Nonlinear Analysis: Modelling and Control, Vol. 22, No. 6, 753–764 ISSN 1392-5113
https://doi.org/10.15388/NA.2017.6.2

Mild solutions of Riemann–Liouville fractional differential equations with fractional impulses

Annamalai Anguraj[a], **Subramaniam Kanjanadevi**[a], **Juan Jose Nieto**[b,c,1]

[a]Department of Mathematics, PSG College of Arts and Science,
Coimbatore-641 014, Tamil Nadu, India
angurajpsg@yahoo.com; kanjanadevimaths@gmail.com

[b]Department of Mathematics, Faculty of Science,
King Abdulaziz University,
Jeddah 21589, Saudi Arabia

[c]Department of Mathematical Analysis, Faculty of Mathematics,
University of Santiago de Compostela,
Santiago de Compostela 15782, Spain
juanjose.nieto.roig@usc.es

Received: November 1, 2016 / **Revised:** June 25, 2017 / **Published online:** October 27, 2017

Abstract. We consider Riemann–Liouville fractional differential equations with fractional-order derivative in the impulsive conditions. We study the existence of the mild solution by applying the Laplace transform method and (a,k)-regularized resolvent operator. We use the contraction mapping principle and fixed point theorem for condensing map to prove our existence results.

Keywords: fractional derivative, fractional impulsive conditions, (a,k)-regularized resolvent operator, Riemann–Liouville operator.

1 Introduction

In recent years, fractional differential equations have been considered to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Since the definition of fractional derivative exhibits the past history of the functions, fractional-order derivative has been found as an excellent mathematical tool for characterizing memory and hereditary properties of complex systems such as viscoelastic deformation, anomalous diffusion, signal processing, stock market, and so on. This is the main advantage of using fractional-order derivatives to real world problems in comparison with integer-order derivatives. For more detailed work, see [5, 7, 17, 21, 23] and the references therein.

[1]The author has been partially supported by the Ministerio de Economía y Competitividad of Spain under grant MTM2013-43014-P, XUNTA under grants R2014/002 and GRC2015/004, and co-financed by the European Community fund FEDER.

Impulsive differential equations have more consideration because of its extensive applications in physics, biology, medical, engineering, and technology [1, 2, 11]. Impulsive differential equations are suitable model for describing processes, which change their state abruptly at a certain moment. In most of the study on fractional impulsive differential equations, the Caputo derivative is used to define the system, and integer-order derivative is used in the impulsive conditions [4, 6, 18, 20, 22]. In [9], Podulbny shown that it is possible to attribute physical meaning to initial conditions defined in the form of Riemann–Liouville fractional derivatives in the field of viscoelasticity. In [10], Kosmatov introduced the fractional-order derivative in the impulsive conditions to study the existence of solutions of fractional impulsive differential equations using both Caputo and the Riemann–Liouville derivatives.

In [4], Fečkan et al. cited some papers in which the mild solutions are defined inappropriatly. But in [19], Wang et al. refuted Fečkan's argument and shown that the example framed by Fečkan in [4] was wrong. But we observed that Fečkan's arguments are correct if the lower limit is taken as zero in equation (1.14) of [19]. The above dispute will not be arised if the lower bound is taken as same in the statement of the problem, definition of fractional derivative and the definition of solution. These comments are also justified by Fečkan et al. in [3] and Liu et al. in [12].

In [8], Hernández et al. have shown that the definition of mild solutions is not appropriate in some recent papers. To make it more appropriate, he introduced the resolvent operator for integral equations to define the mild solutions in [8].

But in [14], Lizama et al. studied that not all the fractional equation can be formulated as an integral equation, so that the concept proposed in [8] fails in the general case. He also derived a suitable variation of constant formula for a large class of fractional differential equations using Laplace transform and with the help of (a, k)-regularized resolvent families. The advantage of this approach is that the domain of $\mathcal{A}$ is not necessarily dense in $\mathscr{X}$ [13].

In the present work, in order to overcome the above disputes, we study the existence of mild solution for Riemann–Liouville fractional differential equations with fractional-order impulsive conditions of the form

$$D^{\alpha}_{0+}u(t) = \mathcal{A}u(t) + f\bigl(t, u(t)\bigr), \quad t \in \mathcal{I} := (0, 1] \setminus \{t_1, \dots, t_m\}, \tag{1}$$

$$D^{\beta}_{0+}u\bigl(t_k^+\bigr) - D^{\beta}_{0+}u\bigl(t_k^-\bigr) = \mathscr{J}_k\bigl(u(t_k)\bigr), \quad k = 1, \dots, m, \tag{2}$$

$$I^{1-\alpha}_{0+}u(0) = u_0, \tag{3}$$

where $u_0 \in \mathscr{X}$, $0 < \beta < \alpha < 1$, the operator $\mathcal{A}$ generates an $(t^{\alpha-1}/\Gamma(\alpha), 1)$-regularized resolvent family $\{\mathscr{S}_\alpha(t)\}_{t \geqslant 0}$ on a Banach space $\mathscr{X}$, $0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1$ are prefixed numbers. The functions $f : \mathcal{I} \times \mathbb{R} \to \mathbb{R}$, $\mathscr{J}_k : \mathbb{R} \to \mathbb{R}$ are continuous, and $f(t_k\pm, x)$ exist for all $x \in \mathbb{R}$, $k = 1, \dots, m.$

Here we derive the mild solution of the above problem (1)–(3) by using the Laplace transform method and $(t^{\alpha-1}/\Gamma(\alpha), 1)$-regularized resolvent operator. The existence results of Riemann–Liouville fractional differential equations with fractional-order impulsive conditions is established by means of a fixed point theorem for condensing map, and the uniqueness result is verified via contraction mapping principle.

2 Preliminaries

Let $\mathcal{L}(\mathscr{X})$ be the space of bounded linear operators from Banach space $\mathscr{X}$ into $\mathscr{X}$ with norm $\|\cdot\|_{\mathcal{L}(\mathscr{X})}$. The domain of $\mathcal{A}$ is provided with the graph norm $\|\cdot\|_{\mathcal{D}(\mathcal{A})} = \|u\|+\|\mathcal{A}u\|$, and denote by $\mathcal{B}_r(u, \mathscr{X})$ the closed ball with center at u and distance r in $\mathscr{X}$. The space $C([0,1]; \mathscr{X})$ symbolizes the space of continuous functions with norm $\|\cdot\|_{C([0,1];\mathscr{X})} = \sup_{t\in[0,1]} \|u(t)\|_{\mathscr{X}}$.

We consider the space $\mathcal{PC}(\mathscr{X})$, which is formed by all the functions $u : [0,1] \to \mathbb{R}$ such that $u(\cdot)$ is continuous at $t \neq t_j$, $u(t_j^-) = u(t_j)$ and $u(t_j^+)$ exists for all $j = 1,2,\dots,N$. The space $\mathcal{PC}(\mathscr{X})$ is a Banach space with respect to the norm $\|u\|_{\mathcal{PC}(\mathscr{X})} = \sup_{t\in[0,1]} \|u(t)\|$.

Here we use the (a,k)-regularized resolvent family, which was initialized in [13] and analyzed in some recent papers [14].

The operator $\{\mathscr{S}_\alpha(t)\}_{t\geqslant 0}$ is of type $(\mathcal{M},\omega)$ if there exist constant $\mathcal{M} \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\|\mathscr{S}_\alpha(t)\| \leqslant \mathcal{M}\mathrm{e}^{\omega t}$ for all $t \geqslant 0$. Suppose if $\omega = 0$, then $\|\mathscr{S}_\alpha(t)\|_{\mathcal{L}(\mathscr{X})} \leqslant \mathcal{M}$, $t \geqslant 0$.

In [13, Prop. 3.1], Lizama established that the Laplace transform of strongly continuous family $\hat{\mathscr{R}}_\alpha(\lambda)$ in $\mathcal{L}(\mathscr{X})$ exists for $\lambda > \omega$. It follows that the Laplace transform of $(t^{\alpha-1}/\Gamma(\alpha), 1)$-regularized resolvent family $\{\mathscr{S}_\alpha(t)\}_{t\geqslant 0}$ is

$$\hat{\mathscr{S}}_\alpha(\lambda) = \lambda^{\alpha-1}\bigl(\lambda^\alpha - \mathcal{A}\bigr)^{-1},$$

see [14].

The Riemann–Liouville fractional integral of a function $u \in L^p(0,1)$, $1 \leqslant p < \infty$, of order $\alpha > 0$ is

$$I_{0+}^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,\mathrm{d}s.$$

The fractional derivative of order $\alpha > 0$ is defined in the Riemann–Liouville sense as

$$D_{0+}^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\,\mathrm{d}s, \quad n = \lceil \alpha \rceil.$$

In general, Riemann–Liouville derivative is a left inverse of the operator I_{0+}^α but not a right inverse. That is, we have $D_{0+}^\alpha I_{0+}^\alpha u(t) = u(t)$ and $I_{0+}^\alpha D_{0+}^\alpha u(t) = u(t) - (I_{0+}^{1-\alpha} u(0)/\Gamma(\alpha))t^{\alpha-1}$ for $0 < \alpha < 1$.

We also recollect the formula for the Laplace transform of Riemann–Liouville derivative given by

$$\hat{D}_{0+}^\alpha u(\lambda) = \lambda^\alpha \hat{u}(\lambda) - \sum_{k=0}^{n-1} \lambda^k D_{0+}^{\alpha-k-1} u(0),$$

and when $0 < \alpha \leqslant 1$,

$$\hat{D}_{0+}^\alpha u(\lambda) = \lambda^\alpha \hat{u}(\lambda) - D_{0+}^{\alpha-1} u(0).$$

Now we derive the mild solution of equations (1)–(3).

First, we split $u(\cdot)$ as $\mathfrak{p}(\cdot) + \mathfrak{q}(\cdot)$ in problem (1)–(3), where $\mathfrak{p}$ is the continuous mild solution for

$$D^{\alpha}_{0^+}\mathfrak{p}(t) = \mathcal{A}\mathfrak{p}(t) + f\big(t, u(t)\big), \quad t \in \mathcal{I}, \tag{4}$$

$$I^{1-\alpha}_{0^+}\mathfrak{p}(0) = u_0, \tag{5}$$

and $\mathfrak{q}$ is the $\mathcal{PC}$-mild solution for

$$D^{\alpha}_{0^+}\mathfrak{q}(t) = \mathcal{A}\mathfrak{q}(t), \quad t \in \mathcal{I}, \tag{6}$$

$$D^{\beta}_{0^+}\mathfrak{q}(t_k^+) - D^{\beta}_{0^+}\mathfrak{q}(t_k^-) = \mathscr{J}_k\big(\mathfrak{u}(t_k)\big), \quad k = 1, \dots, m, \tag{7}$$

$$I^{1-\alpha}_{0^+}\mathfrak{q}(0) = 0. \tag{8}$$

Note that $\mathfrak{p}$ is continuous, so $\mathfrak{p}(t_k^+) = \mathfrak{p}(t_k^-)$, $k = 1, \dots, m$, and any solution of (1)–(3) can be decomposed to the solutions of (4)–(5) and (6)–(8).

Applying Laplace transform and inverse Laplace transform technique to (4), the mild solution of (4)–(5) can be written as

$$\begin{aligned}\mathfrak{p}(t) = {}& \frac{1}{\Gamma(\alpha-1)}\int_0^t \mathscr{S}_\alpha(t-s)s^{\alpha-2}u_0\,\mathrm{d}s \\ &+ \frac{1}{\Gamma(\alpha-1)}\int_0^t \mathscr{S}_\alpha(t-s)\int_0^s (s-\tau)^{\alpha-2} f\big(\tau, u(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s \end{aligned}\tag{9}$$

on $\mathcal{I}$.

Next, we rewrite (6)–(8) into the equivalent integral equation. For that, we operate fractional integral operator $I^{\alpha}_{0^+}$ on each side of (6), which gives

$$\mathfrak{q}(t) = I^{\alpha}_{0^+}\mathcal{A}\mathfrak{q}(t), \quad t \in (0, t_1].$$

Now,

$$D^{\beta}_{0^+}\mathfrak{q}(t) = I^{\alpha-\beta}_{0^+}\mathcal{A}\mathfrak{q}(t), \quad t \in (0, t_1].$$

In particular,

$$D^{\beta}_{0^+}\mathfrak{q}\big(t_1^-\big) = I^{\alpha-\beta}_{0^+}\mathcal{A}\mathfrak{q}(\cdot)(t_1), \quad t \in (0, t_1].$$

Since the impulsive condition (7) is a discontinuity condition, we have

$$\mathfrak{q}(t) = c_1 t^{\alpha-1} + I^{\alpha}_{0^+}\mathcal{A}\mathfrak{q}(t), \quad t \in (t_1, t_2].$$

Then, for $t \in (t_1, t_2]$, we get

$$D^{\beta}_{0^+}\mathfrak{q}(t) = \frac{c_1\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} + I^{\alpha-\beta}_{0^+}\mathcal{A}\mathfrak{q}(t), \quad t \in (t_1, t_2].$$

So that

$$D^{\beta}_{0+}\mathfrak{q}(t_1^+) = \frac{c_1\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t_1^{\alpha-\beta-1} + I_{0+}^{\alpha-\beta}\mathcal{A}\mathfrak{q}(\cdot)(t_1), \quad t \in (t_1, t_2].$$

From (7) we obtain

$$c_1 = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t_1^{1+\beta-\alpha}\mathscr{J}_1\big(u(t_1)\big).$$

Hence,

$$\mathfrak{q}(t) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}t_1^{1+\beta-\alpha}\mathscr{J}_1\big(u(t_1)\big)t^{\alpha-1} + I_{0+}^{\alpha}\mathcal{A}\mathfrak{q}(t), \quad t \in (t_1, t_2].$$

Proceeding in this way, we can derive for the remaining interval $\mathcal{I}$, and we get the integral equation as

$$\mathfrak{q}(t) = \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\bigg(\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\mathscr{J}_k\big(u(t_k)\big)\bigg)t^{\alpha-1} + I_{0+}^{\alpha}\mathcal{A}\mathfrak{q}(t), \quad t \in \mathcal{I}. \tag{10}$$

Taking Laplace transform on both sides, we get

$$\begin{aligned}\hat{\mathfrak{q}}(\lambda) &= \Gamma(\alpha-\beta)\bigg(\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\mathscr{J}_k\big(u(t_k)\big)\bigg)\big(\lambda^{\alpha}-\mathcal{A}\big)^{-1}\\ &= \hat{\mathscr{S}}_{\alpha}(\lambda)\lambda^{1-\alpha}\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\mathscr{J}_k\big(u(t_k)\big)\Gamma(\alpha-\beta),\end{aligned}$$

where $\hat{\mathscr{S}}_{\alpha}(\lambda) = \lambda^{\alpha-1}(\lambda^{\alpha}-\mathcal{A})^{-1}$.

Taking the inverse Laplace transform on each side of the above equation, we get the mild solution of (6)–(8) as

$$\mathfrak{q}(t) = \Gamma(\alpha-\beta)\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\mathscr{J}_k\big(u(t_k)\big)\int_0^t \frac{\mathscr{S}_{\alpha}(t-s)}{\Gamma(\alpha-1)}s^{\alpha-2}\,\mathrm{d}s, \quad t \in \mathcal{I}. \tag{11}$$

Finally, from (9) and (11) we can obtain the mild solution of problem (1)–(3) as

$$\begin{aligned}u(t) &= \frac{1}{\Gamma(\alpha-1)}\int_0^t \mathscr{S}_{\alpha}(t-s)s^{\alpha-2}u_0\,\mathrm{d}s\\ &\quad+ \Gamma(\alpha-\beta)\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\mathscr{J}_k\big(u(t_k)\big)\int_0^t \frac{\mathscr{S}_{\alpha}(t-s)}{\Gamma(\alpha-1)}s^{\alpha-2}\,\mathrm{d}s\\ &\quad+ \frac{1}{\Gamma(\alpha-1)}\int_0^t \mathscr{S}_{\alpha}(t-s)\int_0^s (s-\tau)^{\alpha-2}f\big(\tau, u(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s, \quad t \in \mathcal{I}.\end{aligned} \tag{12}$$

Definition 1. A function u in $\mathcal{PC}(\mathscr{X})$ is a mild solution of problem (1)–(3) if

$$
\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha-1)}\int_0^t \mathscr{S}_\alpha(t-s)s^{\alpha-2}u_0\,\mathrm{d}s \\
&\quad + \Gamma(\alpha-\beta)\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\mathscr{J}_k\big(u(t_k)\big)\int_0^t \frac{\mathscr{S}_\alpha(t-s)}{\Gamma(\alpha-1)}s^{\alpha-2}\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t \mathscr{S}_\alpha(t-s)\int_0^s (s-\tau)^{\alpha-2}f\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s, \quad t\in\mathcal{I}, \qquad (13)
\end{aligned}
$$

is satisfied.

3 Existence and uniqueness results

We assume the hypotheses given below:

(H1) $f:\mathcal{I}\times\mathbb{R}\to\mathbb{R}$ is continuous, and there exists a function $L_f\in C([0,1];\mathbb{R}^+)$ such that $\|f(t,u)-f(t,v)\|\leqslant L_f(t)\|u-v\|$, $t\in\mathcal{I}$ and u,v in $\mathbb{R}$.

(H2) There exists a non-decreasing function $\mathcal{W}\in C([0,\infty)\to(0,\infty))$ and $m_f\in C([0,1];\mathbb{R}^+)$ such that $\|f(t,u)\|\leqslant m_f(t)\mathcal{W}(\|u\|)$, t in $\mathcal{I}$ and u in $\mathbb{R}$.

(H3) The function $\mathscr{J}_k:\mathbb{R}\to\mathbb{R}$ is continuous, and the constants $K,K_c>0$ such that $\|\mathscr{J}_k(u)-\mathscr{J}_k(v)\|\leqslant K\|u-v\|$ and $\|\mathscr{J}_k(u)\|\leqslant K_c\|u\|$, $u,v\in\mathbb{R}$, $k=1,\dots,m$.

Theorem 1. *Assume that* (H1) *and* (H3) *are satisfied and that*

$$
\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}K\mathcal{M}\sum_{0<t_k<t} t_k^{1+\beta-\alpha} + \frac{\mathcal{M}}{\Gamma(\alpha+1)}\big\|L_f(t)\big\| < 1.
$$

Then there exists a unique mild solution of (1)–(3).

Proof. Define the fixed point operator $\mathcal{T}:\mathcal{PC}(\mathscr{X})\to\mathcal{PC}(\mathscr{X})$ by

$$
\begin{aligned}
\mathcal{T}u(t) &= \int_0^t \mathscr{S}_\alpha(t-s)\frac{s^{\alpha-2}}{\Gamma(\alpha-1)}u_0\,\mathrm{d}s \\
&\quad + \Gamma(\alpha-\beta)\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\mathscr{J}_k\big(u(t_k)\big)\int_0^t \mathscr{S}_\alpha(t-s)\frac{s^{\alpha-2}}{\Gamma(\alpha-1)}\,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha-1)}\int_0^t \mathscr{S}_\alpha(t-s)\int_0^s (s-\tau)^{\alpha-2}f\big(\tau,u(\tau)\big)\,\mathrm{d}\tau\,\mathrm{d}s, \quad t\in\mathcal{I}.
\end{aligned}
$$

From the assumption we easily conclude that $\mathcal{T}$ is well defined. Now we prove that $\mathcal{T}$ is contraction.

Let u and v in $\mathcal{PC}(\mathscr{X})$ and $t \in \mathcal{I}$, we get

$$\begin{aligned}
&\big\|\mathcal{T}u(t) - \mathcal{T}v(t)\big\| \\
&\quad\leqslant \Gamma(\alpha-\beta) \sum_{0<t_k<t} t_k^{1+\beta-\alpha} \big\| \mathscr{I}_k\big(u(t_k)\big) - \mathscr{I}_k\big(v(t_k)\big)\big\| \int_0^t \big\|\mathscr{S}_\alpha(t-s)\big\| \frac{s^{\alpha-2}}{\Gamma(\alpha-1)}\,\mathrm{d}s \\
&\qquad + \frac{1}{\Gamma(\alpha-1)} \int_0^t \big\|\mathscr{S}_\alpha(t-s)\big\| \int_0^s (s-\tau)^{\alpha-2} \big\| f\big(\tau, u(\tau)\big) - f\big(\tau, v(\tau)\big)\big\|\,\mathrm{d}\tau\,\mathrm{d}s \\
&\quad\leqslant K\mathcal{M}\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \sum_{0<t_k<t} t_k^{1+\beta-\alpha} t^{\alpha-1} \|u-v\|_{\mathcal{PC}} \\
&\qquad + \frac{\mathcal{M}}{\Gamma(\alpha)} \int_0^t s^{\alpha-1}\,\mathrm{d}s \|L_f\| \|u-v\|_{\mathcal{PC}} \\
&\quad\leqslant \left(K\mathcal{M}\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \sum_{0<t_k<t} t_k^{1+\beta-\alpha} + \frac{\mathcal{M}}{\Gamma(\alpha+1)} \|L_f\| \right) \|u-v\|_{\mathcal{PC}},
\end{aligned}$$

which shows that $\mathcal{T}$ is contraction and there exists a unique mild solution of (1)–(3). □

The accompanying theorem provides the existence results by means of fixed point criterion for condensing map.

Theorem 2. *Assume that conditions* (H2)–(H3) *are satisfied and* $\{\mathscr{S}_\alpha(t)\}_{t\geqslant 0}$ *is compact. If*

$$\mathcal{M}K_c\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \sum_{0<t_k<t} t_k^{1+\beta-\alpha} + \frac{\mathcal{M}}{\Gamma(\alpha+1)} \|m_f\| \limsup_{r\to\infty} \frac{1}{r}\mathcal{W}(r) < 1,$$

then problem (1)–(3) *has a mild solution* u *in* $\mathcal{PC}(\mathscr{X})$.

Proof. Take $r > 0$ such that

$$\frac{\mathcal{M}\|u_0\|}{\Gamma(\alpha)} + s\mathcal{M}K_c\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \sum_{0<t_k<t} t_k^{1+\beta-\alpha} + \frac{\mathcal{M}}{\Gamma(\alpha+1)} \|m_f\|\mathcal{W}(s) < s$$

for all $s \geqslant r$.

Here we prove that the operator $\mathcal{T}$ introduced in the previous Theorem 1 is a condensing map from $\mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$ into $\mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$.

We first show that $\mathcal{T}$ has values in $\mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$, that is $\mathcal{T}\mathcal{B}_r(0, \mathcal{PC}(\mathscr{X})) \subset \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$.

For $u \in \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$ and $t \in \mathcal{I}$, we get

$$\begin{aligned}
\|\mathcal{T}u(t)\| &\leqslant \frac{1}{\Gamma(\alpha-1)} \int_0^t \big\|\mathscr{S}_\alpha(t-s)\big\| s^{\alpha-2} \|u_0\| \,\mathrm{d}s \\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-1)} \sum_{0<t_k<t} t_k^{1+\beta-\alpha} \big\|\mathscr{J}_k\big(u(t_k)\big)\big\| \int_0^t \big\|\mathscr{S}_\alpha(t-s)\big\| s^{\alpha-2} \,\mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\alpha-1)} \int_0^t \big\|\mathscr{S}_\alpha(t-s)\big\| \int_0^s (s-\tau)^{\alpha-2} \big\|f\big(\tau, u(\tau)\big)\big\| \,\mathrm{d}\tau \,\mathrm{d}s, \\
&\leqslant \frac{\mathcal{M}t^{\alpha-1}}{\Gamma(\alpha)} \|u_0\| + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \sum_{0<t_k<t} t_k^{1+\beta-\alpha} t^{\alpha-1} \mathcal{M} K_c \big\|u(t_k)\big\| \\
&\quad + \frac{\mathcal{M}}{\Gamma(\alpha)} \int_0^t s^{\alpha-1} \|m_f\| \mathcal{W}\big(\|u\|\big) \,\mathrm{d}s \\
&\leqslant \frac{\mathcal{M}\|u_0\|}{\Gamma(\alpha)} + r\mathcal{M}K_c \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \sum_{0<t_k<t} t_k^{1+\beta-\alpha} + \frac{\mathcal{M}\|m_f\|}{\Gamma(\alpha+1)} \mathcal{W}(r),
\end{aligned}$$

which implies that $\|\mathcal{T}u(t)\| \leqslant r$ and $\mathcal{T}\mathcal{B}_r(0, \mathcal{PC}(\mathscr{X})) \subset \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$.

To continue the remainder of the proof, we introduce the decomposition operator $\mathcal{T}$ by $\mathcal{T}_1 + \mathcal{T}_2$, where

$$\begin{aligned}
\mathcal{T}_1 &= \frac{1}{\Gamma(\alpha-1)} \int_0^t \mathscr{S}_\alpha(t-s) s^{\alpha-2} u_0 \,\mathrm{d}s \\
&\quad + \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-1)} \sum_{0<t_k<t} t_k^{1+\beta-\alpha} \mathscr{J}_k\big(u(t_k)\big) \int_0^t \mathscr{S}_\alpha(t-s) s^{\alpha-2} \,\mathrm{d}s, \\
\mathcal{T}_2 &= \frac{1}{\Gamma(\alpha-1)} \int_0^t \mathscr{S}_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-2} f\big(\tau, u(\tau)\big) \,\mathrm{d}\tau \,\mathrm{d}s.
\end{aligned}$$

Step 1. The map $\mathcal{T}_1$ is contraction on $\mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$.

For $u, v \in \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$, $t \in \mathcal{I}$, from Theorem 1 it is easy to see that

$$\begin{aligned}
&\big\|\mathcal{T}_1 u(t) - \mathcal{T}_1 v(t)\big\| \\
&\quad \leqslant \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-1)} \sum_{0<t_k<t} t_k^{1+\beta-\alpha} \big\|\mathscr{J}_k\big(u(t_k)\big) - \mathscr{J}_k\big(v(t_k)\big)\big\| \int_0^t \big\|\mathscr{S}_\alpha(t-s)\big\| s^{\alpha-2} \,\mathrm{d}s \\
&\quad \leqslant K\mathcal{M} \frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)} \sum_{0<t_k<t} t_k^{1+\beta-\alpha} \|u-v\|_{\mathcal{PC}}.
\end{aligned}$$

Hence, $\mathcal{T}_1$ is contraction on $\mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$ since $K\mathcal{M}(\Gamma(\alpha-\beta)/\Gamma(\alpha)) \times \sum_{0<t_k<t} t_k^{1+\beta-\alpha} < 1$.

Step 2. The map $\mathcal{T}_2$ is completely continuous on $\mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$.

First, we show that $\mathcal{T}_2$ is a continuous operator.

Let the sequence (u_n) in $\mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$ and $u \in \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$. Assume $u_n \to u$ as $n \to \infty$. Then we have

$$\begin{aligned}&\|\mathcal{T}_2 u_n - \mathcal{T}_2 u\| \\ &\quad \leqslant \frac{1}{\Gamma(\alpha-1)} \int_0^t \|\mathscr{S}_\alpha(t-s)\| \int_0^s (s-\tau)^{\alpha-2} \| f(\tau, u_n(\tau)) - f(\tau, u(\tau)) \| \,\mathrm{d}\tau \,\mathrm{d}s.\end{aligned}$$

Since f is continuous, so $\|\mathcal{T}_2 u_n - \mathcal{T}_2 u\| \to 0$ as $n \to \infty$. From this we know $\mathcal{T}_2$ is continuous.

Further, we prove that $\mathcal{T}_2$ is a compact operator.

In continuation to this, we show that $\{\mathcal{T}_2 u(t)\colon u \in \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))\}$ is relatively compact in $\mathscr{X}$ for every $t \in \mathcal{I}$.

Let $0 < \epsilon < t \leqslant 1$. For $u \in \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$, we obtain the result via the mean value theorem, see [16, Thm. II.3.2],

$$\begin{aligned}\mathcal{T}_2 u(t) &= \frac{1}{\Gamma(\alpha-1)} \int_0^\epsilon \mathscr{S}_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-2} f(\tau, u(\tau)) \,\mathrm{d}\tau \,\mathrm{d}s \\ &\quad + \frac{1}{\Gamma(\alpha-1)} \int_\epsilon^t \mathscr{S}_\alpha(t-s) \int_0^s (s-\tau)^{\alpha-2} f(\tau, u(\tau)) \,\mathrm{d}\tau \,\mathrm{d}s \\ &\in \mathcal{B}_{r_1}(0, \mathscr{X}) + (t-\epsilon)\overline{\mathrm{co}\{\mathscr{S}_\alpha(t-s)y\colon s[\epsilon, t],\ y \in \bar{V}\}},\end{aligned}$$

where $r_1 = (\mathcal{M}\epsilon^\alpha/(\alpha\Gamma(\alpha)))\|m_f\|\mathcal{W}(r)$, and the set $V := \{\int_0^t ((t-s)^{\alpha-2}/\Gamma(\alpha-1)) \times f(s, u(s)) \,\mathrm{d}s\colon s \in [0,1],\ u \in \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))\}$ is relatively compact in $\mathscr{X}$.

Hence, $\{\mathcal{T}_2 u(t)\colon u \in \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))\} \subset K_\epsilon + \mathcal{B}_{r_1}(0, \mathscr{X})$, where K_ϵ is compact and $\mathrm{diam}(\mathcal{B}_{r_1}(0, \mathscr{X})) \to 0$ as $\epsilon \to 0$. From this concept the set $\{\mathcal{T}_2 u(t)\colon u \in \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))\}$ is relatively compact in $\mathscr{X}$.

Finally, we prove that $\{\mathcal{T}_2 u(t)\colon u \in \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))\}$ is equicontinuous on $\mathcal{I}$.

Let $0 < t < 1$ and $\epsilon > 0$. For $u \in \mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$ and $0 < l < \epsilon$ such that $t < t + l$, we obtain

$$\begin{aligned}&\|\mathcal{T}_2 u(t+l) - \mathcal{T}_2 u(t)\| \\ &\quad \leqslant \int_0^t \|\mathscr{S}_\alpha(t+l-s) - \mathscr{S}_\alpha(t-s)\| \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \|f(\tau, u(\tau))\| \,\mathrm{d}\tau \,\mathrm{d}s \\ &\qquad + \int_t^{t+l} \|\mathscr{S}_\alpha(t+l-s)\| \int_0^s \frac{(s-\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \|f(\tau, u(\tau))\| \,\mathrm{d}\tau \,\mathrm{d}s\end{aligned}$$

$$\leqslant \frac{\epsilon t^{\alpha}}{\Gamma(\alpha+1)}\|m_f\|\mathcal{W}(r) + \mathcal{M}\int_t^{t+l}\frac{s^{\alpha-1}}{\Gamma(\alpha)}\|m_f\|\mathcal{W}\big(\|u\|\big)\,\mathrm{d}s$$
$$\leqslant \frac{(\epsilon+\mathcal{M}l^{\alpha})}{\Gamma(\alpha+1)}\|m_f\|\mathcal{W}(r),$$

which proves that $\mathcal{T}(\mathcal{B}_r(0,\mathcal{PC}(\mathscr{X})))$ is right equicontinuous at the point t in $(0,1)$. Likewise, it shows that $\mathcal{T}(\mathcal{B}_r(0,\mathcal{PC}(\mathscr{X})))$ is right equicontinuous at 0 and left equicontinuous at $t\in(0,1]$. Then the set $\mathcal{T}(\mathcal{B}_r(0,\mathcal{PC}(\mathscr{X})))$ is equicontinuous on $\mathcal{I}$.

From Step 2 we deduce that $\mathcal{T}_2$ is completely continuous.

Therefore, we conclude that $\mathcal{T}=\mathcal{T}_1+\mathcal{T}_2$ is a condensing operator from $\mathcal{B}_r(0,\mathcal{PC}(\mathscr{X}))$ into $\mathcal{B}_r(0,\mathcal{PC}(\mathscr{X}))$, and from [16, Thm. IV.3.2] we infer that problem (1)–(3) has a mild solution. □

4 Application

We consider the partial fractional differential equations with fractional impulsive conditions

$$D^{\alpha}_{0+}x(t,y) = \frac{\partial^2}{\partial y^2}x(t,y) + a_1(t)x(t,y), \quad \alpha\in(0,1),\ y\in[0,\pi],\ t\in\mathcal{I}, \tag{14}$$
$$x(t,0) = x(t,\pi) = 0, \tag{15}$$
$$x(0,y) = x_0(y)\in\mathscr{X}, \tag{16}$$
$$D^{\beta}_{0+}x\big(t_k^+\big) - D^{\beta}_{0+}x\big(t_k^-\big) = \mathscr{I}_k\big(x(t_k)\big), \quad k=1,\dots,m. \tag{17}$$

Here, the space $\mathscr{X} = L^2[0,\pi]$ and the function $a_1\in C(\mathcal{I}\times\mathbb{R};\mathbb{R})$, $0=t_0\leqslant t_1\leqslant \cdots < t_{m+1}=1$ are prefixed numbers in $[0,1]$.

The operator $\mathcal{A}:\mathcal{D}(\mathcal{A})\subset\mathscr{X}\to\mathscr{X}$ is determined by $\mathcal{A}x = x''$ with domain $\mathcal{D}(\mathcal{A}) := \{x\in\mathscr{X}\colon x''\in\mathscr{X},\ x(0)=x(\pi)=0\}$. The class of operator $\{\mathscr{T}(t)\}_{t\geqslant 0}$ is an analytic semigroup in $\mathscr{X}$ with the infinitesimal generator $\mathcal{A}$. Here $\mathcal{A}$ has a discrete spectrum with eigenvalues $-n^2$, $n\in\mathbb{N}$, and the corresponding normalized eigenfunction is defined by $u_n(\zeta) = (2/\pi)^{1/2}\sin(n\zeta)$. The value $\mathscr{T}(t)x = \sum_{n=1}^{\infty}\mathrm{e}^{-n^2t}\langle x,u_n\rangle u_n$ for $x\in\mathscr{X}$, and the set $\{u_n\colon n\in\mathbb{N}\}$ is an orthonormal basis for $\mathscr{X}$. From these expressions it follows that $\{\mathscr{T}(t)\}_{t\geqslant0}$ is a compact, so that $(\lambda-\mathcal{A})^{-1}$ is a compact operator for all $\lambda\in\rho(\mathcal{A})$.

The operator $\mathcal{A}$ generates an $(t^{\alpha-1}/\Gamma(\alpha),1)$-regularized resolvent family $\{\mathscr{S}_{\alpha}(t)\}_{t\geqslant0}$, where

$$\mathscr{S}_{\alpha}(t) = \frac{1}{2\pi\mathrm{i}}\int_{\Gamma_R}\mathrm{e}^{\lambda t}\lambda^{\alpha-1}\big(\lambda^{\alpha}I-\mathcal{A}\big)^{-1}\,\mathrm{d}\lambda,$$

Γ_R refers the path consisting of the rays $\{r\mathrm{e}^{\mathrm{i}\vartheta}\}$ and $\{r\mathrm{e}^{-\mathrm{i}\vartheta}\}$ with $r\geqslant R$ and for some $\vartheta\in(\pi,\pi/2)$. Since $(\lambda-\mathcal{A})^{-1}$ is compact, from this result $\{\mathscr{S}_{\alpha}(t)\}_{t\geqslant0}$ is a compact operator. In [15], the compactness properties about $(t^{\alpha-1}/\Gamma(\alpha),1)$- resolvent operators are described.

We reformulate the fractional partial differential equations (14)–(17) in the abstract form (1)–(3) by choosing the appropriate functions f. We define the function $f : C(\mathcal{I} \times \mathbb{R}; \mathbb{R}) \to \mathbb{R}$ by $f(t, x) = a_1(t)x(t, y)$ and $\mathscr{I}_k \in C(\mathbb{R})$, $k = 1, 2, \dots, m$.

If the subsequent condition is satisfied

$$\mathcal{M}\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha)}\sum_{0<t_k<t} t_k^{1+\beta-\alpha}\big\|\mathscr{I}_k\big(u(t_k)\big)\big\| + \frac{\mathcal{M}}{\Gamma(\alpha+1)}\|a_1\|_{C(\mathcal{I}\times\mathbb{R};\mathbb{R})} < 1,$$

then, as per Theorem 2, problem (14)–(17) has at least one mild solution on $\mathcal{B}_r(0, \mathcal{PC}(\mathscr{X}))$.

References

1. A. Anguraj, K. Karthikeyan, Existence of solutions for impulsive neutral functional differential equations with nonlocal conditions, *Nonlinear Anal., Theory Methods Appl.*, **70**(7):2717–2721, 2009.
2. D.N. Chalishajar, K. Karthikeyan, A. Anguraj, Existence results for impulsive perturbed partial neutral functional differential equations in Fréchet spaces, *Dyn. Contin. Discrete Impuls. Syst., Ser. A, Math. Anal.*, **22**(1):25–45, 2015.
3. M. Fečkan, Y. Zhou, J. Wang, Response to "Comments on the concept of existence of solution for impulsive fractional differential equations [Commun Nonlinear Sci Numer Simul 2014;19:401–3.]", *Commun. Nonlonear Sci. Numer. Simul.*, **19**(12):4213–4215, 2014.
4. M. Fečkan, Y. Zhou, J. Wang, On the concept and existence of solution for impulsive fractional differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **17**(7):3050–3060, 2012.
5. C.S. Goodrich, Continuity of solutions to discrete fractional initial value problems, *Comput. Math. Appl.*, **59**(11):3489–3499, 2010.
6. T.L. Guo, K. Zhang, Impulsive fractional partial differential equations, *Electron. J. Qual. Theory Differ. Equ.*, **257**:581–590, 2015.
7. N. Hayashi, E.I. Kaikina, P.I. Naumkin, Asymptotics for fractional nonlinear heat equations, *J. Lond. Math. Soc.*, **72**(3):663–688, 2005.
8. E. Hernández, D. O'Regan, K. Balachandran, On recent developments in the theory of abstract differential equations with fractional derivatives, *Nonlinear Anal., Theory Methods Appl.*, **73**(10):3462–3471, 2010.
9. N. Heymans, I. Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann–Liouville fractional derivatives, *Rheol. Acta*, **45**(5):765–771, 2006.
10. N. Kosmatov, Initial value problems of fractional order with fractional impulsive conditions, *Results Math.*, **63**(3–4):1289–1310, 2013.
11. J.H. Liu, Nonlinear impulsive evolution equations, *Dyn. Contin. Discrete Impulsive Syst.*, **6**:77–85, 1999.
12. Y. Liu, B. Ahmad, A study of impulsive multiterm fractional differential equations with single and multiple base points and applications, *Sci. World J.*, **2014**:194346, 2014.
13. C. Lizama, Regularized solutions for abstract Volterra equations, *J. Math. Anal. Appl.*, **243**(2):278–292, 2000.

14. C. Lizama, G.M. N'Guérékata, Mild solutions for abstract fractional differential equations, *Appl. Anal.*, **98**(8):1731–1754, 2013.

15. C. Lizama, A. Pereira, R. Ponce, On the compactness of fractional resolvent operator function, *Semigroup Forum*, **93**(2):363–374, 2016.

16. R.H. Martin, *Nonlinear Operators and Differential Equations in Banach Spaces*, Robert E. Krieger Publishing Co., Melbourne, FL, 1987.

17. I. Podlubny, *Fractional Differential Equations*, Math. Sci. Eng., Vol. 198, Academic Press, San Diego, CA, 1999.

18. X.B. Shu, Y. Lai, Y. Chen, The existence of mild solutions for impulsive fractional partial differential equations, *Nonlinear Anal., Theory Methods Appl.*, **74**(5):2003–2011, 2011.

19. G. Wang, B. Ahmad, L. Zhang, J.J. Nieto, Comments on the concept of existence of solution for impulsive fractional differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, **19**(3):401–403, 2014.

20. J. Wang, M. Fečkan, Y. Zhou, On the new concept of solutions and existence results for impulsive fractional evolution equations, *Dyn. Partial Differ. Equ.*, **8**(4):345–361, 2011.

21. Z. Wei, Q. Li, J. Che, Initial value problems for fractional differential equations involving Riemann–Liouville sequential fractional derivative, *J. Math. Anal. Appl.*, **367**(1):260–272, 2010.

22. Z. Yan, X. Jia, Impulsive problems for fractional partial neutral functional integro-differential inclusions with infinite delay and analytic resolvent operators, *Mediterr. J. Math.*, **11**(2):393–428, 2014.

23. Y. Zhou, F. Jiao, J. Pecaric, Abstract cauchy problem for fractional functional differential equations, *Topol. Methods Nonlinear Anal.*, **42**(1):119–136, 2013.