for cyclic mappings of Ω-distance*

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Received: October 25, 2016 / **Revised:** July 16, 2017 / **Published online:** October 27, 2017

Abstract. In this paper, we derive and formulate many fixed point results of cyclic form under contractive conditions based on implicit relations in the setting of Ω-distance. Our results improve and generalize many existing results in the literature. Also, we introduce an example to show the validity of our main result.

Keywords: nonlinear contractions, G-metric space, common fixed point, Ω-distance, C-class function.

1 Introduction

Fixed point theory plays an important role to solve many problems in applied analysis. For this reason, many researchers work in the area of fixed point theory. Some authors extended the notion of metric space to many notions such as a partial metric space, a cone metric space and a D-metric space. After that, they extended and improved many fixed point theorems in such spaces. Mustafa and Sims [22] introduced the notion of G-metric space as a generalization of the standard metric space, and they studied the topological structure of the G-metric space. Moreover, Mustafa and Sims [22] gave the definitions of a G-convergent sequence, a G-Cauchy sequence and a G-complete space. Then after, they utilized the notion of the G-metric space to establish and prove many fixed point theorems in this space. Saadati et al. [27] initiated the notion of Ω-distance as a generalization of the notion of G-metric spaces. For some works in fixed and common fixed point theorems

*This research was supported by Prince Sultan University.

in the setting of G-metric space, see [8–11, 21, 23, 30, 35]. Also, for some fixed point results based on distance mappings, see [1,6,7,15,28,29,31,33,34]. Recently, Ansari [4] introduced the concept of a C-class function and studied some fixed point theorems. For some works in a C-class function, we refer the reader to [5, 14, 16, 20]. In 2003, Kirk, Srinivasan, and Veeramani [19] introduced the notion of a cyclic mapping and established some fixed point theorems. Recently, many authors studied many fixed and common fixed point theorems for mappings of cyclic form in different metric spaces. For more details, we refer the reader to [2, 3, 12, 13, 17, 18, 24–26, 32, 36, 37]. In this paper, we utilize the notions of Ω-distance and C-class function to formulate and prove many fixed point theorems of cyclic form.

2 Preliminaries

We start this section by recalling the definition of a cyclic mapping.

Definition 1. Let A and B be two nonempty subsets of a space X. A mapping $T: A \cup B \to A \cup B$ is called cyclic if $T(A) \subseteq B$ and $T(B) \subseteq A$.

The notion of a G-metric space is given as follows:

Definition 2. (See [22].) Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following conditions:

(G1) $G(x, y, z) = 0$ if $x = y = z$,
(G2) $G(x, x, y) > 0$ for all $x, y \in X$ with $x \neq y$,
(G3) $G(x, y, y) \leqslant G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G4) $G(x, y, z) = G(p\{x, y, z\})$ for each permutation of x,y,z (the symmetry), and
(G5) $G(x, y, z) \leqslant G(x, a, a)+G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then G is called a *generalized metric space* or, more specifically, G-metric on X, and the pair (X, G) is called a G-metric space.

The notion of a convergent sequence in the setting of a G-metric space is given as follows:

Definition 3. (See [22].) Let (X, G) be a G-metric space, and (x_n) be a sequence in X. We say that (x_n) is G-convergent to x if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x, x_n, x_m) < \epsilon$ for all $n, m \geqslant k$.

The notion of a G-Cauchy sequence in the setting of a G-metric space is defined as follows:

Definition 4. (See [22].) Let (X, G) be a G-metric space, and (x_n) be a sequence in X. We say that (x_n) is a G-Cauchy sequence if for every $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \epsilon$ for all $n, m, l \geqslant k$.

Definition 5. (See [23].) A G-metric space (X, G) is said to be G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).

The definition of Ω-distance is given as follows:

Definition 6. (See [27].) Let (X,G) be a G-metric space. Then a function $\Omega : X \times X \times X \to [0,\infty)$ is called an Ω-distance on X if the following conditions are satisfied:

(i) $\Omega(x,y,z) \leqslant \Omega(x,a,a) + \Omega(a,y,z)$ for all $x,y,z,a \in X$,
(ii) for any $x,y \in X$, $\Omega(x,y,\cdot), \Omega(x,\cdot,y) : X \to X$ are lower semicontinuous, and
(iii) for each $\epsilon > 0$, there exists a $\delta > 0$ such that $\Omega(x,a,a) \leqslant \delta$ and $\Omega(a,y,z) \leqslant \delta$ imply $G(x,y,z) \leqslant \epsilon$.

Definition 7. (See [27].) Let (X,G) be a G-metric space and Ω be an Ω-distance on X. Then we say that X is Ω-bounded if there exists $M \geqslant 0$ such that $\Omega(x,y,z) \leqslant M$ for all $x,y,x \in X$.

Moreover, Saadati et al. [27] proved the following important lemma in the setting of Ω-distance.

Lemma 1. (*See* [27].) *Let X be a G-metric space, and Ω be an Ω-distance on X. Let (x_n), (y_n) be sequences in X, and (α_n), (β_n) be sequences in $[0,\infty)$ converging to zero, and let $x,y,z,a \in X$. Then we have the following*:

(i) *If $\Omega(y,x_n,x_n) \leqslant \alpha_n$ and $\Omega(x_n,y,z) \leqslant \beta_n$ for all $n \in \mathbb{N}$, then $G(y,y,z) < \epsilon$ and hence $y = z$.*
(ii) *If $\Omega(y_n,x_n,x_n) \leqslant \alpha_n$ and $\Omega(x_n,y_m,z) \leqslant \beta_n$ for any $m > n \in \mathbb{N}$, then $G(y_n,y_m,z) \to 0$ and hence $y_n \to z$.*
(iii) *If $\Omega(x_n,x_m,x_l) \leqslant \alpha_n$ for any $m,n,l \in \mathbb{N}$ with $n \leqslant m \leqslant l$, then (x_n) is a G-Cauchy sequence.*
(iv) *If $\Omega(x_n,a,a) \leqslant \alpha_n$ for any $n \in \mathbb{N}$, then (x_n) is a G-Cauchy sequence.*

The concept of C-class function is given as follows:

Definition 8. (See [4].) A continuous function $F : [0,\infty)^2 \to \mathbb{R}$ is called a C*-class* function if it satisfies following axioms:

(i) $F(s,t) \leqslant s$, and
(ii) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s,t \in [0,\infty)$.

Note that for some F, we have that $F(0,0) = 0$.
We denote the set of C-class functions by $\mathcal{C}$.

Example 1. (See [4].) For $s,t \in [0,+\infty)$, define the functions $F : [0,\infty)^2 \to \mathbb{R}$ by

(a) $F(s,t) = s - t$.
(b) $F(s,t) = ms$ for some $m \in (0,1)$.
(c) $F(s,t) = s/(1+t)^r$ for some $r \in (0,\infty)$.
(d) $F(s,t) = \log(t + a^s)/(1+t)$ for some $a > 1$.
(e) $F(s,t) = \ln(1+a^s)/2$ for $e > a > 1$. Indeed, $f(s,t) = s$ implies that $s = 0$.
(f) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1$, for $r \in (0,\infty)$.

(g) $F(s,t) = s \log_{t+a} a$ for $a > 1$.
(h) $F(s,t) = s - (1+s)/(2+s) \cdot t/(1+t)$.
(i) $F(s,t) = s\beta(s)$, where $\beta : [0,\infty) \to [0,1)$is a continuous function.
(j) $F(s,t) = s - t/(k+t)$.
(k) $F(s,t) = s - \varphi(s)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.
(l) $F(s,t) = sh(s,t)$, where $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function with $h(s,t) < 1$ for all $s,t > 0$.
(m) $F(s,t) = s - (2+t)/(1+t) \cdot t$.
(n) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$.
(o) $F(s,t) = \phi(s)$, where $\phi : [0,\infty) \to [0,\infty)$ is an upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.
(p) $F(s,t) = s/(1+s)^r$, $r \in (0,\infty)$.
(q) $F(s,t) = s/\Gamma(1/2) \int_0^\infty \mathrm{e}^{-x}/(\sqrt{x}+t)\,\mathrm{d}x$, where Γ is the Euler gamma function.

Then the above functions are elements of $\mathcal{C}$.

Definition 9. (See [4].) Let $\varphi : [0,\infty) \to [0,\infty)$ be a continuous, nondecreasing function. Then φ is called an ultra altering distance function if $\varphi(t) > 0$ for all $t > 0$.

Remark 1. We denote the set of ultra altering distance functions by Φ_u.

3 Main result

In this section, we introduce some common fixed point results for mappings of cyclic form by utilizing the notion of Ω-distance in the sense of Saddati et al. [27].

Theorem 1. *Let (X,G) be a complete G-metric space, and Ω be an Ω-distance on X such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X such that $A \cap B \neq \emptyset$ and $X = A \cup B$. Let $f,g : A \cup B \to A \cup B$ be two mappings such that $f(A) \subseteq B$ and $g(B) \subseteq A$. Suppose that there exist $F \in \mathcal{C}$ and $\phi \in \Phi_u$ such that the following conditions hold*:

$$\Omega(fx,gy,gz) \leqslant F\bigl(\Omega(x,y,z), \phi\bigl(\Omega(x,y,z)\bigr)\bigr) \quad \forall x \in A,\ \forall y,z \in B, \tag{1}$$

$$\Omega(gx,fy,fz) \leqslant F\bigl(\Omega(x,y,z), \phi\bigl(\Omega(x,y,z)\bigr)\bigr) \quad \forall y,z \in A,\ \forall x \in B, \tag{2}$$

and

$$\Omega(fx,fy,fz) \leqslant F\bigl(\Omega(x,y,z), \phi\bigl(\Omega(x,y,z)\bigr)\bigr) \quad \forall x,y,z \in A. \tag{3}$$

Also, assume that if $fu \neq u$ or $gu \neq u$, then

$$\inf\bigl\{\Omega(fx,gfx,u)\colon x \in X\bigr\} > 0.$$

If f or g is continuous, then f and g have a unique common fixed point in $A \cap B$.

Proof. Let $x_0 \in A$. Since $f(A) \subseteq B$, then $fx_0 = x_1 \in B$. Also, since $g(B) \subseteq A$, then $gx_1 = x_2 \in B$. Continuing this process, we obtain a sequence (x_n) in X such that $fx_{2n} = x_{2n+1}$ with $x_{2n} \in A$ and $gx_{2n+1} = x_{2n+2}$ with $x_{2n+1} \in B$ for all $n \in \mathbb{N}$.

First, we want to show that

$$\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = \lim_{n\to\infty} \Omega(x_{n+1}, x_n, x_n) = 0.$$

Let $n \in \mathbb{N}$.

If n is even, then $n = 2t$, $t \in \mathbb{N}$. By (2) we get

$$\begin{aligned}\Omega(x_n, x_{n+1}, x_{n+1}) &= \Omega(x_{2t}, x_{2t+1}, x_{2t+1}) \\ &= \Omega(gx_{2t-1}, fx_{2t}, fx_{2t}) \\ &\leqslant F\big(\Omega(x_{2t-1}, x_{2t}, x_{2t}), \phi\big(\Omega(x_{2t-1}, x_{2t}, x_{2t})\big)\big) \\ &= F\big(\Omega(x_{n-1}, x_n, x_n), \phi\big(\Omega(x_{n-1}, x_n, x_n)\big)\big).\end{aligned} \tag{4}$$

If n is odd, then $n = 2t + 1$, $t \in \mathbb{N}$. By (1) we get

$$\begin{aligned}\Omega(x_n, x_{n+1}, x_{n+1}) &= \Omega(x_{2t+1}, x_{2t+2}, x_{2t+2}) \\ &= \Omega(fx_{2t}, gx_{2t+1}, gx_{2t+1}) \\ &\leqslant F\big(\Omega(x_{2t}, x_{2t+1}, x_{2t+1}), \phi\big(\Omega(x_{2t}, x_{2t+1}, x_{2t+1})\big)\big) \\ &= F\big(\Omega(x_{n-1}, x_n, x_n), \phi\big(\Omega(x_{n-1}, x_n, x_n)\big)\big).\end{aligned} \tag{5}$$

From (4) and (5) we have

$$\begin{aligned}\Omega(x_n, x_{n+1}, x_{n+1}) &\leqslant F(\Omega(x_{n-1}, x_n, x_n), \phi\big(\Omega(x_{n-1}, x_n, x_n)\big)) \\ &\leqslant \Omega(x_{n-1}, x_n, x_n) \quad \forall n \in \mathbb{N}.\end{aligned} \tag{6}$$

This shows that $\{\Omega(x_n, x_{n+1}, x_{n+1})\}$ is non-increasing. Thus, there exists $r \geqslant 0$ such that

$$\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = r.$$

If $r > 0$, then $\phi(r) > 0$. Letting $n \to \infty$ in (6), we obtain

$$F\big(r, \phi(r)\big) = r.$$

So, $r = 0$ or $\phi(r) = 0$. Since $\phi(r) > 0$, we have $r = 0$, which is a contradiction. Hence, $r = 0$ and so

$$\lim_{n\to\infty} \Omega(x_n, x_{n+1}, x_{n+1}) = 0. \tag{7}$$

In a similar, way we can show that

$$\lim_{n\to\infty} \Omega(x_{n+1}, x_n, x_n) = 0. \tag{8}$$

We claim that $\{x_{2n}\}$ is a G-Cauchy sequence. Suppose that $\{x_{2n}\}$ is not a G-Cauchy sequence. Then there exists $\varepsilon > 0$ and subsequences $\{x_{2n_k}\}$ and $\{x_{2m_k}\}$ such that n_k is the smallest integer with $2n_k > 2m_k > 2k$ and

$$\Omega(x_{2n_k}, x_{2m_k}, x_{2m_k}) \geqslant \varepsilon \quad \text{and} \quad \Omega(x_{2n_k-2}, x_{2m_k}, x_{2m_k}) \leqslant \varepsilon.$$

Then

$$\begin{aligned}\varepsilon &\leqslant \Omega(x_{2n_k}, x_{2m_k}, x_{2m_k}) \\ &\leqslant \Omega(x_{2n_k}, x_{2n_k-1}, x_{2n_k-1}) + \Omega(x_{2n_{k-1}}, x_{2n_k-2}, x_{2n_k-2}) \\ &\quad + \Omega(x_{2n_k-2}, x_{2m_k}, x_{2m_k}) \\ &\leqslant \Omega(x_{2n_k}, x_{2n_k-1}, x_{2n_k-1}) + \Omega(x_{2n_{k-1}}, x_{2n_k-2}, x_{2n_k-2}) + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ in above inequalities and using (8), we get

$$\lim_{k\to\infty} \Omega(x_{2n_k}, x_{2m_k}, x_{2m_k}) = \varepsilon. \tag{9}$$

Also,

$$\begin{aligned}\varepsilon &\leqslant \Omega(x_{2n_k}, x_{2m_k}, x_{2m_k}) \\ &\leqslant \Omega(x_{2n_k}, x_{2n_k+1}, x_{2n_k+1}) + \Omega(x_{2n_{k+1}}, x_{2m_k+1}, x_{2m_k+1}) \\ &\quad + \Omega(x_{2m_k+1}, x_{2m_k}, x_{2m_k}).\end{aligned}$$

By (3) we have

$$\begin{aligned}&\Omega(x_{2n_k+1}, x_{2m_k+1}, x_{2m_k+1}) \\ &\quad = \Omega(fx_{2n_k}, fx_{2m_k}, fx_{2m_k}) \\ &\quad \leqslant F\big(\Omega(x_{2n_k}, x_{2m_k}, x_{2m_k}), \phi\big(\Omega(x_{2n_k-1}, x_{2m_k}, x_{2l_k})\big)\big) \\ &\quad \leqslant \Omega(x_{2n_k}, x_{2m_k}, x_{2l_k}).\end{aligned}$$

So,

$$\begin{aligned}\varepsilon &\leqslant \Omega(x_{2n_k}, x_{2n_k+1}, x_{2n_k+1}) + \Omega(x_{2n_{k+1}}, x_{2m_k+1}, x_{2m_k+1}) \\ &\quad + \Omega(x_{2m_k+1}, x_{2m_k}, x_{2m_k}) \\ &\leqslant \Omega(x_{2n_k}, x_{2n_k+1}, x_{2n_k+1}) + \Omega(x_{2n_k}, x_{2m_k}, x_{2m_k}) \\ &\quad + \Omega(x_{2m_k+1}, x_{2m_k}, x_{2m_k}).\end{aligned}$$

As k$\to \infty$ in the above inequalities and using (7), (8) and (9), we get

$$\lim_{k\to\infty} \Omega(x_{2n_{k+1}}, x_{2m_k+1}, x_{2m_k+1}) = \varepsilon.$$

Again, by (3) we have

$$\begin{aligned}&\Omega(x_{2n_{k+1}}, x_{2m_k+1}, x_{2m_k+1}) \\ &\quad = \Omega(fx_{2n_k}, fx_{2m_k}, fx_{2m_k}) \\ &\quad \leqslant F\big(\Omega(x_{2n_k}, x_{2m_k}, x_{2m_k}), \phi\big(\Omega(x_{2n_k}, x_{2m_k}, x_{2m_k})\big)\big).\end{aligned}$$

As $k \to \infty$ in the above inequality, we have

$$\varepsilon \leqslant F\bigl(\varepsilon, \phi(\varepsilon)\bigr).$$

So, $\varepsilon = 0$ or $\phi(\varepsilon) = 0$. Since $\phi(\varepsilon) > 0$, we get $\varepsilon = 0$, which is a contradiction.

Hence, (x_{2n}) is a G-Cauchy sequence. So, there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Since (x_n) is G-convergent to u, then every subsequence of (x_n) is also G-convergent to u. So that the subsequences $(x_{2n+1}) = (fx_{2n})$ and $(x_{2n+2}) = (gx_{2n+1})$ are G-convergent to u.

With out loss of generality, we assume that f is continuous. So,

$$\lim_{n\to\infty} fx_{2n} = fu.$$

Since

$$\lim_{n\to\infty} x_{2n+1} = u,$$

then by uniqueness of the limit we have $fu = u$.

By the lower semicontinuity of Ω we get

$$\Omega(x_n, x_m, u) \leqslant \liminf_{p\to\infty} \Omega(x_n, x_m, x_p) \leqslant \epsilon$$

for all $m \geqslant n$.

Now, suppose that $gu \neq u$, then we get

$$0 < \inf\bigl\{\Omega(fx, gfx, u)\colon x \in X\bigr\} \leqslant \inf\bigl\{\Omega(x_n, x_{n+1}, u)\colon n \text{ is odd}\bigr\} \leqslant \epsilon$$

for every $\epsilon > 0$, which is a contradiction. Therefore, $fu = gu = u$.

Since $(x_{2n}) \subseteq A$ and A is closed, we have $u \in A$. Also, since $(x_{2n+1}) \subseteq B$ and B is closed, we have $u \in B$. Hence, u is a common fixed point of f and g in $A \cap B$.

To prove the uniqueness of u, we assume that there exists $v \in X$ such that $fv = gv = v$. Then by (1) we have

$$\begin{aligned}\Omega(u, v, v) = \Omega(fu, gv, gv) &\leqslant F\bigl(\Omega(u, v, v), \phi\bigl(\Omega(u, v, v)\bigr)\bigr)\\ &\leqslant \Omega(u, v, v).\end{aligned}$$

So, $\Omega(u, v, v) = 0$ or $\phi(\Omega(u, v, v)) = 0$. Thus, $\Omega(u, v, v) = 0$. Also,

$$\begin{aligned}\Omega(v, u, v) = \Omega(fv, gu, gv) &\leqslant F\bigl(\Omega(v, u, v), \phi\bigl(\Omega(v, u, v)\bigr)\bigr)\\ &\leqslant \Omega(v, u, v).\end{aligned}$$

So, $\Omega(u, v, v) = 0$ or $\phi(\Omega(u, v, v)) = 0$. Thus, $\Omega(v, u, v) = 0$. According to the definition of the Ω-distance, we conclude that $G(u, u, v) = 0$ and hence $u = v$. Thus, f and g have a unique common fixed point in $A \cap B$. □

By choosing $A = B = X$ in Theorem 1, we get the following result:

Corollary 1. *Let (X, G) be a complete G-metric space, and Ω be an Ω-distance on X such that X is Ω-bounded. Let $f, g : X \to X$ be two mappings. Suppose that there exist $F \in \mathcal{C}$ and $\phi \in \Phi_u$ such that the following condition holds for all $x, y, z \in X$:*

$$\max\{\Omega(fx, gy, gz), \Omega(gx, fy, fz), \Omega(fx, fy, fz)\} \\ \leqslant F\bigl(\Omega(x, y, z), \phi\bigl(\Omega(x, y, z)\bigr)\bigr).$$

Moreover, assume that if $fu \neq u$ or $gu \neq u$, then

$$\inf\{\Omega(fx, gfx, u)\colon x \in X\} > 0.$$

If f or g is continuous, then f and g have a unique common fixed point.

Corollary 2. *Let (X, G) be a complete G-metric space, and Ω be an Ω-distance on X such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X such that $A \cap B \neq \emptyset$ and $X = A \cup B$. Let $f, g : A \cup B \to A \cup B$ be two mappings such that $f(A) \subseteq B$ and $g(B) \subseteq A$. Suppose that there exists $\alpha \in [0, 1)$ such that the following conditions hold:*

$$\Omega(fx, gy, gz) \leqslant \alpha\Omega(x, y, z) \quad \forall x \in A,\ \forall y, z \in B,$$
$$\Omega(gx, fy, fz) \leqslant \alpha\Omega(x, y, z) \quad \forall y, z \in A,\ \forall x \in B,$$

and

$$\Omega(fx, fy, fz) \leqslant \alpha\Omega(x, y, z) \quad \forall x, y, z \in A.$$

Also, assume that if $fu \neq u$ or $gu \neq u$, then

$$\inf\{\Omega(fx, gfx, u)\colon x \in X\} > 0.$$

If f or g is continuous, then f and g have a unique common fixed point in $A \cap B$.

Proof. Define $F : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ by $F(s, t) = \alpha s$. Note that $F \in \mathcal{C}$. The result follows from Theorem 1. □

By choosing $A = B = X$ in Corollary 2 we get the following result:

Corollary 3. *Let (X, G) be a complete G-metric space, and Ω be an Ω-distance on X such that X is Ω-bounded. Let $f, g : X \to X$ be two mappings. Assume that there exists $\alpha \in [0, 1)$ such that the following condition holds for all $x, y, z \in X$:*

$$\max\{\Omega(fx, gy, gz), \Omega(gx, fy, fz), \Omega(fx, fy, fz)\} \leqslant \alpha\Omega(x, y, z).$$

Moreover, assume that if $fu \neq u$ or $gu \neq u$, then

$$\inf\{\Omega(fx, gfx, u)\colon x \in X\} > 0.$$

If f or g is continuous, then f and g have a unique common fixed point in X.

It is worth mentioning that the condition: If $fu \neq u$ or $gu \neq u$, then

$$\inf\{\Omega(fx, gfx, u)\colon x \in X\} > 0$$

in Theorem 1 can be dropped if g is replaced by f. So, we have the following result:

Theorem 2. *Let (X, G) be a complete G-metric space, and Ω be an Ω-distance on X such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X such that $A \cap B \neq \emptyset$ and $X = A \cup B$. Let $f : A \cup B \to A \cup B$ be a cyclic mapping. Suppose that there exist $F \in \mathcal{C}$ and $\phi \in \Phi_u$ such that the following conditions hold*:

$$\Omega(fx, fy, fz) \leqslant F\bigl(\Omega(x, y, z), \phi\bigl(\Omega(x, y, z)\bigr)\bigr) \quad \forall x \in A,\ \forall y, z \in B,$$
$$\Omega(fx, fy, fz) \leqslant F\bigl(\Omega(x, y, z), \phi\bigl(\Omega(x, y, z)\bigr)\bigr) \quad \forall y, z \in A,\ \forall x \in B,$$

and

$$\Omega(fx, fy, fz) \leqslant F\bigl(\Omega(x, y, z), \phi\bigl(\Omega(x, y, z)\bigr)\bigr) \quad \forall x, y, z \in A.$$

If f is continuous, then f has a unique fixed point in $A \cap B$.

Proof. Following the proof of Theorem 1 word by word, we can deduce the proof of this theorem. □

By choosing $A = B = X$ in Theorem 2 we get the following result:

Corollary 4. *Let (X, G) be a complete G-metric space, and Ω be an Ω-distance on X such that X is Ω-bounded. Let $f : X \to X$ be a mapping. Suppose that there exist $F \in \mathcal{C}$ and $\phi \in \Phi_u$ such that the following condition holds for all $x, y, z \in X$*:

$$\Omega(fx, fy, fz) \leqslant F\bigl(\Omega(x, y, z), \phi\bigl(\Omega(x, y, z)\bigr)\bigr).$$

If f is continuous, then f has a unique fixed point.

Corollary 5. *Let (X, G) be a complete G-metric space, and Ω be an Ω-distance on X such that X is Ω-bounded. Let A and B be two nonempty closed subsets of X such that $A \cap B \neq \emptyset$ and $X = A \cup B$. Let $f : A \cup B \to A \cup B$ be a cyclic mappings. Assume that there exists $\alpha \in [0, 1)$ such that the following conditions hold*:

$$\Omega(fx, fy, fz) \leqslant \alpha\Omega(x, y, z) \quad \forall x \in A,\ \forall y, z \in B,$$
$$\Omega(fx, fy, fz) \leqslant \alpha\Omega(x, y, z) \quad \forall y, z \in A,\ \forall x \in B,$$

and

$$\Omega(fx, fy, fz) \leqslant \alpha\Omega(x, y, z) \quad \forall x, y, z \in A.$$

If f is continuous, then f has a unique fixed point in $A \cap B$.

Proof. Define $F : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ by $F(s, t) = \alpha s$. Note that $F \in \mathcal{C}$. So, the result follows from Corollary 4. □

By choosing $A = B = X$ in Corollary 5 we have the following result:

Corollary 6. *Let (X, G) be a complete G-metric space, and Ω be an Ω-distance on X such that X is Ω-bounded. Let $f : X \to X$ be a mapping. Suppose that there exists $\alpha \in [0, 1)$ such that the following condition holds for all $x, y, z \in X$:*

$$\Omega(fx, fy, fz) \leqslant \alpha\Omega(x, y, z).$$

If f is continuous, then f has a unique fixed point in X.

We introduce the following example to support our main result:

Example 2. Let $X = [-1, 1]$. Define $G : X \times X \times X \to [0, \infty)$ by $G(x, y, z) = |x-y| + |y-z| + |x-z|$, and define $\Omega : X \times X \times X \to [0, \infty)$ by $\Omega(x, y, z) = |x-y| + |x-z|$. Let $A = [-1, 0]$, $B = [0, 1]$, and define $f, g : A \cup B \to A \cup B$ by $fx = -x/10$ and $gx = -x/5$. Also, define $\phi : [0, \infty) \to [0, \infty)$ and $F(s, t) : [0, \infty)^2 \to \mathbb{R}$ by $\phi(x) = 4$ and $F(s, t) = s/(1 + t)$. Then

(a) (X, G) is a complete G-metric space.
(b) Ω is an Ω-distance on X, and X is Ω-bounded.
(c) A and B are closed subsets of X with respect to the topology induced by G.
(d) f and g are continuous.
(e) $f(A) \subseteq B$ and $g(B) \subseteq A$.
(f) $\phi \in \Phi_u$ and $F \in \mathcal{C}$.
(g) f and g satisfy the following inequalities:

$$\Omega(fx, gy, gz) \leqslant F\big(\Omega(x, y, z), \phi\big(\Omega(x, y, z)\big)\big) \quad \forall x \in A,\ \forall y, z \in B,$$
$$\Omega(gx, fy, fz) \leqslant F\big(\Omega(x, y, z), \phi\big(\Omega(x, y, z)\big)\big) \quad \forall y, z \in A,\ \forall x \in B,$$

and

$$\Omega(fx, fy, fz) \leqslant F\big(\Omega(x, y, z), \phi\big(\Omega(x, y, z)\big)\big) \quad \forall x, y, z \in A.$$

(h) If $fu \neq u$ or $gu \neq u$, then $\inf\{\Omega(fx, gfx, u)\colon\ x \in X\} > 0$.

Proof. The proof of (a)–(f) is clear. To prove (g), let $x \in A$ and $y, z \in B$. Then

$$\begin{aligned}\Omega(fx, gy, gz) &= \Omega\left(-\frac{x}{10}, -\frac{y}{5}, -\frac{z}{5}\right) = \left|-\frac{x}{10} - \frac{y}{5}\right| + \left|-\frac{x}{10} - \frac{z}{5}\right| \\ &= \left|-\frac{x}{10} + \frac{y}{5}\right| + \left|-\frac{x}{10} + \frac{z}{5}\right| \leqslant \left|-\frac{x}{5} + \frac{y}{5}\right| + \left|-\frac{x}{5} + \frac{z}{5}\right| \\ &\leqslant \frac{|x-y| + |x-z|}{5} = \frac{\Omega(x, y, z)}{1+4} = \frac{\Omega(x, y, z)}{1 + \phi(\Omega(x, y, z))} \\ &= F\big(\Omega(x, y, z), \phi\big(\Omega(x, y, z)\big)\big).\end{aligned}$$

Again, let $x \in B$ and $y, z \in A$. Then

$$\begin{aligned}\Omega(gx, fy, fz) &= \Omega\left(-\frac{x}{5}, -\frac{y}{10}, -\frac{z}{10}\right) = \left|-\frac{x}{5} - \frac{y}{10}\right| + \left|-\frac{x}{5} - \frac{z}{10}\right| \\ &\leqslant \left|-\frac{x}{5} + \frac{y}{5}\right| + \left|-\frac{x}{5} + \frac{z}{5}\right| \leqslant \frac{|x-y|}{5} + \frac{|x-z|}{5} \\ &\leqslant \frac{|x-y| + |x-z|}{5} = \frac{\Omega(x,y,z)}{1 + \phi(\Omega(x,y,z))} \\ &= F\big(\Omega(x,y,z), \phi\big(\Omega(x,y,z)\big)\big).\end{aligned}$$

Finally, let $x, y, z \in A$. Then

$$\begin{aligned}\Omega(fx, fy, fz) &= \Omega\left(-\frac{x}{10}, -\frac{y}{10}, -\frac{z}{10}\right) = \left|-\frac{x}{10} - \frac{y}{10}\right| + \left|-\frac{x}{10} - \frac{z}{10}\right| \\ &= \frac{|x-y|}{10} + \frac{|x-z|}{10} = \frac{|x-y| + |x-z|}{10} \\ &\leqslant \frac{|x-y| + |x-z|}{5} = \frac{\Omega(x,y,z)}{1 + \phi(\Omega(x,y,z))} \\ &= F\big(\Omega(x,y,z), \phi\big(\Omega(x,y,z)\big)\big).\end{aligned}$$

To prove (h), assume that $fu \neq u$ or $gu \neq u$. Then $u \neq 0$. Therefore,

$$\begin{aligned}&\inf\big\{\Omega(fx, gfx, u)\colon x \in X\big\} \\ &\quad= \inf\left\{\Omega\left(-\frac{x}{10}, \frac{x}{50}, u\right)\colon x \in X\right\} \\ &\quad= \inf\left\{\left|-\frac{x}{10} - \frac{x}{50}\right| + \left|-\frac{x}{10} - u\right|\colon x \in X\right\} \\ &\quad= \inf\left\{\frac{6|x|}{50} + \left|\frac{x}{10} + u\right|\colon x \in X\right\} = |u| > 0.\end{aligned}$$

The above example satisfies all the hypotheses of Theorem 1. Hence, f and g have a unique common fixed point in $A \cap B$. Here 0 is the unique common fixed point of f and g in $A \cap B$. $\square$

4 Conclusion

In their research [27], Saadati et al. introduced the notion of Ω-distance as a generalization of a G-metric space in the sense of Mustafa and Sims [22] and studied many interesting fixed and common fixed point theorems. While, Ansari [4] introduced the notion of the C-class function and studied some fixed point theorems. In this paper, we utilized the notion of C-class function to formulate and prove a common fixed point theorem for two mappings of cyclic form under some contractive conditions. Also, we utilized our main theorem to derive a fixed point theorem for a cyclic mapping under some contractive conditions. Moreover, we derived many fixed and common fixed point results from our main theorems.

Acknowledgment. We would like to express our sincere thanks for the reviewers for their valuable comments on this paper that made this paper complete and significant.

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