Nonlinear Analysis: Modelling and Control, Vol. 22, No. 6, 785–792 ISSN 1392-5113
https://doi.org/10.15388/NA.2017.6.4

On the areas under the oscillatory curves

Sergey Smirnov

Institute of Mathematics and Computer Science, University of Latvia,
Raina bulvaris 29, Riga, LV-1459, Latvia

Faculty of Physics and Mathematics, University of Latvia,
Zellu iela 25, Riga LV-1002, Latvia
srgsm@inbox.lv

Received: March 2, 2017 / **Revised:** September 8, 2017 / **Published online:** October 27, 2017

Abstract. We prove that the third-order Emden–Fowler-type equation has oscillatory non-extendable solutions with specific behavior of antiderivative.

Keywords: third-order Emden–Fowler-type equation, oscillatory behavior of solutions, self-similar solutions, matching of the areas.

1 Introduction

Behavior of antiderivatives of solutions plays an important role in the theory of boundary value problems with integral conditions (see [6, 8] and references therein). Results on the estimation of the number of solutions to boundary value problems with integral conditions often are related with the oscillatory properties of solutions and its antiderivatives [5]. Interest in nonlocal boundary value problems for differential equations involving integral boundary conditions is due to the fact that they often appear in physics and in various branches of applied mathematics. Integral conditions are connected not only with the values of a solution on the boundary, but also with the values inside the domain. Let us begin with an example.

Example 1. Consider the function $f(t) = t\sin t^2$ for $t \geqslant 0$. Function $f(t)$ is not periodic and has zeros at the points $t_0 = 0$, $t_1 = \sqrt{\pi}$, $t_2 = \sqrt{2\pi}$, $\ldots$, $t_k = \sqrt{k\pi}$, $\ldots$. As we can see, the distances $(t_k - t_{k-1})$ between two consecutive zeros form decreasing sequence, but the absolute values of the function in extreme points increase (see Fig. 1). If we consider the areas between the curve and axis, we get

$$S_k = \left| \int\limits_{t_{k-1}}^{t_k} t\sin t^2 \,\mathrm{d}t \right| = |-\cos k\pi| = 1.$$

Thus, $S_1 = S_2 = \cdots = S_k = \cdots$.

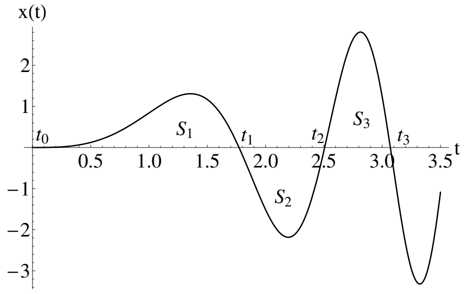

Figure 1. Function $f(t) = t \sin t^2$.

Definition 1. We say that the continuous function $f(t)$ has property (A) if:

(i) There exists $t = t_0$ such that $f(t_0) = 0$ and $f(t)$ has infinitely many simple zeros t_k $(k = 1, 2, \dots)$ for $t > t_0$;

(ii) The distances $(t_k - t_{k-1})$ between two consecutive zeros form decreasing sequence, and the absolute values of the function in extreme points increase;

(iii) $S_1 = S_2 = \cdots = S_k = \cdots$, where $S_k = |\int_{t_{k-1}}^{t_k} f(t)\,\mathrm{d}t|$.

It is our goal to prove that the third-order Emden–Fowler-type equation

$$x''' = -|x|^p \operatorname{sign} x, \tag{1}$$

where $p > 1$, has a solution with property (A).

Also we will show that equation (1) has non-extendable solutions of oscillatory type. Similar result on non-extendable solutions for the third-order Emden–Fowler-type equation was obtained in [1]. The example of the second-order equation with oscillatory solution that has “finite escape time” was constructed in [4].

Equation (1) is a generalization of the second-order Emden–Fowler equation

$$u'' + t^\nu |u|^\mu \operatorname{sign} u = 0, \tag{2}$$

where ν, μ are real constants. Such equations appear in the problems of polytropic gas spheres of finite radius or finite mass [3, 7]. The study of Emden–Fowler equation (2) has been one of the main objects in the field of nonlinear analysis in recent years since the appearance of the Bellman monograph [2].

The paper is organized as follows. In Section 2, we establish the results, which describe the oscillatory behavior of solutions of equation (1). In Section 3, we consider conditions of the self-similar solutions. Section 4 is devoted to the comparison of the areas, and in this section, we provide our main result.

2 Oscillatory behavior of solutions

Proposition 1. *If $x(t)$ is a nontrivial solution of* (1), $x(t_0) = 0$*, then for* $t > t_0$,

$$x(t)x''(t) - \frac{1}{2}x'(t)^2 + \frac{1}{2}x'(t_0)^2 < 0. \tag{3}$$

Proof. In view of equation (1), $x(t)x'''(t) \leqslant 0$, and since $x(t)$ is a nontrivial solution of (1), we get

$$0 > \int_{t_0}^{t} x x''' \, \mathrm{d}s = x(t)x''(t) - \int_{t_0}^{t} x' x'' \, \mathrm{d}s$$
$$= x(t)x''(t) - \frac{1}{2}x'(t)^2 + \frac{1}{2}x'(t_0)^2. \qquad \square$$

Proposition 2. *If $x(t)$ is a nontrivial solution of* (1)*,* $x(t_0) = x(t_1) = 0$ ($x(t) \neq 0$ *for* $t \in (t_0, t_1)$)*, then* $x'(t_1) \neq 0$.

Proof. If $x'(t_1) = 0$, in view of inequality (3), we have the contradiction $x'(t_0)^2 < 0$. $\square$

Remark 1. For a nontrivial solution of (1), a simple zero cannot exist on the left of a double zero.

Proposition 3. *If $x(t)$ is a nontrivial solution of* (1)*,* $x(t_0) = x'(t_1) = 0$ ($x(t) \neq 0$ *for* $t \in (t_0, t_1)$)*, then* $x(t_1)x''(t_1) < 0$.

Proof. The proof follows from inequality (3) if $t = t_1$. $\square$

Proposition 4. *If $x(t)$ is a nontrivial solution of* (1)*,* $x(t_0) = x'(t_1) = 0$ ($x(t) \neq 0$ *for* $t \in (t_0, t_1)$)*, then there exists $t = a > t_1$ such that $x(a) = 0$.*

Proof. First, note that in view of Proposition 2, $x(t_1) \neq 0$. Without loss of generality, let $x(t) > 0$ for $t > t_1$. Since $x(t_1) \neq 0$, then $x(t) > 0$ for $t > t_0$. Therefore, by Proposition 3 we have $x''(t_1) < 0$. Since $x(t) > 0$, then $x'''(t) < 0$ and $x''(t)$ is strictly decreasing for $t > t_0$. Since $x''(t_1) < 0$, then we have $x''(t) < 0$ and $x'(t)$ is strictly decreasing for $t > t_1$. Since $x'(t_1) = 0$, then we have $x'(t) < 0$ and $x(t)$ is strictly decreasing for $t > t_1$. If the first and the second derivatives of $x(t)$ are negative for $t > t_1$, then $x(t)$ must eventually be negative. Hence the proof. $\square$

Proposition 5. *If $x(t)$ is a nontrivial solution of* (1)*,* $x(t_0) = 0$*, then there exists $t = a > t_0$ such that $x(a) = 0$.*

Proof. Without loss of generality, let $x(t) > 0$ for $t > t_0$. If there exists $t_1 > t_0$ such that $x'(t_1) = 0$, then the proof follows from Proposition 4 above. Therefore, assume that $x'(t)$ does not vanish for $t > t_0$. Since $x'(t) > 0$ for t immediately to the right of t_0, it follows that $x'(t) > 0$ for $t > t_0$. As $x(t) > 0$, then $x'''(t) < 0$ and $x''(t)$ is strictly decreasing for $t > t_0$.

First, suppose there exists $t_2 \geqslant t_0$ such that $x''(t_2) = 0$, then $x''(t) < 0$ for $t > t_2$. If two consecutive derivatives of $x'(t)$ are negative, then $x'(t)$ must ultimately be negative.

Now assume that $x''(t) > 0$ for $t > t_0$. So, $x'(t)$ is strictly increasing for $t > t_0$. Integrating equation (1) between $b > t_0$ and t, we obtain

$$\int_{b}^{t} x'''(s) \, \mathrm{d}s = -\int_{b}^{t} \left|x(s)\right|^p \operatorname{sign} x(s) \, \mathrm{d}s,$$

or eliminating nonnegative term and taking into account our assumption that $x(t) > 0$ for $t > t_0$, we get

$$x''(b) = x''(t) + \int_b^t x(s)^p \, \mathrm{d}s \geqslant \int_b^t x(s)^p \, \mathrm{d}s.$$

The left side is independent of t, and thus, the integral on the right-hand side must converge as $t \to +\infty$. This contradiction proves the proposition. □

Corollary 1. *If $x(t)$ is a nontrivial solution of* (1)*, $x(t_0) = 0$, then $x(t)$ has an infinity of simple zeros in $(t_0, +\infty)$. Moreover, if $x'(t_0) = 0$, then $x(t)$ does not vanish in $(-\infty, t_0)$.*

Proposition 6. *Let $x(t)$ be a nontrivial solution of* (1) *and $x(t_0) = 0$. If t'_1 and t'_2 are arbitrary consecutive extreme points of $x(t)$, then $|x(t'_2)| > |x(t'_1)|$.*

Proof. Let t'_1 and t'_2 be arbitrary consecutive extreme points. Consider

$$\int_{t'_1}^{t'_2} x'x''' \, \mathrm{d}t = -\int_{t'_1}^{t'_2} (x'')^2 \, \mathrm{d}t < 0.$$

We obtain the last inequality because $x(t)$ is a nontrivial solution. On the other hand,

$$\begin{aligned}\int_{t'_1}^{t'_2} x'x''' \, \mathrm{d}t &= -\int_{t'_1}^{t'_2} x'|x|^p \operatorname{sign} x \, \mathrm{d}t \\ &= -\frac{1}{p+1}\left(\left|x(t'_2)\right|^{p+1} - \left|x(t'_1)\right|^{p+1}\right) < 0.\end{aligned}$$

Thus, $|x(t'_2)| > |x(t'_1)|$. □

Proposition 7. *If $x(t)$ is a nontrivial solution of* (1) *and $x(t_0) = x(t_1) = 0$ ($t_0 < t_1$), then $|x'(t_1)| > |x'(t_0)|$.*

Proof. Consider

$$0 > \int_{t_0}^{t_1} xx''' \, \mathrm{d}t = -\int_{t_0}^{t_1} x' \, \mathrm{d}x' = -\frac{x'(t_1)^2}{2} + \frac{x'(t_0)^2}{2}.$$

Thus, $|x'(t_1)| > |x'(t_0)|$. □

Proposition 8. *If $x(t)$ is a nontrivial solution of* (1) *and $x(t_0) = x(t_1) = 0$ ($t_0 < t_1$), then $|x''(t_1)| > |x''(t_0)|$.*

Proof. Consider

$$\int_{t_0}^{t_1} x''x''' \, \mathrm{d}t = -\int_{t_0}^{t_1} x''|x|^p \operatorname{sign} x \, \mathrm{d}t = \int_{t_0}^{t_1} p|x|^{p-1}(x')^2 \, \mathrm{d}t > 0.$$

On the other hand,

$$\int_{t_0}^{t_1} x''x'''\,\mathrm{d}t = \frac{x''(t_1)^2}{2} - \frac{x''(t_0)^2}{2}.$$

Thus, $|x''(t_1)| > |x''(t_0)|$. □

Proposition 9. *If $x(t)$ is a nontrivial solution of* (1)*, $x(t_0) = x(t_2) = 0$ ($x(t) \neq 0$ for $t \in (t_0, t_2)$), then $x'(t_2)x''(t_2) > 0$.*

Proof. Let $t_1 \in (t_0, t_2)$ be such that $x'(t_1) = 0$. Multiplying equation (1) by x' and integrating from t_1 to t_2, we get

$$\int_{t_1}^{t_2} x'x'''\,\mathrm{d}s = -\int_{t_1}^{t_2} x'|x|^p \operatorname{sign} x\,\mathrm{d}s,$$

$$x'(t_2)x''(t_2) - \int_{t_1}^{t_2} (x'')^2\,\mathrm{d}s = -\frac{1}{p+1}\bigl(\bigl|x(t_2)\bigr|^{p+1} - \bigl|x(t_1)\bigr|^{p+1}\bigr),$$

$$x'(t_2)x''(t_2) = \int_{t_1}^{t_2} (x'')^2\,\mathrm{d}s + \frac{1}{p+1}\bigl|x(t_1)\bigr|^{p+1} > 0.$$ □

Corollary 2. *If $x(t)$ is a nontrivial solution of* (1)*, $x(t_0) = x(t_2) = 0$ and $x(t) > 0$ for $t \in (t_0, t_2)$, then $x'(t_2) < 0$ and $x''(t_2) < 0$.*

Remark 2. Let $x(t)$ be a nontrivial solution of (1) with initial data

$$x(t_0) = 0, \qquad x'(t_0) = \alpha_0, \qquad x''(t_0) = \beta_0, \tag{4}$$

where $\alpha_0\beta_0 > 0$. In view of Proposition 5, there exists $t = t_1$ such that $x(t_1) = 0$. Let us denote $x'(t_1) = \alpha_1$ and $x''(t_1) = \beta_1$. Obviously, α_1 and β_1 depend on α_0 and β_0 or $\alpha_1 = \alpha_1(\alpha_0, \beta_0)$ and $\beta_1 = \beta_1(\alpha_0, \beta_0)$. Moreover, in view of Propositions 7–9, $\alpha_1 = -l\alpha_0$ and $\beta_1 = -m\beta_0$, where $l > 1$ and $m > 1$ are some constants.

Proposition 10. *There exists a nontrivial solution of initial value problem* (1), (4) *such that if $-\alpha_1 = l\alpha_0$ ($l > 1$), then $-\beta_1 = l^{(2p+1)/(p+2)}\beta_0$.*

Proof. Consider the function

$$\varphi(\alpha_0, \beta_0) = -\beta_1\alpha_0^{(2p+1)/(p+2)} - (-\alpha_1)^{(2p+1)/(p+2)}\beta_0$$

for $\alpha_0 \geqslant 0$ and $\beta_0 \geqslant 0$.

If $\alpha_0 = 0$, then $\varphi(\alpha_0, \beta_0) \leqslant 0$, and if $\beta_0 = 0$, then $\varphi(\alpha_0, \beta_0) \geqslant 0$. Thus, there exist $\overline{\alpha}_0$ and $\overline{\beta}_0$ such that $\varphi(\overline{\alpha}_0, \overline{\beta}_0) = 0$ or

$$-\beta_1\overline{\alpha}_0^{(2p+1)/(p+2)} = (-\alpha_1)^{(2p+1)/(p+2)}\overline{\beta}_0.$$

Therefore,

$$-\beta_1 = \left(-\frac{\alpha_1}{\overline{\alpha}_0}\right)^{(2p+1)/(p+2)} \overline{\beta}_0.$$

Let $-\alpha_1/\overline{\alpha}_0 = l$, then $-\beta_1 = l^{(2p+1)/(p+2)}\beta_0$. □

Definition 2. We say that the solution of initial value problem (1), (4) has property (A_0) if for $-\alpha_1 = l\alpha_0$ ($l > 1$), we have $-\beta_1 = l^{(2p+1)/(p+2)}\beta_0$. We denote this solution by $\overline{x}(t)$.

Remark 3. In view of Corollary 1, $\overline{x}(t)$ has an infinity of simple zeros in $(t_0, +\infty)$.

3 Self-similar solutions

Proposition 11. *If $x(t)$ is a solution of equation* (1)*, then the function*

$$y(t) = \pm B^{3/(p-1)} x(Bt + C), \tag{5}$$

where $B > 0$ and C are arbitrary constants, is also a solution of equation (1)*.*

Remark 4. A similar statement for higher-order Emden–Fowler-type equation can be found in [1].

Proof. The proposition can be proved by direct substitution. So,

$$y''' = \pm B^{3/(p-1)+3} x'''(Bt + C),$$
$$|y|^p \operatorname{sign} y = \left| \pm B^{3/(p-1)} x(Bt + C)\right|^p \operatorname{sign} y.$$

Then

$$\pm B^{3/(p-1)+3} x'''(Bt + C) = -\left|\pm B^{3/(p-1)} x(Bt + C)\right|^p \operatorname{sign} y.$$

We get

$$B^{3/(p-1)+3} = B^{3p/(p-1)} \quad \text{or} \quad B^{3p/(p-1)} = B^{3p/(p-1)}.$$ □

Corollary 3. *Let $x(t)$ be a solution of initial value problem* (1), (4)*. Every solution of initial value problem* (1)*, $x(t_1) = 0$, $x'(t_1) = \alpha$, $x''(t_1) = \beta$ with α, β satisfying $\alpha = B^{(p+2)/(p-1)}\alpha_0$, $\beta = B^{(2p+1)/(p-1)}\beta_0$ (or $\alpha = -B^{(p+2)/(p-1)}\alpha_0$, $\beta = -B^{(2p+1)/(p-1)}\beta_0$), where $B > 0$ is a constant, can be expressed via solution $x(t)$ as* (5)*.*

Corollary 4. *If the solution of initial value problem* (1), (4) *$\overline{x}(t)$ has property* (A_0)*, then the distances $(t_k - t_{k-1})$ between two consecutive zeros generate geometric sequence with quotient less then one.*

Proof. Let t_0, t_1, and t_2 be three arbitrary consecutive zeros of $\overline{x}(t)$. Since $\overline{x}(t)$ has property (A_0), then $-\overline{x}'(t_1) = l\overline{x}'(t_0)$ and $-\overline{x}''(t_1) = l^{(2p+1)/(p+2)}\overline{x}''(t_0)$. Now consider

$$y(t) = -B^{3/(p-1)}\overline{x}(Bt + C)$$

with $C = -Bt_1 + t_0$. In view of Corollary 3,

$$y(t) = -B^{3/(p-1)}\overline{x}(Bt - Bt_1 + t_0)$$

is also a solution of (1) with initial data $y(t_1) = 0$, $y'(t_1) = -B^{(p+2)/(p-1)}\overline{x}'(t_0)$, $y''(t_1) = -B^{(2p+1)/(p-1)}\overline{x}''(t_0)$.

Let $l = B^{(p+2)/(p-1)}$, then $\overline{x}'(t_1) = y'(t_1)$ and $\overline{x}''(t_1) = y''(t_1)$. It means that $y(t) = \overline{x}(t)$. Since $y(t) = -B^{3/(p-1)}\overline{x}(Bt - Bt_1 + t_0)$, then $t_2 - t_1 = (t_1 - t_0)/B$, where $B > 1$. Hence, the distances $(t_k - t_{k-1})$ between two consecutive zeros of $\overline{x}(t)$ generate geometric sequence with quotient less then one. $\square$

Remark 5. Since the distances $(t_k - t_{k-1})$ between two consecutive zeros of $\overline{x}(t)$ generate geometric sequence with quotient less then one, then

$$\sum_{k=1}^{+\infty}(t_k - t_{k-1}) < +\infty.$$

It means that $\overline{x}(t)$ is not defined for all $t > t_0$. Thus, equation (1) has non-extendable solutions of oscillatory type.

4 On the matching of the areas

Corollary 5. *If $\overline{x}(t)$ is a solution of initial value problem* (1), (4)*, which has property* (A_0)*, $p = 4$, then $S_1 = S_2 = \cdots = S_k = \cdots$, where $S_k = |\int_{t_{k-1}}^{t_k}\overline{x}(t)\,\mathrm{d}t|$.*

Proof. Let t_0, t_1, and t_2 be three arbitrary consecutive zeros of $\overline{x}(t)$. Since $y(t) = -B^{3/(p-1)}\overline{x}(Bt - Bt_1 + t_0)$, we have

$$\begin{aligned}\left|\int_{t_1}^{t_2}\overline{x}(s)\,\mathrm{d}s\right| &= \left|\int_{t_1}^{t_2}B^{3/(p-1)}\overline{x}(Bs - Bt_1 + t_0)\,\mathrm{d}s\right| \\ &= B^{3/(p-1)-1}\left|\int_{t_1}^{t_2}\overline{x}(Bs - Bt_1 + t_0)\,\mathrm{d}(Bs - Bt_1 + t_0)\right| \\ &= B^{(4-p)/(p-1)}\left|\int_{t_0}^{t_1}\overline{x}(s)\,\mathrm{d}s\right|.\end{aligned}$$

If $p = 4$, we have

$$\left|\int_{t_0}^{t_1}\overline{x}(s)\,\mathrm{d}s\right| = \left|\int_{t_1}^{t_2}\overline{x}(s)\,\mathrm{d}s\right|. \qquad \square$$

Remark 6. If $1 < p < 4$, then $S_1 < S_2 < \cdots < S_k < \cdots$. If $p > 4$, then $S_1 > S_2 > \cdots > S_k > \cdots$.

Theorem 1. *If $\overline{x}(t)$ is a solution of initial value problem* (1), (4)*, which has property* (A_0)*, $p = 4$, then $\overline{x}(t)$ has property* (A)*.*

Proof. (i) In view of Corollary 1, $\overline{x}(t)$ has an infinity of simple zeros in $(t_0, +\infty)$.

(ii) In view of Corollary 4 and Proposition 6, the distances $(t_k - t_{k-1})$ between two consecutive zeros form decreasing sequence, and the absolute values of the function in extreme points increase.

(iii) In view of Corollary 5, $S_1 = S_2 = \cdots = S_k = \cdots$, where $S_k = |\int_{t_{k-1}}^{t_k} \overline{x}(t)\,\mathrm{d}t|$. $\square$

Conclusions:

- We have shown that the third-order Emden–Fowler-type equation has oscillatory non-extendable solutions.
- If degree of nonlinearity $p = 4$, then $S_1 = S_2 = \cdots = S_k = \cdots$, where S_k are the areas under the curve between two consecutive zeros.
- If $1 < p < 4$, then $S_1 < S_2 < \cdots < S_k < \cdots$.
- If $p > 4$, then $S_1 > S_2 > \cdots > S_k > \cdots$.

References

1. I.V. Astashova, Application of dynamical systems to the study of asymptotic properties of solutions to nonlinear higher-order differential equations, *J. Math. Sci., New York*, **126**(5):1361–1391, 2005.
2. R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill, New York, 1953.
3. R. Emden, *Gaskugeln*, B. Teubner, Leipzig, 1907.
4. S.P. Hastings, Boundary value problems in one differential equation with a discontinuity, *J. Differ. Equations*, **1**(3):346–369, 1965.
5. R. Ma, A survey on nonlocal boundary value problems, *Appl. Math. E-notes*, **7**:257–279, 2007.
6. S.K. Ntouyas, Nonlocal initial and boundary value problems: A survey, in A. Cañada, P. Drábek, A. Fonda (Eds.), *Handbook of Differential Equations: Ordinary Differential Equations, Vol. 2*, Elsevier, North Holland, 2005, pp. 461–558.
7. G. Sansone, *Equazioni Differenziali nel Campo Reale, Parte seconda*, Zanichelli, Bologna, 1949.
8. A. Štikonas, A survey on stationary problems, Green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions, *Nonlinear Anal. Model. Control*, **19**(3):301–334, 2014.