Nonlinear Analysis: Modelling and Control, Vol. 22, No. 6, 765–784 ISSN 1392-5113
https://doi.org/10.15388/NA.2017.6.3

On a Hadamard-type fractional turbulent flow model with deviating arguments in a porous medium*

Taoli Wang[a], **Guotao Wang**[a,1], **Xiao-Jun Yang**[b,c]

[a]School of Mathematics and Computer Science,
Shanxi Normal University,
Linfen, Shanxi 041004, China
17835066293@163.com; wgt2512@163.com

[b]School of Mechanics and Civil Engineering,
China University of Mining and Technology,
Xuzhou 221116, China

[c]State Key Laboratory for Geomechanics
and Deep Underground Engineering,
China University of Mining and Technology,
Xuzhou 221116, China
dyangxiaojun@163.com

Received: January 15, 2017 / **Revised:** May 24, 2017 / **Published online:** October 27, 2017

Abstract. In this paper, we concern a Hadamard-type fractional-order turbulent flow model with deviating arguments. By using some standard fixed point theorems, the uniqueness, existence and nonexistence of solutions of the fractional turbulent flow model are investigated. Our results are new and are well illustrated with the aid of three examples.

Keywords: fractional turbulent flow model, Hadamard fractional derivation, deviating arguments.

1 Introduction

Fractional differential equation models are important and useful in a number of fields such as blood flow phenomena, earthquake, rheology, electrodynamics of a complex medium, aerodynamics, viscoelasticity, and optics and signal processing, and so on [10, 21, 23, 28, 38, 47]. With the development of fractional differential equation models in theory and practice, the subject has acquired great achievements. For some recent works on the subject, readers can see [4–9, 12, 13, 15, 19, 25, 27, 29, 31, 33, 36, 37, 40–42, 44–46] and the references therein.

*This research was partially supported by National Natural Science Foundation of China (No. 11501342) and the Scientific and Technological Innovation Programs of Higher Education Institutions in Shanxi (Nos. 2014135 and 2014136).

[1]Corresponding author.

Differential equations with deviating arguments are often applied to investigations connected with economic, mechanics, mathematical physics, etc. [1, 16, 32]. In recent years, many researchers are taking notice of Hadamard-type fractional differential equation, not only Riemann–Liouville or Caputo-type fractional differential equation. It is another kind of fractional derivatives and be introduced by Hadamard in 1892 [18]. Hadamard derivative has predominant properties, and its definition includes logarithmic function of arbitrary. For the development of Hamamard fractional calculus, we refer to [2, 3, 11, 17, 20–22, 26, 34, 35].

As we all know, turbulent flow in a porous medium is a basic mechanics problem. In [24], in order to study this type of problem, Leibenson put forward the p-Laplacian equation as follows:

$$\big(\phi_p\big(x'(t)\big)\big)' = f\big(t, x(t), x'(t)\big),$$

where $\phi_p(s) = |s|^{p-2}s$, which is defined as p-Laplacian operator, and $\phi_p^{-1}(s) = \phi_q(s)$, $1/p + 1/q = 1$.

In [14], Chen, Liu and Hu studied the existence of solutions of the following fractional differential equation with p-Laplacian operator at resonance:

$$\begin{aligned} &D^{\beta}_{0+}\phi_p\big(D^{\alpha}_{0+}x(t)\big) = f\big(t, x(t), D^{\alpha}_{0+}x(t)\big), \quad t \in [0,1],\\ &D^{\alpha}_{0+}x(0) = D^{\alpha}_{0+}x(1) = 0, \end{aligned}$$

where $0 < \alpha, \beta \leqslant 1$, $1 < \alpha + \beta \leqslant 2$, D^{α}_{0+} is a Caputo fractional derivative. By using the coincidence degree theory, a new existence result for the above problem is given.

In [39], Zhang studied the existence of solutions of the following two point boundary value problems with the fractional p-Laplacian operator:

$$\begin{aligned} &D^{\beta}_{0+}\phi_p\big(D^{\alpha}_{0+}x(t)\big) = f\big(t, x(t), D^{\alpha}_{0+}x(t)\big), \quad t \in [0,1],\\ &D^{\alpha}_{0+}x(0) = x(1) = 0, \end{aligned}$$

where $0 < \alpha, \beta \leqslant 1$, D^{α}_{0+} is a Caputo fractional derivative, and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous. With the help of Schauder fixed point theorem, author establishes two theorems on the existence of solutions for two point boundary value problems with the fractional p-Laplacian operator.

In [43], Zhang, Liu and Wu investigated the fractional-order model of turbulent flow in a porous medium

$$\begin{aligned} &-D^{\beta}_t\varphi_p\big(-D^{\alpha}_t x\big)(t) = f\big(x(t), D^{\gamma}_t x(t)\big), \quad t \in (0,1),\\ &D^{\alpha}_t x(0) = D^{\alpha+1}_t x(1) = D^{\alpha}_t x(1) = 0,\\ &D^{\gamma}_t x(0) = 0, \qquad D^{\gamma}_t x(1) = \int_0^1 D^{\gamma}_t x(s)\,\mathrm{d}A(s), \end{aligned}$$

where $D^{\alpha}_t, D^{\beta}_t, D^{\gamma}_t$ are the standard Riemann–Liouville derivatives, $\int_0^1 x(s)\,\mathrm{d}A(s)$ is denoted by a Riemann–Stieltjes integral, and $0 < \gamma \leqslant 1 < \alpha \leqslant 2 < \beta < 3$, $\alpha - \gamma > 1$.

A is a function of bounded variation, and dA can be a signed measure. By employing the fixed point theorem of the mixed monotone operator, authors obtain the uniqueness of positive solution of the above fractional-order turbulent flow model.

Inspired by the above mentioned work, in this paper, we study the uniqueness, the existence and nonexistence of solutions for a fractional-order turbulent flow model with deviating arguments

$$\begin{aligned}
&-{}^{H}D^{\beta}\phi_p\bigl(-{}^{H}D^{\alpha}u(t)\bigr) = f\bigl(t, u(t), u\bigl(\theta(t)\bigr)\bigr),\\
&{}^{H}D^{\alpha}u(1) = {}^{H}D^{\alpha}u(\mathrm{e}) = 0, \qquad u(1) = 0,\\
&{}^{H}D^{\alpha-1}u(1) = \eta\,{}^{H}D^{\alpha-1}u(\mathrm{e}),
\end{aligned} \tag{1}$$

where $1 < \alpha, \beta \leqslant 2$, $\eta \in \mathbb{R}$, $J = [1, \mathrm{e}]$, $\theta \in C[J, J]$, $t \in J$, $f \in C(J \times \mathbb{R}^2, \mathbb{R})$, ${}^{H}D^{\alpha}$, ${}^{H}D^{\beta}$ denote Hadamard fractional derivative of order α and β, respectively.

Since the fractional-order turbulent flow model involves Hadamard fractional derivative and deviating arguments has been seldom studied, in this paper, we study the uniqueness, the existence and nonexistence of solutions of the Hadamard-type fractional-order turbulent flow model with deviating arguments, i.e. problem (1). By using the Schauder's fixed point theorem and the contraction mapping principle, three new results on the Hadamard-type fractional-order turbulent flow model (1) are obtained.

We also use the following properties of the p-Laplacian operator:

(L1) If $1 < q < 2$, $uv > 0$, $|u|, |v| \geqslant m > 0$, then

$$\bigl|\phi_q(u) - \phi_q(v)\bigr| \leqslant (q-1)m^{q-2}|u - v|.$$

(L2) If $q \geqslant 2$, $|u|, |v| \leqslant M$, then

$$\bigl|\phi_q(u) - \phi_q(v)\bigr| \leqslant (q-1)M^{q-2}|u - v|.$$

2 Preliminaries

For the convenience of the reader, we recall some basic definitions about Hadamard fractional calculus and important lemmas.

Definition 1. (See [21].) The Hadamard fractional integral of order q for a function g is defined as

$${}^{H}I^{q}g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{g(s)}{s}\,\mathrm{d}s, \quad q > 0,$$

provided the integral exists.

Definition 2. (See [21].) The Hadamard derivation of fractional order q for a function $g : [1, \infty) \to \mathbb{R}$ is defined as

$${}^{H}D^{q}g(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^{n} \int_1^t \left(\log\frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s}\,\mathrm{d}s,$$

$n-1<q<n$, $n=[q]+1$, where $[q]$ denotes the integer part of the real number q, and $\log(\cdot)=\log_{\mathrm{e}}(\cdot)$.

Lemma 1. (*See* [21].) *Let $q>0$ and $x\in C[1,\infty)\bigcap L^1[1,\infty)$. Then the Hadamard fractional differential equation ${}^H D^q x(t)=0$ has the solutions*

$$x(t)=\sum_{i=1}^{n} c_i(\log t)^{q-i},$$

and the following formula holds:

$${}^H I^q\, {}^H D^q x(t)=x(t)+\sum_{i=1}^{n} c_i(\log t)^{q-i},$$

where $c_i\in\mathbb{R}$, $i=1,2,\dots,n$, and $n-1<q<n$.

Lemma 2. *For any $\sigma\in C[1,\mathrm{e}]$, the unique solution of the linear Hadamard fractional-order turbulent flow model*

$$\begin{aligned}
&-{}^H D^\beta\phi_p\bigl(-{}^H D^\alpha u(t)\bigr)=\sigma(t),\quad t\in J,\\
&{}^H D^\alpha u(1)={}^H D^\alpha u(\mathrm{e})=0,\qquad u(1)=0,\\
&{}^H D^{\alpha-1}u(1)=\eta\,{}^H D^{\alpha-1}u(\mathrm{e})
\end{aligned}\tag{2}$$

is given by

$$\begin{aligned}
u(t)={}&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\sigma(\tau)\frac{\mathrm{d}\tau}{\tau}\right.\\
&\left.-\frac{(\log s)^{\beta-1}}{\Gamma(\beta)}\int_1^s(1-\log s)^{\beta-1}\sigma(\tau)\frac{\mathrm{d}\tau}{\tau}\right]\frac{\mathrm{d}s}{s}\\
&+\frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)}\int_1^{\mathrm{e}}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\sigma(\tau)\frac{\mathrm{d}\tau}{\tau}\right.\\
&\left.-\frac{(\log s)^{\beta-1}}{\Gamma(\beta)}\int_1^s(1-\log s)^{\beta-1}\sigma(\tau)\frac{\mathrm{d}\tau}{\tau}\right]\frac{\mathrm{d}s}{s}.
\end{aligned}\tag{3}$$

Proof. By Lemma 1, we get

$$\phi_p\bigl(-{}^H D^\alpha u(t)\bigr)=-\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}\sigma(s)\frac{\mathrm{d}s}{s}+c_0(\log t)^{\beta-1}+c_1(\log t)^{\beta-2},$$

where constants $c_0,c_1\in\mathbb{R}$.

Thus,

$$ {}^{H}D^{\alpha}u(t) = -\phi_q\left[-\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}\sigma(s)\frac{\mathrm{d}s}{s} + c_0(\log t)^{\beta-1} + c_1(\log t)^{\beta-2}\right], $$

where constants $c_0, c_1 \in \mathbb{R}$.

Employing the boundary value condition ${}^{H}D^{\alpha}u(1) = {}^{H}D^{\alpha}u(\mathrm{e}) = 0$, we have

$$ c_1 = 0, \qquad c_0 = \frac{1}{\Gamma(\beta)}\int_1^{\mathrm{e}}(1-\log s)^{\beta-1}\sigma(s)\frac{\mathrm{d}s}{s}. $$

Thus, one has

$$ {}^{H}D^{\alpha}u(t) = \phi_q\left[\frac{1}{\Gamma(\beta)}\int_1^t\left(\log\frac{t}{s}\right)^{\beta-1}\sigma(s)\frac{\mathrm{d}s}{s} - \frac{(\log t)^{\beta-1}}{\Gamma(\beta)}\int_1^{\mathrm{e}}(1-\log s)^{\beta-1}\sigma(s)\frac{\mathrm{d}s}{s}\right]. $$

Applying Lemma 1 again, we have

$$ \begin{aligned} u(t) = {}&\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\sigma(\tau)\frac{\mathrm{d}\tau}{\tau}\right.\\ &\left.-\frac{(\log s)^{\beta-1}}{\Gamma(\beta)}\int_1^s(1-\log s)^{\beta-1}\sigma(\tau)\frac{\mathrm{d}\tau}{\tau}\right]\frac{\mathrm{d}s}{s} + c_2(\log t)^{\alpha-1}\\ &+ c_3(\log t)^{\alpha-2}, \end{aligned} \tag{4} $$

where constants $c_2, c_3 \in \mathbb{R}$.

By condition $u(1) = 0$, we get $c_3 = 0$. This, together with (4), gives

$$ \begin{aligned} {}^{H}D^{\alpha-1}u(t) = {}&\int_1^t\phi_q\left[\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\sigma(\tau)\frac{\mathrm{d}\tau}{\tau}\right.\\ &\left.-\frac{(\log s)^{\beta-1}}{\Gamma(\beta)}\int_1^s(1-\log s)^{\beta-1}\sigma(\tau)\frac{\mathrm{d}\tau}{\tau}\right]\frac{\mathrm{d}s}{s} + c_2\Gamma(\alpha). \end{aligned} $$

The boundary value condition ${}^{H}D^{\alpha-1}u(1) = \eta\,{}^{H}D^{\alpha-1}u(\mathrm{e})$ implies

$$ \begin{aligned} c_2 = {}&\frac{1}{(1-\eta)\Gamma(\alpha)}\int_1^{\mathrm{e}}\phi_q\left[\frac{1}{\Gamma(\beta)}\int_1^s\left(\log\frac{s}{\tau}\right)^{\beta-1}\sigma(\tau)\frac{\mathrm{d}\tau}{\tau}\infty\right.\\ &\left.-\frac{(\log s)^{\beta-1}}{\Gamma(\beta)}\int_1^s(1-\log s)^{\beta-1}\sigma(\tau)\frac{\mathrm{d}\tau}{\tau}\right]\frac{\mathrm{d}s}{s}. \end{aligned} $$

Then, substituting c_2 and c_3 into (4), we get equation (3).

The proof is complete. $\square$

3 Uniqueness results

To transform problem (1) into a fixed point problem, we define the operator $T : C[1,\mathrm{e}] \to C[1,\mathrm{e}]$ as

$$
\begin{aligned}
(Tu)(t) = {}& \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \phi_q\Biggl[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\frac{\mathrm{d}\tau}{\tau} \\
&- \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr]\frac{\mathrm{d}s}{s} \\
&+ \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \phi_q\Biggl[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\frac{\mathrm{d}\tau}{\tau} \\
&- \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr]\frac{\mathrm{d}s}{s}.
\end{aligned}
$$

To state and prove the uniqueness results, we pose the following conditions:

(H1) The function $f : [1,\mathrm{e}] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is jointly continuous.

(H2) There exists a function $k_f \in L([1,\mathrm{e}],\mathbb{R})$ such that

$$
\bigl|f(t,u_1,u_2) - f(t,v_1,v_2)\bigr| \leqslant k_f(t)\bigl(|u_1-v_1| + |u_2-v_2|\bigr)
$$

for all $(t,u_1,u_2), (t,v_1,v_2) \in [1,\mathrm{e}] \times \mathbb{R} \times \mathbb{R}$.

For convenience, we define an operator $T_0 : C[1,\mathrm{e}] \to C[1,\mathrm{e}]$ as follows:

$$
\begin{aligned}
T_0u(s) = \phi_q\Biggl[&\frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\frac{\mathrm{d}\tau}{\tau} \\
&- \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\frac{\mathrm{d}\tau}{\tau}\Biggr].
\end{aligned}
$$

Lemma 3. *Assume conditions* (H1), (H2) *hold and $q > 2$. Then there exists a constant $k > 0$ such that*

$$
\bigl|T_0u(s) - T_0v(s)\bigr| \leqslant k\|u-v\|,
$$

where $k = (q-1)2^q\|k_f\|^{q-1}(2C+M)^{q-2}/(\Gamma(\beta+1))^{q-1}$.

Proof. Define a ball $B_C = \{u \in C[1,\mathrm{e}]:\ \|u\| \leqslant C\}$, and suppose $q > 2$

$$\begin{aligned}
&\left| \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau} \right.\\
&\qquad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau} \right| \\
&\quad \leqslant \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \bigl|f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\bigr| \frac{\mathrm{d}\tau}{\tau} \\
&\qquad + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} \bigl|f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\bigr| \frac{\mathrm{d}\tau}{\tau} \\
&\quad \leqslant \frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} k_f(\tau)\bigl(\bigl|u(\tau)\bigr| + \bigl|u\bigl(\theta(\tau)\bigr)\bigr| + \bigl|f(\tau,0,0)\bigr|\bigr) \frac{\mathrm{d}\tau}{\tau} \\
&\qquad + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} k_f(\tau)\bigl(\bigl|u(\tau)\bigr| + \bigl|u\bigl(\theta(\tau)\bigr)\bigr| + \bigl|f(\tau,0,0)\bigr|\bigr) \frac{\mathrm{d}\tau}{\tau} \\
&\quad \leqslant \frac{(\log s)^{\beta}}{\Gamma(\beta+1)} \|k_f\|\bigl(2\|u\| + M\bigr) + \frac{(\log s)^{\beta-1}}{\Gamma(\beta+1)} \|k_f\|\bigl(2\|u\| + M\bigr) \\
&\quad \leqslant \frac{2\|k_f\|(2C+M)}{\Gamma(\beta+1)},
\end{aligned}$$

where $M = \max_{\tau\in[1,\mathrm{e}]} |f(\tau,0,0)|$.

Now using the property (L2), we get

$$\begin{aligned}
&\bigl|(T_0u)(s) - (T_0v)(s)\bigr| \\
&\quad = \left| \phi_q\left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau} \right.\right.\\
&\qquad \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau} \right] \\
&\qquad - \phi_q\left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} f\bigl(\tau, v(\tau), v\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau} \right.\\
&\qquad \left.\left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f\bigl(\tau, v(\tau), v\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau} \right] \right|
\end{aligned}$$

$$
\begin{aligned}
&\leqslant (q-1)\left(\frac{2\|k_f\|(2C+M)}{\Gamma(\beta+1)}\right)^{q-2} \\
&\quad\times \left|\frac{1}{\Gamma(\beta)}\int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1}\left[f\big(\tau,u(\tau),u\big(\theta(\tau)\big)\big)-f\big(\tau,v(\tau),v\big(\theta(\tau)\big)\big)\right]\frac{\mathrm{d}\tau}{\tau}\right. \\
&\quad\left.-\frac{(\log s)^{\beta-1}}{\Gamma(\beta)}\int_1^s (1-\log s)^{\beta-1}\left[f\big(\tau,u(\tau),u\big(\theta(\tau)\big)\big)-f\big(\tau,v(\tau),v\big(\theta(\tau)\big)\big)\right]\frac{\mathrm{d}\tau}{\tau}\right| \\
&\leqslant (q-1)\left(\frac{2\|k_f\|(2C+M)}{\Gamma(\beta+1)}\right)^{q-2} \\
&\quad\times \left|\frac{1}{\Gamma(\beta)}\int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1}\left[k_f(\tau)\big(\left|u(\tau)-v(\tau)\right|+\left|u\big(\theta(\tau)\big)-v\big(\theta(\tau)\big)\right|\big)\right]\frac{\mathrm{d}\tau}{\tau}\right. \\
&\quad\left.-\frac{(\log s)^{\beta-1}}{\Gamma(\beta)}\int_1^s (1-\log s)^{\beta-1}\left[k_f(\tau)\big(\left|u(\tau)-v(\tau)\right|+\left|u\big(\theta(\tau)\big)-v\big(\theta(\tau)\big)\right|\big)\right]\frac{\mathrm{d}\tau}{\tau}\right| \\
&\leqslant 2(q-1)\|k_f\|\left(\frac{2\|k_f\|(2C+M)}{\Gamma(\beta+1)}\right)^{q-2} \\
&\quad\times \left|\frac{1}{\Gamma(\beta)}\int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1}\|u-v\|\frac{\mathrm{d}\tau}{\tau}-\frac{(\log s)^{\beta-1}}{\Gamma(\beta)}\int_1^s (1-\log s)^{\beta-1}\|u-v\|\frac{\mathrm{d}\tau}{\tau}\right| \\
&\leqslant \frac{(q-1)2^{q-1}\|k_f\|^{q-1}(2C+M)^{q-2}}{(\Gamma(\beta+1))^{q-2}}\left[\frac{(\log s)^{\beta}}{\Gamma(\beta+1)}\|u-v\|+\frac{(\log s)^{\beta-1}}{\Gamma(\beta+1)}\|u-v\|\right] \\
&\leqslant \frac{(q-1)2^{q}\|k_f\|^{q-1}(2C+M)^{q-2}}{(\Gamma(\beta+1))^{q-1}}\|u-v\| = k\|u-v\|. \qquad \square
\end{aligned}
$$

Theorem 1. *Assume* (H1), (H2) *hold and* $q>2$. *If*

$$
k\left[\frac{1}{\Gamma(\alpha+1)}+\frac{1}{(1-\eta)\Gamma(\alpha)}\right]<1, \tag{5}
$$

then problem (1) *has a unique solution.*

Proof. For any $u, v \in B_C$, in view of (H1) and (H2) combined with Lemma 3, we have

$$
\begin{aligned}
&\left|(Tu)(t)-(Tv)(t)\right| \\
&\quad= \frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} T_0u(s)\frac{\mathrm{d}s}{s}+\frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)}\int_1^{\mathrm{e}} T_0u(s)\frac{\mathrm{d}s}{s} \\
&\qquad-\frac{1}{\Gamma(\alpha)}\int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} T_0v(s)\frac{\mathrm{d}s}{s}-\frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)}\int_1^{\mathrm{e}} T_0v(s)\frac{\mathrm{d}s}{s}
\end{aligned}
$$

$$\leqslant \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} k\|u-v\| \frac{\mathrm{d}s}{s} + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} k\|u-v\| \frac{\mathrm{d}s}{s}$$
$$\leqslant k\left[\frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)}\right]\|u-v\|$$
$$\leqslant k\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{(1-\eta)\Gamma(\alpha)}\right]\|u-v\|.$$

In view of (5), T is a contraction mapping. By the Banach contraction mapping principle, the operator T has a fixed point, which implies that Hadamard-type fractional turbulent flow model (1) has a unique solution. □

4 Existence results

Now, we are in the position to present a existence result, which is based on the following Schaefer fixed point theorem.

Theorem 2. (*See* [30].) *Let X be a Banach space. Assume that Ω is an open bounded subset of X with $\theta \in \Omega$, and let $T : \overline{\Omega} \to X$ be a completely continuous operator such that $\|Tu\| \leqslant \|u\|$ for all $u \in \partial\Omega$. Then T has a fixed point in $\overline{\Omega}$.*

Theorem 3. *Assume that*:

(H3) *There exist non-decreasing functions φ, ψ: $[0,\infty) \to [0,\infty)$ and the functions $a(t), b(t) \in C[1,\mathrm{e}]$ such that*

$$\big|f(t,u,v)\big| \leqslant a(t)\varphi\big(|u|\big) + b(t)\psi\big(|v|\big), \quad t \in [1,\mathrm{e}],\ u,v \in \mathbb{R}.$$

(H4) *There exists a constant $\mathcal{M} > 0$ such that*

$$\frac{\mathcal{M}}{2^{q-1}\big[\|a\|\varphi\big(\|u\|\big) + \|b\|\psi\big(\|u\|\big)\big]^{q-1}} \times \frac{1}{\big[\frac{1}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{1}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}}\big]} > 1.$$

Then problem (1) *has at least one solution.*

Proof. Firstly, we prove that the operator T is completely continuous. Clearly, continuity of the operator T follows from the continuity of f. Let $\Omega \subset C[1,\mathrm{e}]$ be bounded. For any $u, v \in \Omega$, there exist positive constant L such that $|f(t,u,v)| \leqslant L$. We can get

$$\big|(Tu)(t)\big| \leqslant \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q\left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau}\right)^{\beta-1} \big|f\big(\tau,u(\tau),u\big(\theta(\tau)\big)\big)\big| \frac{\mathrm{d}\tau}{\tau} + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} \big|f\big(\tau,u(\tau),u\big(\theta(\tau)\big)\big)\big| \frac{\mathrm{d}\tau}{\tau}\right] \frac{\mathrm{d}s}{s}$$

$$
\begin{aligned}
&\quad + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \phi_q\Biggl[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} \bigl|f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\bigr| \frac{\mathrm{d}\tau}{\tau} \\
&\quad + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} \bigl|f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\bigr| \frac{\mathrm{d}\tau}{\tau}\Biggr] \frac{\mathrm{d}s}{s} \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \phi_q\left(\frac{2L}{\Gamma(\beta+1)}\right) \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \phi_q\left(\frac{2L}{\Gamma(\beta+1)}\right) \frac{\mathrm{d}s}{s} \\
&= \frac{2^{q-1} L^{q-1} (\log t)^{\alpha}}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{2^{q-1} L^{q-1} (\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \\
&\leqslant \frac{2^{q-1} L^{q-1}}{\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \left(\frac{1}{\alpha} + \frac{1}{1-\eta}\right) \\
&= \mathcal{N} \quad \text{(constant)}.
\end{aligned}
$$

Therefore, $\|Tu\| \leqslant \mathcal{N}$. For all $t_1, t_2 \in [1, \mathrm{e}]$ and $1 \leqslant t_1 \leqslant t_2 \leqslant \mathrm{e}$, we have

$$
\begin{aligned}
&\bigl|(Tu)(t_2) - (Tu)(t_1)\bigr| \\
&\quad = \Biggl| \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left[\left(\log\frac{t_2}{s}\right)^{\alpha-1} - \left(\log\frac{t_1}{s}\right)^{\alpha-1}\right] \\
&\qquad \times \phi_q\Biggl[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau} \\
&\qquad - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau}\Biggr] \frac{\mathrm{d}s}{s} \\
&\qquad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log\frac{t_2}{s}\right)^{\alpha-1} \phi_q\Biggl[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau} \\
&\qquad - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau}\Biggr] \frac{\mathrm{d}s}{s} \\
&\qquad + \frac{(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \phi_q\Biggl[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau}
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr) \frac{\mathrm{d}\tau}{\tau} \biggr] \frac{\mathrm{d}s}{s} \biggr| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_1^{t_1} \biggl[\biggl(\log \frac{t_2}{s}\biggr)^{\alpha-1} - \biggl(\log \frac{t_1}{s}\biggr)^{\alpha-1} \biggr] \phi_q \biggl[\frac{2L}{\Gamma(\beta+1)}\biggr] \frac{\mathrm{d}s}{s} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \biggl(\log \frac{t_2}{s}\biggr)^{\alpha-1} \phi_q \biggl[\frac{2L}{\Gamma(\beta+1)}\biggr] \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \phi_q \biggl[\frac{2L}{\Gamma(\beta+1)}\biggr] \frac{\mathrm{d}s}{s} \biggr| \\
&= \biggl[\frac{(\log t_2)^{\alpha} - (\log t_1)^{\alpha}}{\Gamma(\alpha+1)} + \frac{(\log t_2)^{\alpha-1} - (\log t_1)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \biggr] \phi_q \biggl[\frac{2L}{\Gamma(\beta+1)}\biggr].
\end{aligned}
$$

Thus, by the Arzela–Ascoli theorem, the operator T is completely continuous.

Next, we consider the set $V = \{u \in C[1,\mathrm{e}]\colon u = \mu T u, \mu \in (0,1)\}$ and show that the set V is bounded.

$$
\begin{aligned}
|u| &= \bigl|\mu(Tu)(t)\bigr| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_1^t \biggl(\log \frac{t}{s}\biggr)^{\alpha-1} \phi_q \biggl[\frac{1}{\Gamma(\beta)} \int_1^s \biggl(\log \frac{s}{\tau}\biggr)^{\beta-1} \bigl| f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\bigr| \frac{\mathrm{d}\tau}{\tau} \\
&\quad + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} \bigl| f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\bigr| \frac{\mathrm{d}\tau}{\tau} \biggr] \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \phi_q \biggl[\frac{1}{\Gamma(\beta)} \int_1^s \biggl(\log \frac{s}{\tau}\biggr)^{\beta-1} \bigl| f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\bigr| \frac{\mathrm{d}\tau}{\tau} \\
&\quad + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} \bigl| f\bigl(\tau, u(\tau), u\bigl(\theta(\tau)\bigr)\bigr)\bigr| \frac{\mathrm{d}\tau}{\tau} \biggr] \frac{\mathrm{d}s}{s} \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_1^t \biggl(\log \frac{t}{s}\biggr)^{\alpha-1} \phi_q \biggl[\frac{1}{\Gamma(\beta)} \int_1^s \biggl(\log \frac{s}{\tau}\biggr)^{\beta-1} \bigl[a(t)\varphi\bigl(|u|\bigr) + b(t)\psi\bigl(\bigl|u\bigl(\theta(t)\bigr)\bigr|\bigr)\bigr] \frac{\mathrm{d}\tau}{\tau} \\
&\quad + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} \bigl[a(t)\varphi\bigl(|u|\bigr) + b(t)\psi\bigl(\bigl|u\bigl(\theta(t)\bigr)\bigr|\bigr)\bigr] \frac{\mathrm{d}\tau}{\tau} \biggr] \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \phi_q \biggl[\frac{1}{\Gamma(\beta)} \int_1^s \biggl(\log \frac{s}{\tau}\biggr)^{\beta-1} \bigl[a(t)\varphi\bigl(|u|\bigr) + b(t)\psi\bigl(\bigl|u\bigl(\theta(t)\bigr)\bigr|\bigr)\bigr] \frac{\mathrm{d}\tau}{\tau}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int\limits_1^s (1-\log s)^{\beta-1} \bigl[a(t)\varphi\bigl(|u|\bigr) + b(t)\psi\bigl(\bigl|u\bigl(\theta(t)\bigr)\bigr|\bigr)\bigr] \frac{\mathrm{d}\tau}{\tau} \Biggr] \frac{\mathrm{d}s}{s} \\
&= \frac{1}{\Gamma(\alpha)} \int\limits_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \phi_q \Biggl[\frac{(\log s)^{\beta}[a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)]}{\Gamma(\beta+1)} \\
&\quad + \frac{(\log s)^{\beta-1}[a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)]}{\Gamma(\beta+1)} \Biggr] \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int\limits_1^{\mathrm{e}} \phi_q \Biggl[\frac{(\log s)^{\beta}[a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)]}{\Gamma(\beta+1)} \\
&\quad + \frac{(\log s)^{\beta-1}[a(t)\varphi(|u|) + b(t)\psi(|u(\theta(t))|)]}{\Gamma(\beta+1)} \Biggr] \frac{\mathrm{d}s}{s} \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int\limits_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{2^{q-1}[\|a(t)\|\varphi(\|u\|) + \|b(t)\|\psi(\|u\|)]^{q-1}}{(\Gamma(\beta+1))^{q-1}} \frac{\mathrm{d}s}{s} \\
&\quad + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int\limits_1^{\mathrm{e}} \frac{2^{q-1}[\|a(t)\|\varphi(\|u\|) + \|b(t)\|\psi(\|u\|)]^{q-1}}{(\Gamma(\beta+1))^{q-1}} \frac{\mathrm{d}s}{s} \\
&= \frac{(\log t)^{\alpha} 2^{q-1}[\|a(t)\|\varphi(\|u\|) + \|b(t)\|\psi(\|u\|)]^{q-1}}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} \\
&\quad + \frac{(\log t)^{\alpha-1} 2^{q-1}[\|a(t)\|\varphi(\|u\|) + \|b(t)\|\psi(\|u\|)]^{q-1}}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \\
&\leqslant 2^{q-1}\bigl[\|a\|\varphi\bigl(\|u\|\bigr) + \|b\|\psi\bigl(\|u\|\bigr)\bigr]^{q-1} \\
&\quad \times \left[\frac{1}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{1}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \right].
\end{aligned}
$$

Thus,

$$
\frac{\|u\|}{2^{q-1}[\|a\|\varphi(\|u\|) + \|b\|\psi(\|u\|)]^{q-1}[\frac{1}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{1}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}}]} \leqslant 1.
$$

This, together with condition (H4), gives $\|u\| < \mathcal{M}$. So, the set V is bounded. Thus, by the Schaefer's fixed point theorem, the operator T has at least one fixed point, which implies that Hadamard-type fractional turbulent flow model has at least one solution. □

5 Nonexistence results

In this section, we introduce a parameter in problem (1), then we give some sufficient conditions of nonexistence of solution to a class of Hadamard-type fractional turbulent flow model with a parameter. Precisely, we study the following problem:

$$\begin{aligned} &-{}^H D^\beta \phi_p\big(-{}^H D^\alpha u(t)\big) = \lambda f\big(t, u(t), u\big(\theta(t)\big)\big), \\ &{}^H D^\alpha u(1) = {}^H D^\alpha u(\mathrm{e}) = 0, \qquad u(1) = 0, \\ &{}^H D^{\alpha-1} u(1) = \eta\, {}^H D^{\alpha-1} u(\mathrm{e}), \end{aligned} \tag{6}$$

where λ is a positive parameter.

Theorem 4. *If* $\lim_{|u|+|v|\to 0^+} f(t,u,v)/(|u|+|v|)^{p-1}$ *and* $\lim_{|u|+|v|\to\infty} f(t,u,v)/(|u|+|v|)^{p-1}$ *exist, then there exists* $\lambda_0 > 0$ *such that the Hadamard fractional turbulent flow model* (6) *has no solution for* $0 < \lambda < \lambda_0$.

Proof. Since $\lim_{|u|+|v|\to 0^+} f(t,u,v)/(|u|+|v|)^{p-1}$ and $\lim_{|u|+|v|\to\infty} f(t,u,v)/(|u|+|v|)^{p-1}$ exist, then there exist positive numbers ε_1, ε_2, r, N such that

(i) for $0 < \|u\| < r,\ |f(t,u(t),u(\theta(t)))| < 2^{p-1}\varepsilon_1\|u\|^{p-1}$.
(ii) for $\|u\| > N,\ |f(t,u(t),u(\theta(t)))| < 2^{p-1}\varepsilon_2\|u\|^{p-1}$.

Without loss of generality, let $r < N$ and

$$\varepsilon_3 = \max\left\{\varepsilon_1, \varepsilon_2, \max\left\{\frac{f(t,u(t),u(\theta(t)))}{[|u(t)| + |u(\theta(t))|]^{p-1}}\colon r < \|u\| < N\right\}\right\}.$$

Thus, for all $t \in (1,\mathrm{e})$ and $u \in \mathbb{R}$, we have $|f(t,u(t),u(\theta(t)))| < 2^{p-1}\varepsilon_3\|u\|^{p-1}$.

Let us consider

$$\begin{aligned} (T_\lambda u)(t) = {} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \phi_q\left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} \lambda f\big(\tau, u(\tau), u\big(\theta(\tau)\big)\big)\frac{\mathrm{d}\tau}{\tau}\right. \\ & \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} \lambda f\big(\tau, u(\tau), u\big(\theta(\tau)\big)\big)\frac{\mathrm{d}\tau}{\tau}\right]\frac{\mathrm{d}s}{s} \\ & + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \phi_q\left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} \lambda f\big(\tau, u(\tau), u\big(\theta(\tau)\big)\big)\frac{\mathrm{d}\tau}{\tau}\right. \\ & \left. - \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} \lambda f\big(\tau, u(\tau), u\big(\theta(\tau)\big)\big)\frac{\mathrm{d}\tau}{\tau}\right]\frac{\mathrm{d}s}{s}. \end{aligned}$$

If u is a solution of the fractional turbulent flow model (6), then $u = T_\lambda u$. In fact,

$$\begin{aligned} \big|(T_\lambda u)(t)\big| \leqslant {} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} \phi_q\left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log\frac{s}{\tau}\right)^{\beta-1} \big|\lambda f\big(\tau, u(\tau), u\big(\theta(\tau)\big)\big)\big|\frac{\mathrm{d}\tau}{\tau}\right. \\ & \left. + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} \big|\lambda f\big(\tau, u(\tau), u\big(\theta(\tau)\big)\big)\big|\frac{\mathrm{d}\tau}{\tau}\right]\frac{\mathrm{d}s}{s} \end{aligned}$$

$$
\begin{aligned}
&+ \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \phi_q \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau} \right)^{\beta-1} \left| \lambda f\left(\tau, u(\tau), u\left(\theta(\tau)\right)\right) \right| \frac{\mathrm{d}\tau}{\tau} \right. \\
&\left. + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} \left| \lambda f\left(\tau, u(\tau), u\left(\theta(\tau)\right)\right) \right| \frac{\mathrm{d}\tau}{\tau} \right] \frac{\mathrm{d}s}{s} \\
\leqslant{} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \phi_q \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau} \right)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1} \frac{\mathrm{d}\tau}{\tau} \right. \\
&\left. + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1} \frac{\mathrm{d}\tau}{\tau} \right] \frac{\mathrm{d}s}{s} \\
&+ \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \phi_q \left[\frac{1}{\Gamma(\beta)} \int_1^s \left(\log \frac{s}{\tau} \right)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1} \frac{\mathrm{d}\tau}{\tau} \right. \\
&\left. + \frac{(\log s)^{\beta-1}}{\Gamma(\beta)} \int_1^s (1-\log s)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1} \frac{\mathrm{d}\tau}{\tau} \right] \frac{\mathrm{d}s}{s} \\
={} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \phi_q \left[\frac{(\log s)^{\beta} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1}}{\Gamma(\beta+1)} \right. \\
&\left. + \frac{(\log s)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1}}{\Gamma(\beta+1)} \right] \frac{\mathrm{d}s}{s} \\
&+ \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \phi_q \left[\frac{(\log s)^{\beta} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1}}{\Gamma(\beta+1)} \right. \\
&\left. + \frac{(\log s)^{\beta-1} 2^{p-1} (\lambda \varepsilon_3) \|u\|^{p-1}}{\Gamma(\beta+1)} \right] \frac{\mathrm{d}s}{s} \\
\leqslant{} & \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s} \right)^{\alpha-1} \frac{2^q (\lambda \varepsilon_3)^{q-1} \|u\|}{(\Gamma(\beta+1))^{q-1}} \frac{\mathrm{d}s}{s} + \frac{(\log t)^{\alpha-1}}{(1-\eta)\Gamma(\alpha)} \int_1^{\mathrm{e}} \frac{2^q (\lambda \varepsilon_3)^{q-1} \|u\|}{(\Gamma(\beta+1))^{q-1}} \frac{\mathrm{d}s}{s} \\
={} & \frac{(\log t)^{\alpha} 2^q (\lambda \varepsilon_3)^{q-1} \|u\|}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{(\log t)^{\alpha-1} 2^q (\lambda \varepsilon_3)^{q-1} \|u\|}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \\
\leqslant{} & \left[\frac{2^q (\lambda \varepsilon_3)^{q-1}}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}} + \frac{2^q (\lambda \varepsilon_3)^{q-1}}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \right] \|u\| \\
={} & M (\lambda \varepsilon_3)^{q-1} \|u\|,
\end{aligned}
$$

where $M = 2^q(2-\eta)/((1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1})$.

Let $\lambda_0 = (\varepsilon_3 M^{1/(q-1)})^{-1}$. For $0 < \lambda < \lambda_0$, we have

$$\|T_\lambda u\| \leqslant M(\lambda\varepsilon_3)^{q-1}\|u\| < M(\lambda_0\varepsilon_3)^{q-1}\|u\| = \|u\|,$$

which is a contradiction. Therefore, u is not a solution of problem (6). □

6 Examples

Example 1. Consider the following Hadamard fractional boundary value problem:

$$\begin{aligned}
&-{}^H D^{3/2}\bigl[\phi_{3/2}\bigl(-{}^H D^{3/2}u(t)\bigr)\bigr] = \frac{t^2+1}{20}\frac{|u(t)|}{|u(t)|+1} + \frac{t^2}{25}\mathrm{e}^{-u(\theta(t))}, \quad t \in [1,\mathrm{e}],\\
&{}^H D^{3/2}u(1) = {}^H D^{3/2}u(0) = 0, \qquad u(1) = 0, \qquad {}^H D^{1/2}u(1) = \frac{1}{3}{}^H D^{1/2}u(\mathrm{e}),
\end{aligned} \tag{7}$$

where $p = 3/2$, $q = 3$, $\alpha = 3/2$, $\beta = 3/2$, $\eta = 1/3$, $\theta \in C([1,\mathrm{e}],[1,\mathrm{e}])$.

Note

$$\begin{aligned}
&\bigl|f(t,u_1,u_2) - f(t,v_1,v_2)\bigr|\\
&\quad = \left|\frac{t^2+1}{20}\left(\frac{|u_1|}{|u_1|+1} - \frac{|v_1|}{|v_1|+1}\right) + \frac{t^2}{25}\bigl(\mathrm{e}^{-u_2} - \mathrm{e}^{-v_2}\bigr)\right|\\
&\quad \leqslant \frac{t^2+1}{20}\bigl(|u_1 - v_1| + |u_2 - v_2|\bigr),
\end{aligned}$$

we find that $k_f(t) = (t^2+1)/20$, then choose $0 < k < 0.409$, we have

$$k\left[\frac{1}{\Gamma(\alpha+1)} + \frac{1}{(1-\eta)\Gamma(\alpha)}\right] < 2.4449k \leqslant 1,$$

we can choose $k < 0.409$.

Thus, the hypotheses of Theorem 1 hold. Therefore, the conclusion of Theorem 1 applies to Hadamard fractional boundary value problem (7).

Example 2. We consider the following Hadamard fractional boundary value problem:

$$\begin{aligned}
&-{}^H D^{3/2}\bigl[\phi_4\bigl(-{}^H D^{7/4}u(t)\bigr)\bigr] = \frac{|u(t)|^5}{t^2(|u(t)|^5+5)} + \frac{\mathrm{e}^2}{5}\mathrm{e}^{u^2(\sqrt{t})}, \quad t \in [1,\mathrm{e}],\\
&{}^H D^{7/4}u(1) = {}^H D^{7/4}u(0) = 0, \qquad u(1) = 0, \qquad {}^H D^{3/4}u(1) = \frac{1}{4}{}^H D^{3/4}u(\mathrm{e}),
\end{aligned} \tag{8}$$

where $p = 4$, $q = 4/3$, $\alpha = 7/4$, $\beta = 3/2$, $\eta = 1/4$, $\theta(t) = \sqrt{t}$, and $f(t,u,v) = |u|^5/(t^2(|t|^5+5)) + (e^2/5)\mathrm{e}^{v^2}$. We can see

$$\bigl|f(t,u,v)\bigr| \leqslant \frac{|u|^5}{t^2(|u|^5+5)} + \frac{\mathrm{e}^2}{5}\mathrm{e}^{|u|^2},$$

then $a(t) = 1/t^2$, $b(t) = \mathrm{e}^2/5$, and condition (H3) holds.

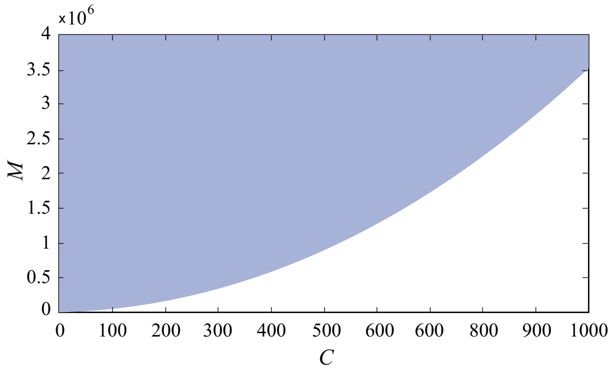

Figure 1

By a simple computation,

$$\frac{\mathcal{M}}{2^{q-1}[\frac{|u|^5}{t^2(|u|^5+5)}+\frac{\mathrm{e}^2}{5}\mathrm{e}^{|u|^2}]^{q-1}[\frac{1}{\Gamma(\alpha+1)(\Gamma(\beta+1))^{q-1}}+\frac{1}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}}]}>1,$$

i.e.

$$\frac{\mathcal{M}}{2.3748[\frac{\|C\|^5}{\|C\|^5+5}+\frac{\mathrm{e}^2}{5}e^{\|C\|^2}]}>1,$$

see Fig. 1.

If we choose appropriate $\mathcal{M}$ and C, condition (H4) is holds. Then all the assumptions of Theorem 3 are hold. Therefore, problem (8) has at least one solution.

Example 3. We consider the following Hadamard fractional turbulent flow model:

$$\begin{aligned}
&-{}^{H}D^{5/4}\bigl[\phi_3\bigl(-{}^{H}D^{3/2}u(t)\bigr)\bigr]=\frac{\lambda t(|u(t)|+|u(t^2)|)^{5/2}}{500[(|u(t)|+|u(t^2)|)^2+1]},\quad t\in[1,\mathrm{e}],\\
&{}^{H}D^{3/2}u(1)={}^{H}D^{3/2}u(0)=0,\qquad u(1)=0,\qquad {}^{H}D^{1/2}u(1)=\frac{1}{2}{}^{H}D^{1/2}u(e),
\end{aligned}\tag{9}$$

where $p=3$, $q=3/2$, $\alpha=3/2$, $\beta=5/4$, $\eta=1/2$, $\theta(t)=t^2$, λ is a positive parameter, and

$$f(t,u,v)=\frac{t(|u(t)|+|u(t^2)|)^{5/2}}{500[(|u(t)|+|u(t^2)|)^2+1]}.$$

We have

$$\lim_{|u|+|v|\to0^+}\frac{f(t,u,v)}{(|u|+|v|)^{p-1}}=\lim_{|u(t)|+|u(t^2)|\to0^+}\frac{t(|u(t)|+|u(t^2)|)^{1/2}}{500[(|u(t)|+|u(t^2)|)^2+1]}=0,$$

$$\lim_{|u|+|v|\to\infty}\frac{f(t,u,v)}{(|u|+|v|)^{p-1}}=\lim_{|u(t)|+|u(t^2)|\to\infty}\frac{t(|u(t)|+|u(t^2)|)^{1/2}}{500[(|u(t)|+|u(t^2)|)^2+1]}=0,$$

i.e.

(i) for all $\varepsilon_1 > 0$ (choose $\varepsilon_1 = 1/500$), there is $r = (500\varepsilon_1)^2/2 = 0.5$ such that if $0 < \|u\| < r$,

$$\left|\frac{t(|u(t)| + |u(t^2)|)^{1/2}}{500[(|u(t)| + |u(t^2)|)^2 + 1]} - 0\right| < \varepsilon_1;$$

(ii) for all $\varepsilon_2 > 0$ (choose $\varepsilon_2 = 1/10^8$), there is $N = (\mathrm{e}/(500\varepsilon_2)^{2/3}/2 \approx 1939$ such that if $\|u\| > N$,

$$\left|\frac{t(|u(t)| + |u(t^2)|)^{1/2}}{500[(|u(t)| + |u(t^2)|)^2 + 1]} - 0\right| < \varepsilon_2,$$

and if $\|u\| \in (r, 1)$,

$$\frac{f(t,u,v)}{(|u| + |v|)^{p-1}} < \frac{\sqrt{2}t}{500},$$

if $\|u\| \in (1, N)$,

$$\frac{f(t,u,v)}{(|u| + |v|)^{p-1}} < \frac{t}{1000\sqrt{2}}.$$

Therefore, $\varepsilon_3 = \max\{1/500, 1/10^8, 0.0076\} = 0.0076$. Since

$$M = \frac{2^q(2-\eta)}{(1-\eta)\Gamma(\alpha)(\Gamma(\beta+1))^{q-1}} \approx 8.99,$$

then $\lambda_0 = (\varepsilon_3 M^{1/(q-1)})^{-1} = 1.6280$.

Thus, all the assumptions of Theorem 4 are satisfied. Consequently, the conclusion of Theorem 4 implies that problem (9) has no solution for $0 < \lambda < 1.6280$.

References

1. R.P. Agarwal, D. O'Regan, P.J.Y. Wong, *Positive Solutions of Differential, Difference and Integral Equations*, Kluwer Academic, Dordrecht, 1999.
2. B. Ahmad, S. Ntouyas, A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations, *Fract. Calc. Appl. Anal.*, **17**(2):348–360, 2014.
3. B. Ahmad, S.K. Ntouyas, J. Tariboon, A study of mixed Hadamard and Riemann–Liouville fractional integro-differential inclusions via endpoint theory, *Appl. Math. Lett.*, **52**:9–14, 2016.
4. İ. Aslan, An analytic approach to a class of fractional differential-difference equations of rational type via symbolic computation, *Math. Methods Appl. Sci.*, **38**(1):27–36, 2014.
5. İ. Aslan, Analytic investigation of a reaction-diffusion brusselator model with the time-space fractional derivative, *Int. J. Nonlinear Sci. Numer. Simul.*, **15**(2):149–155, 2014.
6. İ. Aslan, Symbolic computation of exact solutions for fractional differential-difference equation models, *Nonlinear Anal. Model. Control*, **20**(1):132–144, 2015.
7. İ. Aslan, Exact solution for a local fractional DDE associated with a nonlinear transmission line, *Commun. Theor. Phys.*, **66**(3):315–320, 2016.

8. İ. Aslan, Exact solutions for fractional DDEs via auxiliary equation method coupled with the fractional complex transform, *Math. Methods Appl. Sci.*, **39**(18):5619–5625, 2016.

9. Z. Bai, S. Zhang, S. Sun, C. Yin, Monotone iterative method for fractional differential equations, *Electron. J. Differ. Equ.*, **2016**(6):1–8, 2016.

10. D. Baleanu, K. Diethelm, E. Scalas, J.J. Trujillo, *Fractional Calculus: Models and Numerical Methods*, Ser. Complex. Nonlinearity Chaos, Vol. 3, World Scientific, Boston, 2012.

11. P.L. Butzer, A.A. Kilbas, J.J. Trujillo, Compositions of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.*, **269**(2):387–400, 2002.

12. A. Cabada, G. Wang, Positive solutions of nonlinear fractional differential equations with integral boundary value conditions, *J. Math. Anal. Appl.*, **389**(1):403–411, 2012.

13. A. Chadha, D.N. Pandey, Existence results for an impulsive neutral stochastic fractional integro-differential equation with infinite delay, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **128**:149–175, 2015.

14. T. Chen, W. Liu, Z. Hu, A boundary value problem for fractional differential equation with p-Laplacian operator at resonance, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **75**(6):3210–3217, 2012.

15. Y. Cui, Uniqueness of solution for boundary value problems for fractional differential equations, *Appl. Math. Lett.*, **51**:48–54, 2016.

16. K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, 1985.

17. R. Garra, F. Polito, On some operators involving Hadamard derivatives, *Integral Transforms Spec. Funct.*, **24**(10):773–782, 2013.

18. J. Hadamard, Essai sur l'étude des fonctions données par leur développment de Taylor, *Journ. de Math. (4)*, **8**:101–186, 1892.

19. Z. Han, H. Lu, C. Zhang, Positive solutions for eigenvalue problems of fractional differential equation with generalized p-Laplacian, *Appl. Math. Comput.*, **257**:526–536, 2015.

20. A.A. Kilbas, Hadamard-type fractional calculus, *J. Korean Math. Soc.*, **38**(6):1191–1204, 2001.

21. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud., Vol. 204, Elsevier, Amsterdam, 2006.

22. A.A. Kilbas, J.J. Trujillo, Hadamard-type integrals as G-transforms, *Integral Transform. Spec. Funct.*, **14**(5):413–427, 2003.

23. V. Lakshmikantham, S. Leela, J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cambridge, 2009.

24. L.S. Leibenson, General problem of the movement of a compressible fluid in a porous medium, *Izv. Akad. Nauk SSSR, Ser. Geogr. Geofiz.*, **9**:7–10, 1945 (in Russian).

25. Z. Liu, S. Zeng, Y. Bai, Maximum principles for multi-term space-time variable-order fractional diffusion equations and their applications, *Fract. Calc. Appl. Anal.*, **19**(1):188–211, 2016.

26. Q. Ma, R. Wang, J. Wang, Y. Ma, Qualitative analysis for solutions of a certain more generalized two-dimensional fractional differential system with hadamard derivative, *Appl. Math. Comput.*, **257**:436–445, 2015.

27. M. Mongiovi, M. Zingales, A non-local model of thermal energy transport: The fractional temperature equation, *Int. J. Heat Mass Transfer*, **67**:593–601, 2013.
28. I. Podlubny, *Fractional Differential Equations*, Math. Sci. Eng., Vol. 198, Academic Press, New York, 1999.
29. M. Röckner, R. Zhu, X. Zhu, Existence and uniqueness of solutions to stochastic functional differential equations in infinite dimensions, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **125**:358–397, 2015.
30. D.R. Smart, *Fixed Point Theorems*, Cambridge Univ. Press, Cambridge, 1974.
31. S. Suganya, M.M. Arjunan, J.J. Trujillo, Existence results for an impulsive fractional integro-differential equation with state-dependent delay, *Appl. Math. Comput.*, **266**:54–69, 2015.
32. G. Wang, Monotone iterative technique for boundary value problems of a nonlinear fractional differential equation with deviating arguments, *J. Comput. Appl. Math.*, **236**(9):2425–2430, 2012.
33. G. Wang, Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval, *Appl. Math. Lett.*, **47**:1–7, 2015.
34. G. Wang, K. Pei, D. Baleanu, Explicit iteration to Hadamard fractional integro-differential equations on infinite domain, *Adv. Difference Equ.*, **216**:299, 2016.
35. G. Wang, T. Wang, On a nonlinear hadamard type fractional differential equation with p-Laplacian operator and strip condition, *J. Nonlinear Sci. Appl.*, **9**(7):5073–5081, 2016.
36. L. Wang, X. Zhang, X. Lu, Existence and uniqueness of solutions for a singular system of higher-order nonlinear fractional differential equations with integral boundary conditions, *Nonlinear Anal. Model. Control*, **18**(4):493–518, 2013.
37. X. Yang, J.A. Tenreiro Machado, H.M. Srivastava, A new numerical technique for solving the local fractional diffusion equation: Two-dimensional extended differential transform approach, *Appl. Math. Comput.*, **274**:143–151, 2016.
38. X.-J. Yang, D. Baleanu, H.M. Srivastava, *Local Fractional Integral Transforms and Their Applications*, Academic Press, London, 2015.
39. B. Zhang, Solvability of fractional boundary value problems with p-Laplacian operator, *Adv. Difference Equ.*, **2015**:352, 2015.
40. L. Zhang, B. Ahmad, G. Wang, Explicit iterations and extremal solutions for fractional differential equations with nonlinear integral boundary conditions, *Appl. Math. Comput.*, **268**:388–392, 2015.
41. X. Zhang, Positive solutions for a class of singular fractional differential equation with infinite-point boundary value conditions, *Appl. Math. Lett.*, **39**:22–27, 2015.
42. X. Zhang, L. Liu, B. Wiwatanapataphee, Y. Wu, The eigenvalue for a class of singular p-Laplacian fractional differential equations involving the Riemann–Stieltjes integral boundary condition, *Appl. Math. Comput.*, **235**:412–422, 2014.
43. X. Zhang, L. Liu, Y. Wu, The uniqueness of positive solution for a fractional order model of turbulent flow in a porous medium, *Appl. Math. Lett.*, **37**:26–33, 2014.
44. X. Zhang, L. Liu, Y. Wu, Nontrivial solutions for a fractional advection dispersion equation in anomalous diffusion, *Appl. Math. Lett.*, **66**:1–8, 2017.

45. X. Zhang, L. Wang, Q. Sun, Existence of positive solutions for a class of nonlinear fractional differential equations with integral boundary conditions and a parameter, *Appl. Math. Comput.*, **226**:708–718, 2014.

46. X. Zhang, Y. Wu, L. Caccetta, Nonlocal fractional order differential equations with changing-sign singular perturbation, *Appl. Math. Modelling*, **39**(1):6543–6552, 2015.

47. Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.