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Solutions of stationary Kirchhoff equations involving nonlocal operators with critical nonlinearity in $\mathbb{R}^N$*

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Abstract. In this paper, we consider the existence and multiplicity of solutions for fractional Schrödinger equations with critical nonlinearity in $\mathbb{R}^N$. We use the fractional version of Lions' second concentration-compactness principle and concentration-compactness principle at infinity to prove that (PS_c) condition holds locally. Under suitable assumptions, we prove that it has at least one solution and, for any $m \in \mathbb{N}$, it has at least m pairs of solutions. Moreover, these solutions can converge to zero in some Sobolev space as $\varepsilon \to 0$.

Keywords: fractional Schrödinger equations, critical nonlinearity, variational method, critical points.

1 Introduction

The main purpose of this paper is to study the existence and multiplicity of solutions of the following fractional Schrödinger equations with critical nonlinearity:

$$\begin{aligned} &\varepsilon^{2s}\left[g\left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\right)\right](-\Delta)^s u + V(x)u \\ &\quad = |u|^{2_s^*-2}u + h(x,u), \quad x \in \mathbb{R}^N, \\ &u(x) \to 0 \quad \text{as } |x| \to \infty, \end{aligned} \tag{1}$$

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where $\varepsilon > 0$, $N > 2s$, $0 < s < 1$, $2_s^* = 2N/(N-2s)$, and

$$H^s(\mathbb{R}^N) := \left\{u \in L^2(\mathbb{R}^N)\colon \frac{|u(x)-u(y)|}{|x-y|^{N/2+s}} \in L^2(\mathbb{R}^N \times \mathbb{R}^N)\right\}.$$

We make the following assumptions on $V(x)$, $g(x)$, and $h(x)$ throughout this paper:

- (V) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $V(x_0) = \min V = 0$, and there is $\tau_0 > 0$ such that the set $V^{\tau_0} = \{x \in \mathbb{R}^N\colon V(x) < \tau_0\}$ has finite Lebesgue measure;
- (G) (g_1) There exists $\alpha_0 > 0$ such that nondecreasing function $g(t) \geqslant \alpha_0$ for all $t \geqslant 0$;
  - (g_2) There exists Σ satisfying $2/\mu < \Sigma < 1$ and $G(t) \geqslant \Sigma g(t)t$ for all $t \geqslant 0$, where $G(t) = \int_0^t g(s)\,\mathrm{d}s$;
- (H) (h_1) $h \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and $h(x,t) = o(|t|)$ uniformly in x as $t \to 0$;
  - (h_2) There are $c_0 > 0$ and $q \in (2, 2_s^*)$ such that $|h(x,t)| \leqslant c_0(1 + t^{q-1})$;
  - (h_3) There $l_0 > 0$, $r > 2$, and $2 < \mu < 2_s^*$ such that $H(x,t) \geqslant l_0|t|^r$ and $\mu H(x,t) \leqslant h(x,t)t$ for all (x,t), where $H(x,t) = \int_0^t h(x,s)\,\mathrm{d}s$.

The fractional Laplacian operator $(-\Delta)^s$ (up to normalization constants) may be defined as

$$(-\Delta)^s u := \mathrm{P.V.} \int\limits_{\mathbb{R}^N} \frac{|u(x)-u(y)|}{|x-y|^{N+2s}}\,\mathrm{d}y, \quad x \in \mathbb{R}^N,$$

where P.V. stands for the principal value. It may be viewed as the infinitesimal generators of a Lévy stable diffusion processes [1]. This operator arises in the description of various phenomena in the applied sciences, such as phase transitions, materials science, conservation laws, minimal surfaces, water waves, optimization, plasma physics, and so on; see [21] and references therein for more detailed introduction.

In these last years, a great deal of work has been devoted to the study of semiclassical standing waves for the fractional nonlinear Schrödinger equation of the form

$$\mathrm{i}\varepsilon\frac{\partial\psi}{\partial t} = \varepsilon^{2s}(-\Delta)^s\psi + P(x)\psi - f(x, |\psi|), \quad x \in \mathbb{R}^N, \tag{2}$$

where ε is a small positive constant, which corresponds to the Planck constant, $(-\Delta)^s$, $0 < s < 1$, is the fractional Laplacian, $P(x)$ is a potential function. Problem (2) models naturally many physical problems, such as phase transition, conservation laws, especially in fractional quantum mechanics, etc.; see [14]. It was introduced by Laskin [16, 17] as a fundamental equation of fractional quantum mechanics in the study of particles on stochastic fields modeled by Lévy process. We refer to [21] for more physical background.

To obtain standing waves of the fractional nonlinear Schrödinger equation (2), we set $\psi(x,t) = \mathrm{e}^{-\mathrm{i}Et/\varepsilon}u(x)$ for some function $u \in H^s(\mathbb{R}^N)$, and let $V(x) = P(x) - E$. Then problem (2) is reduced to the following equation:

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{3}$$

In quantum mechanics, when ε tends to zero, the existence and multiplicity of solutions to (3) is of particular importance.

In the nonlocal case, that is, when $s \in (0,1)$, the nonlocal model has attracted much attentions recently. For the case of a bounded domain, Ricceri [24] established a theorem tailor-made for a class of nonlocal problems involving nonlinearities with bounded primitive. In [8], Molica Bisci and Repovš studied a class of nonlocal fractional Laplacian equations depending on two real parameters and obtained the existence of three weak solutions by exploiting the result established by Ricceri in [24]. For more related results, we refer the readers to [3, 4, 6, 7, 11, 13, 18] and the references therein. For the whole space $\mathbb{R}^N$ were also studied by a number of authors. Felmer et al. [14] studied the existence and regularity of positive solution when f has subcritical growth and satisfies the Ambrosetti–Rabinowitz condition. Secchi [25] obtained the existence of ground state solutions of (3) when $V(x) \to \infty$ as $|x| \to \infty$ and Ambrosetti–Rabinowitz condition holds. In [29], the authors obtained the existence of infinitely many weak solutions for (3) by variant fountain theorem when f has subcritical growth. For the case of critical growth, Shang and Zhang [26] studied the existence and multiplicity of solutions for the critical fractional Schrödinger equation

$$\varepsilon^{2\alpha}(-\Delta)^\alpha u + V(x)u = |u|^{2_s^*-2}u + \lambda f(u), \quad x \in \mathbb{R}^N. \tag{4}$$

Based on variational methods, they showed that problem (4) has a nonnegative ground state solution for all sufficiently large λ and small ε. Moreover, Shen and Gao in [28] obtained the existence of nontrivial solutions for problem (4) under various assumptions on f and potential function $V(x)$, in which the authors assumed the well-known Ambrosetti–Rabinowitz condition. See also recent papers [2, 22, 25, 27] on the fractional Schrödinger equations (4). In [32], the fractional Schrödinger equations with a critical nonlinearity considered by using fractional version of concentration-compactness principle and radially decreasing rearrangements, they obtained the existence of a ground state solutions. However, there are no such results on Kirchhoff type problems (1).

The interest in studying problems like problem (1) relies not only on mathematical purposes but also on their significance in real models. For example, in the Appendix of paper [15], the authors construct a stationary Kirchhoff variational problem, which models, as a special significant case, the nonlocal aspect of the tension arising from nonlocal measurements of the fractional length of the string.

In this paper, inspired by [12, 31], we consider the existence and multiplicity of standing wave solutions of the fractional Schrödinger equation (1). To the best of our knowledge, the existence and multiplicity of standing wave solutions to problem (1) on $\mathbb{R}^N$ has not ever been studied by variational methods. To prove all the results, we mainly follow the ideas in [12, 31]. Our proofs are based on variational methods. Let us point out that although the idea was used before for other problems, the adaptation to the procedure to our problem is not trivial at all. Because the appearance of non-local term and the function g, we must reconsider this problem and need more delicate estimates.

Our main result is the following.

Theorem 1. *Let* (V), (G), *and* (H) *be satisfied. Thus*:

(i) *For any* $\kappa > 0$, *there is* $\mathcal{E}_\kappa > 0$ *such that if* $\varepsilon \leqslant \mathcal{E}_\kappa$ *problem* (1) *has at least one solution* u_ε *satisfying*

$$\frac{2\mu - \Sigma}{\Sigma} \int\limits_{\mathbb{R}^N} H(x, u_\varepsilon)\,\mathrm{d}x + \left(\frac{2}{\Sigma} - \frac{1}{2_s^*}\right) \int\limits_{\mathbb{R}^N} |u_\varepsilon|^{2_s^*}\,\mathrm{d}x \leqslant \kappa\varepsilon^N, \tag{5}$$

$$\left(\frac{\Sigma}{2} - \frac{1}{\mu}\right)\alpha_0 \int\limits_{\mathbb{R}^N} \varepsilon^{2s}\left|(-\Delta)^{s/2}u_\varepsilon\right|^2 \mathrm{d}x + \left(\frac{1}{2} - \frac{1}{\mu}\right) \int\limits_{\mathbb{R}^N} V(x)|u_\varepsilon|^2\,\mathrm{d}x \leqslant \kappa\varepsilon^N. \tag{6}$$

Moreover, $u_\varepsilon \to 0$ *in* $H^s(\mathbb{R}^N)$ *as* $\varepsilon \to 0$.

(ii) *Assume additionally that* $h(x,t)$ *is odd in* t, *for any* $m \in \mathbb{N}$ *and* $\kappa > 0$, *there is* $\mathcal{E}_{m\kappa} > 0$ *such that if* $\varepsilon \leqslant \mathcal{E}_{m\kappa}$, *problem* (1) *has at least* m *pairs of solutions* $u_{\varepsilon,i}$, $u_{\varepsilon,-i}$, $i = 1, 2, \dots, m$, *which satisfy estimates* (5) *and* (6). *Moreover,* $u_{\varepsilon,i} \to 0$ *in* $H^s(\mathbb{R}^N)$ *as* $\varepsilon \to 0$, $i = 1, 2, \dots, m$.

Remark 1. We should point out that Theorem 1 is different from the previous results of [12, 31] in two main directions:

(i) $g(t) \not\equiv C$. There exist many functions $g(t)$ satisfying condition (g_1)–(g_2), for example, $g(t) = a + bt$, $a, b > 0$, and $\Sigma = 1/2$.

(ii) Other potentials $V(x)$ guaranteeing compactness of the embedding from $E \hookrightarrow H^s(\mathbb{R}^N)$ can also be used in this paper.

(iii) We use the fractional version of Lions' second concentration-compactness principle and concentration-compactness principle at infinity to prove that (PS$_c$) condition holds, which is different from methods used in [12].

(iv) The method are employed to establish the existence and multiplicity of standing wave solutions for problems (1), which is different from methods used in [31].

This paper is organized as follows: Section 2 is devoted to preliminary. In Section 3, we introduce the variational framework and restate the problem in a equivalent form by replacing ε^{-2s} with λ. Furthermore, we describe the corresponding main results (Theorem 3). In Section 4, we prove the behaviors of the bounded (PS) sequences and then show that the energy functional satisfies (PS$_c$) by using the fractional version of concentration-compactness principle. In Section 4, we give behaviors of (PS$_c$) sequences and its consequences. In Section 4, we verify the geometry of the mountain pass theorem and estimate the minimax value. At last, we give the proof of the main results.

2 Preliminaries

For the convenience of the reader, in this part we recall some definitions and basic properties of fractional Sobolev spaces $H^s(\mathbb{R}^N)$. For a deeper treatment on these spaces and their applications to fractional Laplacian problems of elliptic type, we refer to [9, 21] and references therein.

For any $s \in (0,1)$, the fractional Sobolev space $H^s(\mathbb{R}^N)$ is defined by

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N)\colon [u]_{H^s(\mathbb{R}^N)} < \infty\},$$

where $[u]_{H^s(\mathbb{R}^N)}$ denotes the so-called Gagliardo semi-norm, that is

$$[u]_{H^s(\mathbb{R}^N)} = \left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2},$$

and $H^s(\mathbb{R}^N)$ is endowed with the norm

$$\|u\|_{H^s(\mathbb{R}^N)} = [u]_{H^s(\mathbb{R}^N)} + \|u\|_{L^2(\mathbb{R}^N)}.$$

As it is well known, $H^s(\mathbb{R}^N)$ turns out to be a Hilbert space with scalar product

$$\langle u,v\rangle_{H^s(\mathbb{R}^N)} = \iint\limits_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int\limits_{\mathbb{R}^N} u(x)v(x)\,\mathrm{d}x$$

for any $u,v \in H^s(\mathbb{R}^N)$. The space $H^s(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm $[u]_{H^s(\mathbb{R}^N)}$.

By Proposition 3.6 in [21], we have

$$[u]_{H^s(\mathbb{R}^N)} = \|(-\Delta)^{s/2}\|_{L^2(\mathbb{R}^N)}$$

for any $u \in H^s(\mathbb{R}^N)$, i.e.

$$\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u(x)\right|^2\mathrm{d}x.$$

Thus,

$$\iint\limits_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\mathbb{R}^N} (-\Delta)^{s/2}u(x)\cdot(-\Delta)^{s/2}v(x)\,\mathrm{d}x.$$

Theorem 2. (*See* [14, *Lemma* 2.1].) *The embedding $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ is continuous for any $p \in [2, 2_s^*]$, and the embedding $H^s(\mathbb{R}^N)) \hookrightarrow L^p_{loc}(\mathbb{R}^N))$ is compact for any $p \in [2, 2_s^*)$.*

3 An equivalent variational problem

We set $\lambda = \varepsilon^{-2s}$ and rewrite (1) in the following form:

$$\begin{aligned} &g\left(\iint\limits_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\right)(-\Delta)^s u + \lambda V(x)u \\ &\quad = \lambda|u|^{2_s^*-2}u + \lambda h(x,u), \quad x \in \mathbb{R}^N, \\ &u(x) \to 0 \quad \text{as } |x| \to \infty \end{aligned} \tag{7}$$

for $\lambda \to \infty$.

We introduce the space

$$E = \left\{ u \in H^s(\mathbb{R}^N):\ \int_{\mathbb{R}^N} V(x)|u|^2 \, \mathrm{d}x < \infty \right\},$$

which is a reflexive Banach space under the scalar product

$$\langle v_1, v_2 \rangle_E = \int_{\mathbb{R}^N} (-\Delta)^{s/2} v_1 (-\Delta)^{s/2} v_2 \, \mathrm{d}x + \int_{\mathbb{R}^N} V(x) v_1 v_2 \, \mathrm{d}x.$$

The norm induced by the product $\langle \cdot, \cdot \rangle_E$ is

$$\|u\|_E = \sqrt{\langle u, u \rangle_E} \quad \text{for } u \in H^s(\mathbb{R}^N).$$

By assumption (V), we know that the embedding $E \hookrightarrow H^s(\mathbb{R}^N)$ is continuous. Note that the norm $\|\cdot\|_E$ is equivalent to the one $\|\cdot\|_\lambda$ defined by

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2} u\right|^2 \mathrm{d}x + \lambda \int_{\mathbb{R}^N} V(x)|u|^2 \, \mathrm{d}x \right)^{1/2}$$

for each $\lambda > 0$. It is obvious that for each $s \in [2, 2_s^*]$, there is $c_s > 0$ independent of $\lambda \geqslant 1$ such that

$$|u|_s \leqslant c_s \|u\|_E \leqslant c_s \|u\|_\lambda. \tag{8}$$

In the following, we denote by $|\cdot|_s$ the norm in $L^s(\mathbb{R}^N)$ and by $\|\cdot\|_E$ the norm in $H^s(\mathbb{R}^N)$. Note that the norm $\|\cdot\|_E$ is equivalent to the $\|\cdot\|_\lambda$ for each $\lambda > 0$.

The energy functional $J_\lambda : E \to \mathbb{R}$ associated with problem (7)

$$J_\lambda(u) := \frac{1}{2} G\left(\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2} u\right|^2 \mathrm{d}x \right) + \frac{1}{2} \int_{\mathbb{R}^N} \lambda V(x)|u|^2 \, \mathrm{d}x - \frac{\lambda}{2_s^*} \int_{\mathbb{R}^N} |u|^{2_s^*} \, \mathrm{d}x - \lambda \int_{\mathbb{R}^N} H(x, u) \, \mathrm{d}x$$

is well defined. Define the Nahari manifold

$$\mathcal{N} = \left\{ u \in E:\ \left\langle J_\lambda'(u), u \right\rangle_E = 0 \right\}.$$

Under the assumptions, it is easy to check that as arguments [23, 30] $J_\lambda \in C^1(E, \mathbb{R})$ and its critical points are solutions of (7).

We say that $u \in E$ is a weak solution of (7) if

$$\left\langle J_\lambda'(u), v \right\rangle = g\left(\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2} u\right|^2 \mathrm{d}x \right) \int_{\mathbb{R}^N} (-\Delta)^{s/2} u \cdot (-\Delta)^{s/2} v \, \mathrm{d}x + \lambda \int_{\mathbb{R}^N} V(x) u v \, \mathrm{d}x - \lambda \int_{\mathbb{R}^N} |u|^{2_s^* - 2} u v \, \mathrm{d}x - \lambda \int_{\mathbb{R}^N} h(x, u) v \, \mathrm{d}x,$$

where $v \in E$.

We are going to prove the following result.

Theorem 3. *Let* (V), (G) *and* (H) *be satisfied. Thus*:

(i) *For any* $\kappa > 0$, *there is* $\Lambda_\kappa > 0$ *such that if* $\lambda \geqslant \Lambda_\kappa$ *problem* (7) *has at least one solution* u_λ *satisfying*

$$\frac{2\mu - \Sigma}{\Sigma} \int\limits_{\mathbb{R}^N} H(x, u_\lambda)\,\mathrm{d}x + \left(\frac{2}{\Sigma} - \frac{1}{2_s^*}\right) \int\limits_{\mathbb{R}^N} |u_\lambda|^{2_s^*}\,\mathrm{d}x \leqslant \kappa\lambda^{-N/(2s)}, \tag{9}$$

$$\left(\frac{\Sigma}{2} - \frac{1}{\mu}\right)\alpha_0 \int\limits_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u_\lambda\right|^2\mathrm{d}x + \left(\frac{1}{2} - \frac{1}{\mu}\right)\lambda \int\limits_{\mathbb{R}^N} V(x)|u_\lambda|^2\,\mathrm{d}x \leqslant \kappa\lambda^{1-N/(2s)}. \tag{10}$$

Moreover, $u_\lambda \to 0$ *in* $H^s(\mathbb{R}^N)$ *as* $\lambda \to \infty$.

(ii) *Assume additionally that* $h(x,t)$ *is odd in* t, *for any* $m \in \mathbb{N}$ *and* $\kappa > 0$, *there is* $\Lambda_{m\kappa} > 0$ *such that if* $\lambda \geqslant \Lambda_{m\kappa}$, *problem* (7) *has at least* m *pairs of solutions* $u_{\lambda,i}$, $u_{\lambda,-i}$, $i = 1, 2, \dots, m$, *which satisfy estimates* (9) *and* (10). *Moreover,* $u_{\lambda,i} \to 0$ *in* $H^s(\mathbb{R}^N)$ *as* $\lambda \to \infty$, $i = 1, 2, \dots, m$.

4 Behaviors of (PS) sequences

We recall the fractional version of concentration-compactness principle in the fractional Sobolev space [31, 32], which due to Lions [19, 20].

Lemma 1. (*See* [31, 32].) *Let* $\Omega \subseteq \mathbb{R}^N$ *be an open subset, and let* $\{u_n\}$ *be a weakly convergent sequence to* u *in* $H^s(\mathbb{R}^N)$ *weakly converging to* u *as* $n \to \infty$ *and such that* $|u_n|^{2_s^*} \rightharpoonup \nu$ *and* $|(-\Delta)^{s/2}u_n|^2 \rightharpoonup \mu$ *in the sense of measures. Then, either* $u_n \to u$ *in* $L^{2_s^*}_{\mathrm{loc}}(\mathbb{R}^N)$ *or there exists a* (*at most countable*) *set of distinct points* $\{x_j\}_{j\in I} \subseteq \overline{\Omega}$ *and positive numbers* $\{\nu_j\}_{j\in I}$ *such that*

$$\nu = |u|^{2_s^*} + \sum_{j\in I} \delta_{x_j}\nu_j, \quad \nu_j > 0.$$

If, in addition, Ω *is bounded, then there exist a positive measure* $\widetilde{\mu} \in \mathcal{M}(\mathbb{R}^N)$ *with* $\operatorname{supp}\widetilde{\mu} \subseteq \overline{\Omega}$ *and positive numbers* $\{\mu_j\}_{j\in I}$ *such that*

$$\mu = \left|(-\Delta)^{s/2}u\right|^2 + \widetilde{\mu} + \sum_{j\in I} \delta_{x_j}\mu_j, \quad \mu_j > 0,$$

and

$$\nu_j \leqslant \left(S^{-1}\mu\left(\{x_j\}\right)\right)^{2_s^*/2},$$

where S *is the best Sobolev constant, i.e.*

$$S = \inf_{u\in H^s(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,\mathrm{d}x}{\int_{\mathbb{R}^N} |u|^{2_s^*}\,\mathrm{d}x},$$

$x_j \in \mathbb{R}^N$, δ_{x_j} *are Dirac measures at* x_j, *and* μ_j, ν_j *are constants.*

Remark 2. In the case $\Omega = \mathbb{R}^N$, the above principle of concentration-compactness does not provide any information about the possible loss of mass at infinity. The following result expresses this fact in quantitative terms.

Lemma 2. (*See* [31, 32].) *Let* $\{u_n\} \subset H^s(\mathbb{R}^N)$ *such that* $u_n \rightharpoonup u$ *weakly in* $H^s(\mathbb{R}^N)$, $|u_n|^{2_s^*} \rightharpoonup \nu$ *and* $|(-\Delta)^{s/2}u_n|^2 \rightharpoonup \mu$ *weakly-$*$ in* $\mathcal{M}(\mathbb{R}^N)$*, and define*

$$\text{(i)}\ \mu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int\limits_{\{x\in\mathbb{R}^N:\,|x|>R\}} \left|(-\Delta)^{s/2}u_n\right|^2 \mathrm{d}x,$$

$$\text{(ii)}\ \nu_\infty = \lim_{R\to\infty}\limsup_{n\to\infty}\int\limits_{\{x\in\mathbb{R}^N:\,|x|>R\}} |u_n|^{2_s^*} \mathrm{d}x.$$

The quantities ν_∞ *and* μ_∞ *exist and satisfy*

$$\text{(iii)}\ \limsup_{n\to\infty}\int\limits_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u_n\right|^2 \mathrm{d}x = \int\limits_{\mathbb{R}^N} \mathrm{d}\mu + \mu_\infty,$$

$$\text{(iv)}\ \limsup_{n\to\infty}\int\limits_{\mathbb{R}^N} |u_n|^{2_s^*} \mathrm{d}x = \int\limits_{\mathbb{R}^N} \mathrm{d}\nu + \nu_\infty,$$

$$\text{(v)}\ \nu_\infty \leqslant \left(S^{-1}\nu_\infty\right)^{2_s^*/2}.$$

We recall that a C^1 functional J on Banach space X is said to satisfy the Palais–Smale condition at level c ((PS$_c$) in short) if every sequence $\{u_n\} \subset X$ satisfying $\lim_{n\to\infty} J_\lambda(u_n) = c$ and $\lim_{n\to\infty} \|J'_\lambda(u_n)\|_{X^*} = 0$ has a convergent subsequence.

Lemma 3. *Suppose that* (V) *and* (H) *hold. Then any* (PS$_c$) *sequence* $\{u_n\}$ *is bounded in* E *and* $c \geqslant 0$.

Proof. Let $\{u_n\}$ be a sequence in E such that

$$\begin{aligned} c + o(1) &= J_\lambda(u_n) \\ &= \frac{1}{2}G\biggl(\int\limits_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u_n\right|^2 \mathrm{d}x\biggr) + \frac{1}{2}\int\limits_{\mathbb{R}^N} \lambda V(x)|u_n|^2 \mathrm{d}x \\ &\quad - \frac{\lambda}{2^*(s)}\int\limits_{\mathbb{R}^N} |u_n|^{2_s^*} \mathrm{d}x - \lambda\int\limits_{\mathbb{R}^N} H(x,u_n)\,\mathrm{d}x, \end{aligned} \tag{11}$$

$$\begin{aligned} \left\langle J'_\lambda(u_n), v\right\rangle &= g\biggl(\int\limits_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u_n\right|^2 \mathrm{d}x\biggr)\int\limits_{\mathbb{R}^N} (-\Delta)^{s/2}u_n\cdot(-\Delta)^{s/2}v\,\mathrm{d}x \\ &\quad + \lambda\int\limits_{\mathbb{R}^N} V(x)u_n v\,\mathrm{d}x - \lambda\int\limits_{\mathbb{R}^N} |u|^{2_s^*-2}u_n v\,\mathrm{d}x - \lambda\int\limits_{\mathbb{R}^N} h(x,u_n)v\,\mathrm{d}x \\ &= o(1)\|u_n\|. \end{aligned} \tag{12}$$

By (11), (12), and condition (h_3), we have

$$\begin{aligned}
&c+o(1)\|u_n\| \\
&\quad = J_\lambda(u_n) - \frac{1}{\mu}\langle J'_\lambda(u_n), u_n\rangle \\
&\quad = \frac{1}{2}G\biggl(\int\limits_{\mathbb{R}^N} \bigl|(-\Delta)^{s/2}u_n\bigr|^2\,\mathrm{d}x\biggr) - g\biggl(\int\limits_{\mathbb{R}^N} \bigl|(-\Delta)^{s/2}u_n\bigr|^2\,\mathrm{d}x\biggr)\int\limits_{\mathbb{R}^N} \bigl|(-\Delta)^{s/2}u_n\bigr|^2\,\mathrm{d}x \\
&\qquad + \biggl(\frac{1}{2}-\frac{1}{\mu}\biggr)\lambda\int\limits_{\mathbb{R}^N} V(x)|u_n|^2\,\mathrm{d}x + \biggl(\frac{1}{\mu}-\frac{1}{2_s^*}\biggr)\lambda\int\limits_{\mathbb{R}^N}|u_n|^{2_s^*}\,\mathrm{d}x \\
&\qquad + \lambda\int\limits_{\mathbb{R}^N}\biggl[\frac{1}{\mu}h(x,u_n)u_n - H(x,u_n)\biggr]\mathrm{d}x \\
&\quad \geqslant \biggl(\frac{\Sigma}{2}-\frac{1}{\mu}\biggr)\alpha_0\int\limits_{\mathbb{R}^N}\bigl|(-\Delta)^{s/2}u_n\bigr|^2\,\mathrm{d}x + \biggl(\frac{1}{2}-\frac{1}{\mu}\biggr)\lambda\int\limits_{\mathbb{R}^N} V(x)|u_n|^2\,\mathrm{d}x. \qquad (13)
\end{aligned}$$

Therefore, inequality (13) imply that $\{u_n\}$ is bounded in E. Taking the limit in (13), we show that $c \geqslant 0$. This completes the proof of Lemma 3. □

The main result in this section is the following compactness result.

Lemma 4. *Suppose that* (V), (G), *and* (H) *hold. For any* $\lambda \geqslant 1$, J_λ *satisfies* (PS_c) *condition for all* $c \in (0, \Sigma_0\lambda^{1-N/(2s)})$, *where* $\Sigma_0 := (1/\mu - 1/2_s^*)S^{N/(2s)}$, *that is any* (PS_c)*-sequence* $(u_n) \subset E$ *has a strongly convergent subsequence in* E.

Proof. Let $\{u_n\}$ be a (PS_c) sequence. By Lemma 3, $\{u_n\}$ is bounded in E. Hence, up to a subsequence, we may assume that

$$\begin{aligned}
u_n &\rightharpoonup u \quad \text{weakly in } E, \\
u_n &\to u \quad \text{a.e. in } \mathbb{R}^N, \\
u_n &\to u \quad \text{in } L^t_{\mathrm{loc}}(\mathbb{R}^N),\ 1 \leqslant t < 2_s^*.
\end{aligned}$$

Moreover, by Prokhorov's theorem (see [7, Thm. 8.6.2]) there exist $\mu, \nu \in \mathcal{M}(\mathbb{R}^N)$ such that

$$\begin{aligned}
\bigl|(-\Delta)^{s/2}u_n\bigr|^2 &\rightharpoonup \mu \quad (\text{weak*-sense of measures}), \\
|u_n|^{2_s^*} &\rightharpoonup \nu \quad (\text{weak*-sense of measures}),
\end{aligned}$$

where μ and ν are a nonnegative bounded measures on $\mathbb{R}^N$. It follows from Lemma 1 that $u_n \to u$ in $L^{2_s^*}_{\mathrm{loc}}(\mathbb{R}^N)$ or $\nu = |u|^{2_s^*} + \sum_{j\in I}\delta_{x_j}\nu_j$ as $n \to \infty$, where I is a countable set, $\{\nu_j\} \subset [0,\infty)$, $\{x_j\} \subset \mathbb{R}^N$.

Take $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leqslant \phi \leqslant 1$; $\phi \equiv 1$ in $B(x_j, \varepsilon)$, $\phi(x) = 0$ in $\mathbb{R}^N \setminus B(x_j, 2\varepsilon)$. For any $\varepsilon > 0$, define $\phi_\varepsilon = \phi((x - x_j)/\varepsilon)$, where $j \in I$. It follows that

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)\phi_\varepsilon(x) - u_n(y)\phi_\varepsilon(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\leqslant 2\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2\phi_\varepsilon^2(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + 2\iint\limits_{\mathbb{R}^{2N}} \frac{|\phi_\varepsilon(x) - \phi_\varepsilon(y)|^2|u_n(x)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\leqslant 2\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + 2\iint\limits_{\mathbb{R}^{2N}} \frac{|\phi_\varepsilon(x) - \phi_\varepsilon(y)|^2|u_n(x)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y. \qquad (14)
\end{aligned}$$

Similarly to the proof of Lemma 3.4 in [31], we have

$$\iint\limits_{\mathbb{R}^{2N}} \frac{|\phi_\varepsilon(x) - \phi_\varepsilon(y)|^2|u_n(x)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \leqslant C\varepsilon^{-2s} \int\limits_{B(x_i, K\varepsilon)} \left|u_n(x)\right|^2 \mathrm{d}x + CK^{-N}, \qquad (15)$$

where $K > 4$. As $\{u_n\}$ is bounded in $H^s(\mathbb{R}^N)$, it follows from (14) and (15) that $\{u_n\phi_\varepsilon\}$ is bounded in $H^s(\mathbb{R}^N)$. Then $\langle J_\lambda'(u_n), u_n\phi_\varepsilon\rangle \to 0$, which implies

$$\begin{aligned}
&g\biggl(\int\limits_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u_n\right|^2 \mathrm{d}x\biggr) \int\limits_{\mathbb{R}^N} (-\Delta)^{s/2}u_n \cdot (-\Delta)^{s/2}(u_n\phi_\varepsilon)\,\mathrm{d}x \\
&\quad= -\lambda\int\limits_{\mathbb{R}^N} V(x)u_n^2\phi_\varepsilon\,\mathrm{d}x + \lambda\int\limits_{\mathbb{R}^N} |u|^{2_s^*}\phi_\varepsilon\,\mathrm{d}x + \lambda\int\limits_{\mathbb{R}^N} h(x,u)\phi_\varepsilon\,\mathrm{d}x + o_n(1). \qquad (16)
\end{aligned}$$

Since

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} (-\Delta)^{s/2}u_n \cdot (-\Delta)^{s/2}(u_n\phi_\varepsilon)\,\mathrm{d}x \\
&\quad= \iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(u_n(x)\phi_\varepsilon(x) - u_n(y)\phi_\varepsilon(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))^2\phi_\varepsilon(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad+ \iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\phi_\varepsilon(x) - \phi_\varepsilon(y))u_n(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

it is easy to verify that

$$\iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))^2\phi_\varepsilon(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \to \int\limits_{\mathbb{R}^N} \phi_\varepsilon\,\mathrm{d}\mu$$

as $n \to \infty$ and

$$\int\limits_{\mathbb{R}^N} \phi_\varepsilon \,\mathrm{d}\mu \to \mu\bigl(\{x_i\}\bigr)$$

as $\varepsilon \to 0$. Note that the Hölder's inequality implies

$$\begin{aligned}
&\left| \iint\limits_{\mathbb{R}^{2N}} \frac{(u_n(x) - u_n(y))(\phi_\varepsilon(x) - \phi_\varepsilon(y))u_n(x)}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \right| \\
&\quad \leqslant \iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)| \cdot |\phi_\varepsilon(x) - \phi_\varepsilon(y)| \cdot |u_n(x)|}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leqslant C\biggl(\iint\limits_{\mathbb{R}^{2N}} \frac{|u_n(x)|^2 |\phi_\varepsilon(x) - \phi_\varepsilon(y)|^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \biggr)^{1/2}. \qquad (17)
\end{aligned}$$

Similarly to the proof of Lemma 3.4 in [31], we have

$$\lim_{\varepsilon \to 0} \lim_{n \to \infty} \iint\limits_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\phi_\varepsilon(x) - \phi_\varepsilon(y))^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y = 0. \qquad (18)$$

In the following, we just give a sketch of the proof for the reader's convenience.

On the one hand, we have

$$\begin{aligned}
\mathbb{R}^N \times \mathbb{R}^N &= \bigl(\bigl(\mathbb{R}^N \setminus B(x_i, 2\varepsilon)\bigr) \cup B(x_i, 2\varepsilon)\bigr) \times \bigl(\bigl(\mathbb{R}^N \setminus B(x_i, 2\varepsilon)\bigr) \cup B(x_i, 2\varepsilon)\bigr) \\
&= \bigl(\bigl(\mathbb{R}^N \setminus B(x_i, 2\varepsilon)\bigr) \times \bigl(\mathbb{R}^N \setminus B(x_i, 2\varepsilon)\bigr)\bigr) \cup \bigl(B(x_i, 2\varepsilon) \times \mathbb{R}^N\bigr) \\
&\quad \cup \bigl(\bigl(\mathbb{R}^N \setminus B(x_i, 2\varepsilon)\bigr) \times B(x_i, 2\varepsilon)\bigr).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\phi_\varepsilon(x) - \phi_\varepsilon(y))^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \iint\limits_{B(x_i,2\varepsilon) \times \mathbb{R}^N} \frac{u_n^2(x)(\phi_\varepsilon(x) - \phi_\varepsilon(y))^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \iint\limits_{(\mathbb{R}^N \setminus B(x_i,2\varepsilon)) \times B(x_i,2\varepsilon)} \frac{u_n^2(x)(\phi_\varepsilon(x) - \phi_\varepsilon(y))^2}{|x-y|^{N+2s}} \,\mathrm{d}x\,\mathrm{d}y \\
&\quad \leqslant C\varepsilon^{-2s} \int\limits_{B(x_i,K\varepsilon)} u_n^2(x) \,\mathrm{d}x + CK^{-N} \biggl(\int\limits_{\mathbb{R}^N \setminus B(x_i,K\varepsilon)} \bigl|u_n(x)\bigr|^{2_s^*} \,\mathrm{d}x \biggr)^{2/2_s^*} \\
&\quad \leqslant C\varepsilon^{-2s} \int\limits_{B(x_i,K\varepsilon)} u_n^2(x) \,\mathrm{d}x + CK^{-N}.
\end{aligned}$$

Note that $u_n \rightharpoonup u$ weakly in E. By Theorem 1, we obtain $u_n \to u$ in $L^t_{\mathrm{loc}}(\mathbb{R}^N)$, $1 \leqslant t < 2^*_s$, which implies

$$C\varepsilon^{-2s} \int\limits_{B(x_i,K\varepsilon)} u_n^2(x)\,\mathrm{d}x + CK^{-N} \to C\varepsilon^{-2s} \int\limits_{B(x_i,K\varepsilon)} u^2(x)\,\mathrm{d}x + CK^{-N}$$

as $n \to \infty$. Then

$$\begin{aligned}
&C\varepsilon^{-2s} \int\limits_{B(x_i,K\varepsilon)} u^2(x)\,\mathrm{d}x + CK^{-N} \\
&\quad \leqslant C\varepsilon^{-2s} \biggl(\int\limits_{B(x_i,K\varepsilon)} \bigl|u_n(x)\bigr|^{2^*_s}\,\mathrm{d}x \biggr)^{2/2^*_s} \biggl(\int\limits_{B(x_i,K\varepsilon)} \mathrm{d}x \biggr)^{1-2/2^*_s} + CK^{-N} \\
&\quad = CK^{2s} \biggl(\int\limits_{B(x_i,K\varepsilon)} \bigl|u_n(x)\bigr|^{2^*_s}\,\mathrm{d}x \biggr)^{2/2^*_s} + CK^{-N} \to CK^{-N}
\end{aligned}$$

as $\varepsilon \to 0$. Furthermore, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \iint\limits_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\phi_\varepsilon(x) - \phi_\varepsilon(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \lim_{K\to\infty} \limsup_{\varepsilon\to 0} \limsup_{n\to\infty} \iint\limits_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\phi_\varepsilon(x) - \phi_\varepsilon(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = 0.
\end{aligned}$$

It follows from the definition of ϕ_ε and $u_n \to u$ in $L^t_{\mathrm{loc}}(\mathbb{R}^N)$, $1 \leqslant t < 2^*_s$, that

$$\lim_{\varepsilon\to 0} \lim_{n\to\infty} \int\limits_{\mathbb{R}^N} h(x,u_n)u_n\phi_\varepsilon\,\mathrm{d}x = 0. \tag{19}$$

Since ϕ_ε has compact support, letting $n \to \infty$ in (16), we deduce from (17), (18), and (19) that

$$\alpha_0\mu\bigl(\{x_j\}\bigr) \leqslant \lambda\nu_j.$$

Combing this with Lemma 1, we obtain $\nu_j \geqslant \alpha_0\lambda^{-1}S\nu_j^{2/2^*_s}$. This result implies that

$$\text{either}\quad \text{(I)}\ \nu_j = 0 \quad \text{or} \quad \text{(II)}\ \nu_j \geqslant \bigl(\alpha_0\lambda^{-1}S\bigr)^{N/(2s)}.$$

To obtain the possible concentration of mass at infinity, similarly, we define a cut off function $\phi_R \in C_0^\infty(\mathbb{R}^N)$ such that $\phi_R(x) = 0$ on $|x| < R$ and $\phi_R(x) = 1$ on $|x| > R+1$. We could verify that $\{u_n\phi_R\}$ is bounded in E, hence, $\langle J'_\lambda(u_n), u_n\phi_R\rangle \to 0$ as $n \to \infty$, which implies

$$g\left(\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u_n\right|^2 \mathrm{d}x\right) \int_{\mathbb{R}^N} (-\Delta)^{s/2}u_n \cdot (-\Delta)^{s/2}(u_n\phi_R)\,\mathrm{d}x$$
$$= -\lambda \int_{\mathbb{R}^N} V(x)u_n^2\phi_R\,\mathrm{d}x + \lambda \int_{\mathbb{R}^N} |u|^{2_s^*}\phi_R\,\mathrm{d}x + \lambda \int_{\mathbb{R}^N} h(x,u)\phi_R\,\mathrm{d}x + o_n(1). \tag{20}$$

Notice that

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-\Delta)^{s/2}u_n \cdot (-\Delta)^{s/2}(u_n\phi_R)\,\mathrm{d}x \\
&\quad= \iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(u_n(x)\phi_R(x)-u_n(y)\phi_R(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))^2\phi_R(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad+ \iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\phi_R(x)-\phi_R(y))u_n(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

It is easy to verify that

$$\limsup_{R\to\infty}\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))^2\phi_R(y)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \mu_\infty$$

and

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^{2N}} \frac{(u_n(x)-u_n(y))(\phi_R(x)-\phi_R(y))u_n(x)}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\right| \\
&\quad\leqslant C\left(\iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)|\phi_R(x)-\phi_R(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\limsup_{R\to\infty}\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)(\phi_R(x)-\phi_R(y))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad= \limsup_{R\to\infty}\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)((1-\phi_R(x))-(1-\phi_R(y)))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Similarly to the proof of Lemma 3.4 in [31], we have

$$\limsup_{R\to\infty}\limsup_{n\to\infty} \iint_{\mathbb{R}^{2N}} \frac{u_n^2(x)((1-\phi_R(x))-(1-\phi_R(y)))^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = 0.$$

It follows from the definition of ϕ_ε that

$$\lim_{R\to\infty}\lim_{n\to\infty}\int\limits_{\mathbb{R}^N} h(x,u_n)u_n\phi_R\,\mathrm{d}x = 0.$$

Letting $R\to\infty$ in (20), we obtain

$$\alpha_0\mu_\infty \leqslant \lambda\nu_\infty.$$

By Lemma 2, we obtain $\nu_\infty \geqslant \alpha_0\lambda^{-1}S\nu_\infty^{2/2_s^*}$. This result implies that

$$\text{either} \quad \text{(III)}\ \nu_\infty = 0 \quad \text{or} \quad \text{(IV)}\ \nu_\infty \geqslant \left(\alpha_0\lambda^{-1}S\right)^{N/(2s)}.$$

Next, we claim that (II) and (IV) cannot occur. If case (IV) holds for some $j\in I$, then by using Lemma 2 and condition (h_3), we have that

$$\begin{aligned}
c &= \lim_{n\to\infty}\left(J_\lambda(u_n) - \frac{1}{\mu}\langle J_\lambda'(u_n),u_n\rangle\right)\\
&\geqslant \left(\frac{\Sigma}{2}-\frac{1}{\mu}\right)g\left(\int\limits_{\mathbb{R}^N}\left|(-\Delta)^{s/2}u_n\right|^2\mathrm{d}x\right)\int\limits_{\mathbb{R}^N}\left|(-\Delta)^{s/2}u_n\right|^2\mathrm{d}x\\
&\quad + \left(\frac{1}{2}-\frac{1}{\mu}\right)\lambda\int\limits_{\mathbb{R}^N}V(x)|u_n|^2\,\mathrm{d}x + \left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)\lambda\int\limits_{\mathbb{R}^N}|u_n|^{2_s^*}\,\mathrm{d}x\\
&\quad + \lambda\int\limits_{\mathbb{R}^N}\left[\frac{1}{\mu}h(x,u_n)u_n - H(x,u_n)\right]\mathrm{d}x\\
&\geqslant \left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)\lambda\int\limits_{\mathbb{R}^N}|u_n|^{2_s^*}\,\mathrm{d}x \geqslant \left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)\lambda\int\limits_{\mathbb{R}^N}|u|^{2_s^*}\phi_R\,\mathrm{d}x\\
&= \left(\frac{1}{\mu}-\frac{1}{2_s^*}\right)\lambda\nu_\infty \geqslant \Sigma_0\lambda^{1-N/(2s)},
\end{aligned}$$

where $\Sigma_0 = (1/\mu – 1/2_s^*)S^{N/(2s)}$. This is impossible. Consequently, $\nu_j = 0$ for all $j\in J$. Similarly, we can prove that (II) cannot occur for each j. Thus,

$$\int\limits_{\mathbb{R}^N}|u_n|^{2_s^*}\,\mathrm{d}x \to \int\limits_{\mathbb{R}^N}|u|^{2_s^*}\,\mathrm{d}x. \tag{21}$$

As $|u_n-u|^{2_s^*} \leqslant 2^{2_s^*}(|u_n|^{2_s^*}+|u|^{2_s^*})$, it follows from the Fatou lemma that

$$\begin{aligned}
\int\limits_{\mathbb{R}^N}2^{2_s^*+1}|u|^{2^{2_s^*}}\,\mathrm{d}x &= \int\limits_{\mathbb{R}^N}\liminf_{n\to\infty}\left(2^{2_s^*}|u_n|^{2^{2_s^*}} + 2^{2_s^*}|u|^{2^{2_s^*}} - |u_n-u|^{2^{2_s^*}}\right)\mathrm{d}x\\
&\leqslant \liminf_{n\to\infty}\int\limits_{\mathbb{R}^N}\left(2^{2_s^*}|u_n|^{2^{2_s^*}} + 2^{2_s^*}|u|^{2^{2_s^*}} - |u_n-u|^{2^{2_s^*}}\right)\mathrm{d}x\\
&= \int\limits_{\mathbb{R}^N}2^{2_s^*+1}|u|^{2^{2_s^*}}\,\mathrm{d}x - \limsup_{n\to\infty}\int\limits_{\mathbb{R}^N}|u_n-u|^{2^{2_s^*}}\,\mathrm{d}x,
\end{aligned}$$

which implies that $\limsup_{n\to\infty}\int_{\mathbb{R}^N}|u_n-u|^{2_s^*}\,\mathrm{d}x=0$. Then

$$u_n\to u\quad\text{in } L^{2_s^*}(\mathbb{R}^N) \text{ as } n\to\infty.$$

Thus, from the weak lower semicontinuity of the norm, conditon (g_1), and Brezis–Lieb lemma [10] we have

$$\begin{aligned}
o(1)\|u_n\| &= \langle J'_\lambda(u_n),u_n\rangle\\
&= g\biggl(\int_{\mathbb{R}^N}\bigl|(-\Delta)^{s/2}u_n\bigr|^2\,\mathrm{d}x\biggr)\int_{\mathbb{R}^N}\bigl|(-\Delta)^{s/2}u_n\bigr|^2\,\mathrm{d}x\\
&\quad+\lambda\int_{\mathbb{R}^N}V(x)|u_n|^2\,\mathrm{d}x-\lambda\int_{\mathbb{R}^N}|u_n|^{2^*(s)}\,\mathrm{d}x-\lambda\int_{\mathbb{R}^N}h(x,u_n)u_n\,\mathrm{d}x\\
&\geqslant \alpha_0\int_{\mathbb{R}^N}\bigl(\bigl|(-\Delta)^{s/2}u_n\bigr|^2-\bigl|(-\Delta)^{s/2}u\bigr|^2\bigr)\,\mathrm{d}x+\lambda\int_{\mathbb{R}^N}V(x)\bigl(|u_n|^2-|u|^2\bigr)\,\mathrm{d}x\\
&\quad+g\biggl(\int_{\mathbb{R}^N}\bigl|(-\Delta)^{s/2}u\bigr|^2\,\mathrm{d}x\biggr)\int_{\mathbb{R}^N}\bigl|(-\Delta)^{s/2}u\bigr|^2\,\mathrm{d}x+\lambda\int_{\mathbb{R}^N}V(x)|u|^2\,\mathrm{d}x\\
&\quad-\lambda\int_{\mathbb{R}^N}|u|^{2^*(s)}\,\mathrm{d}x-\lambda\int_{\mathbb{R}^N}h(x,u)u\,\mathrm{d}x\\
&= \min\{\alpha_0,1\}\|u_n-u\|_\lambda^2+o(1)\|u\|_\lambda.
\end{aligned}$$

Here we use $J'_\lambda(u)=0$. Thus, we prove that $\{u_n\}$ strongly converges to u in E. This completes the proof of Lemma 4. □

5 Proof of Theorem 3

In the following, we always consider $\lambda\geqslant 1$. By assumptions (V), (G), and (H), one can see that $J_\lambda(u)$ has mountain pass geometry.

Lemma 5. *Assume* (V), (G), *and* (H) *hold. There exist* $\alpha_\lambda,\rho_\lambda>0$ *such that* $J_\lambda(u)>0$ *if* $u\in B_{\rho_\lambda}\setminus\{0\}$ *and* $J_\lambda(u)\geqslant\alpha_\lambda$ *if* $u\in\partial B_{\rho_\lambda}$, *where* $B_{\rho_\lambda}=\{u\in E\colon \|u\|_\lambda\leqslant\rho_\lambda\}$.

Proof. By (h_1)–(h_3), for $\delta\leqslant(2\min\{\Sigma\alpha_0/2,1/2\}\lambda\mu_2^2)^{-1}$, there is $C_\delta>0$ such that

$$\frac{1}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}\,\mathrm{d}x+\int_{\mathbb{R}^N}H(x,u)\,\mathrm{d}x\leqslant\delta|u|_2^2+C_\delta|u|_{2_s^*}^{2_s^*},$$

where c_s is the embedding constant of (8). It follows that

$$\begin{aligned}
J_\lambda(u) &:= \frac12 G\biggl(\int_{\mathbb{R}^N}\bigl|(-\Delta)^{s/2}u\bigr|^2\,\mathrm{d}x\biggr)+\frac12\int_{\mathbb{R}^N}\lambda V(x)|u|^2\,\mathrm{d}x\\
&\quad-\frac{\lambda}{2_s^*}\int_{\mathbb{R}^N}|u|^{2_s^*}\,\mathrm{d}x-\lambda\int_{\mathbb{R}^N}H(x,u)\,\mathrm{d}x
\end{aligned}$$

$$\geqslant \min\left\{\frac{\Sigma\alpha_0}{2}, \frac{1}{2}\right\}\|u\|_\lambda^2 - \lambda\delta|u|_2^2 - \lambda C_\delta|u|_{2_s^*}^{2_s^*}$$
$$\geqslant \frac{1}{2}\min\left\{\frac{\Sigma\alpha_0}{2}, \frac{1}{2}\right\}\|u\|_\lambda^2 - \lambda C_\delta|u|_{2_s^*}^{2_s^*}$$
$$\geqslant \frac{1}{2}\min\left\{\frac{\Sigma\alpha_0}{2}, \frac{1}{2}\right\}\|u\|_\lambda^2 - \lambda C_\delta\mu_{2_s^*}^{2_s^*}\|u\|_\lambda^{2_s^*}.$$

Since $2_s^* > 2$, we know that the conclusion of Lemma 5 holds. This completes the proof of Lemma 5. □

Lemma 6. *Under the assumption of Lemma* 5*, for any finite dimensional subspace* $F \subset E$*,*

$$J_\lambda(u) \to -\infty \quad \textit{as } u \in F,\ \|u\|_\lambda \to \infty.$$

Proof. On the one hand, by integrating (g_2), we obtain

$$G(t) \leqslant \frac{G(t_0)}{t_0^{1/\Sigma}} t^{1/\Sigma} = C_0 t^{1/\Sigma} \tag{22}$$

for all $t \geqslant t_0 > 0$. Using conditions (V) and (h_1)–(h_3), we can get

$$J_\lambda(u) \leqslant \frac{C_0}{2}\|u\|_\lambda^{2/\Sigma} + \frac{1}{2}\|u\|_\lambda^2 - \frac{\lambda}{p^*}|u|_{2_s^*}^{2_s^*} - \lambda l_0|u|_r^r$$

for all $u \in F$. Since all norms in a finite-dimensional space are equivalent and $2/\Sigma < 2_s^*$, $2 < 2_s^*$. This completes the proof of Lemma 6. □

Since $J_\lambda(u)$ does not satisfy condition (PS_c) for all $c > 0$, in the following, we will find a special finite-dimensional subspaces by which we construct sufficiently small minimax levels.

Recall that assumption (V) implies that there exists $x_0 \in \mathbb{R}^N$ such that $V(x_0) = \min_{x\in\mathbb{R}^N} V(x) = 0$. Without loss of generality, we assume from now on that $x_0 = 0$.

Observe that, by (h_3),

$$\frac{\lambda}{2_s^*}\int\limits_{\mathbb{R}^N} K(x)|u|^{2_s^*}\,\mathrm{d}x\lambda\int\limits_{\mathbb{R}^N} H(x,u)\,\mathrm{d}x \geqslant l_0\lambda\int\limits_{\mathbb{R}^N} |u|^r\,\mathrm{d}x.$$

Define the function $I_\lambda \in C^1(E,\mathbb{R})$ by

$$I_\lambda(u) := \frac{1}{2}G\left(\int\limits_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u\right|^2\mathrm{d}x\right) + \frac{1}{2}\int\limits_{\mathbb{R}^N} \lambda V(x)|u|^2\,\mathrm{d}x - l_0\lambda\int\limits_{\mathbb{R}^N} |u|^r\,\mathrm{d}x.$$

Then $J_\lambda(u) \leqslant I_\lambda(u)$ for all $u \in E$, and it suffices to construct small minimax levels for I_λ.

Note that

$$\inf\left\{ \int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}\phi\right|^2 \mathrm{d}x\colon\ \phi \in C_0^\infty\left(\mathbb{R}^N\right),\ |\phi|_r = 1 \right\} = 0.$$

For any $1 > \zeta > 0$, one can choose $\phi_\zeta \in C_0^\infty(\mathbb{R}^N)$ with $|\phi_\zeta|_r = 1$ and $\operatorname{supp}\phi_\zeta \subset B_{r_\zeta}(0)$ so that $|(-\Delta)^{s/2}\phi_\zeta|_2^2 < \zeta$. Set

$$f_\lambda = \phi_\zeta(\lambda^{1/(2s)}x), \tag{23}$$

then

$$\operatorname{supp} f_\lambda \subset B_{\lambda^{-1/(2s)}r_\zeta}(0).$$

Observe that

$$\begin{aligned}
I_\lambda(tf_\lambda) &\leqslant \frac{C_0}{2}t^{2/\Sigma}\left(\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}f_\lambda\right|^2 \mathrm{d}x\right)^{1/\Sigma} + \frac{t^2}{2}\int_{\mathbb{R}^N} \lambda V(x)|f_\lambda|^2\,\mathrm{d}x \\
&\quad - t^r l_0 \lambda \int_{\mathbb{R}^N} |f_\lambda|^r\,\mathrm{d}x \\
&= \lambda^{1-N/(2s)}\left[\frac{C_0}{2}t^{2/\Sigma}\left(\lambda^{1-N/(2s)}\right)^{1/\Sigma-1}\left(\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}\phi_\zeta\right|^2 \mathrm{d}x\right)^{1/\Sigma}\right. \\
&\quad \left. + \frac{t^2}{2}\int_{\mathbb{R}^N} V\left(\lambda^{-1/(2s)}x\right)|\phi_\delta|^p\,\mathrm{d}x - t^r l_0 \int_{\mathbb{R}^N} |\phi_\delta|^r\,\mathrm{d}x\right] \\
&\leqslant \lambda^{1-N/(2s)}\left[\frac{C_0}{2}t^{2/\Sigma}\left(\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}\phi_\zeta\right|^2 \mathrm{d}x\right)^{1/\Sigma}\right. \\
&\quad \left. + \frac{t^2}{2}\int_{\mathbb{R}^N} V\left(\lambda^{-1/(2s)}x\right)|\phi_\delta|^2\,\mathrm{d}x - t^r l_0 \int_{\mathbb{R}^N} |\phi_\delta|^r\,\mathrm{d}x\right] \\
&= \lambda^{1-N/(2s)}\Psi_\lambda(t\phi_\zeta),
\end{aligned}$$

where $\Psi_\lambda \in C^1(E,\mathbb{R})$ is defined by

$$\begin{aligned}
\Psi_\lambda(u) := \frac{C_0}{2}\left(\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2}u\right|^2 \mathrm{d}x\right)^{1/\Sigma} \\
+ \frac{1}{2}\int_{\mathbb{R}^N} V\left(\lambda^{-1/(2s)}x\right)|u|^2\,\mathrm{d}x - l_0 \int_{\mathbb{R}^N} |u|^r\,\mathrm{d}x.
\end{aligned}$$

Since $s > 2/\Sigma$, there exists finite number $t_0 \in [0, +\infty)$ such that

$$\begin{aligned}\max_{t\geqslant 0} \Psi_\lambda(t\phi_\zeta) &= \frac{C_0}{2} t_0^{2/\Sigma} \biggl(\int\limits_{\mathbb{R}^N} \bigl|(-\Delta)^{s/2}\phi_\zeta\bigr|^2 \,\mathrm{d}x \biggr)^{1/\Sigma} + \frac{t_0^2}{2} \int\limits_{\mathbb{R}^N} V\bigl(\lambda^{-1/(2s)}x\bigr)|\phi_\zeta|^2 \,\mathrm{d}x \\ &\quad - t_0^r l_0 \int\limits_{\mathbb{R}^N} |\phi_\zeta|^r \,\mathrm{d}x \\ &\leqslant \frac{C_0}{2} t_0^{2/\Sigma} \biggl(\int\limits_{\mathbb{R}^N} \bigl|(-\Delta)^{s/2}\phi_\zeta\bigr|^2 \,\mathrm{d}x \biggr)^{1/\Sigma} + \frac{t_0^2}{2} \int\limits_{\mathbb{R}^N} V\bigl(\lambda^{-1/(2s)}x\bigr)|\phi_\zeta|^2 \,\mathrm{d}x.\end{aligned}$$

On the one hand, since $V(0) = 0$ and note that $\operatorname{supp}\phi_\zeta \subset B_{r_\zeta}(0)$, there is $\Lambda_\zeta > 0$ such that

$$V\bigl(\lambda^{-1/(2s)}x\bigr) \leqslant \frac{\zeta}{|\phi_\delta|_2^2} \quad \text{for all } |x| \leqslant r_\zeta \text{ and } \lambda \geqslant \Lambda_\zeta.$$

This implies that

$$\max_{t\geqslant 0} \Psi_\lambda(t\phi_\delta) \leqslant \frac{C_0}{2} t_0^{2/\Sigma} \zeta^{1/\Sigma} + \frac{t_0^2}{2}\zeta \leqslant T^*\zeta, \tag{24}$$

where $T^* := (C_0 t_0^{2/\Sigma} + t_0^2)/2$. Therefore, for all $\lambda \geqslant \Lambda_\zeta$,

$$\max_{t\geqslant 0} J_\lambda(t\phi_\delta) \leqslant T^*\zeta\lambda^{1-N/(2s)}. \tag{25}$$

Thus, we have the following lemma.

Lemma 7. *Under the assumption of Lemma* 5*, for any* $\kappa > 0$*, there exists* $\Lambda_\kappa > 0$ *such that for each* $\lambda \geqslant \Lambda_\kappa$*, there is* $\widehat{f_\lambda} \in E$ *with* $\|\widehat{f_\lambda}\| > \rho_\lambda$*,* $J_\lambda(\widehat{f_\lambda}) \leqslant 0$*, and*

$$\max_{t\in[0,1]} J_\lambda(t\widehat{f_\lambda}) \leqslant \kappa\lambda^{1-N/(2s)}. \tag{26}$$

Proof. Choose $\zeta > 0$ so small that $T^*\zeta \leqslant \kappa$. Let $f_\lambda \in E$ be the function defined by (23). Taking $\Lambda_\kappa = \Lambda_\delta$. Let $\widehat{t_\lambda} > 0$ be such that $\widehat{t_\lambda}\|f_\lambda\|_\lambda > \rho_\lambda$ and $J_\lambda(tf_\lambda) \leqslant 0$ for all $t \geqslant \widehat{t_\lambda}$. Let $\widehat{f_\lambda} = \widehat{t_\lambda} f_\lambda$, the conclusion of Lemma 7 holds by (25). □

For any $m^* \in \mathbb{N}$, one can choose m^* functions $\phi_\zeta^i \in C_0^\infty(\mathbb{R}^N)$ such that $\operatorname{supp}\phi_\zeta^i \cap \operatorname{supp}\phi_\zeta^k = \emptyset$, $i \neq k$, $|\phi_\zeta^i|_s = 1$, and $|(-\Delta)^{s/2}\phi_\zeta^i|_2^2 < \zeta$. Let $r_\zeta^{m^*} > 0$ be such that $\operatorname{supp}\phi_\zeta^i \subset B_{r_\zeta}^i(0)$ for $i = 1, 2, \dots, m^*$. Set

$$f_\lambda^i(x) = \phi_\zeta^i\bigl(\lambda^{1/(2s)}x\bigr) \quad \text{for } i = 1, 2, \dots, m^* \tag{27}$$

and

$$H_{\lambda\zeta}^{m^*} = \operatorname{span}\bigl\{f_\lambda^1, f_\lambda^2, \dots, f_\lambda^{m^*}\bigr\}.$$

Observe that for each $u = \sum_{i=1}^{m^*} c_i f_\lambda^i \in H_{\lambda\zeta}^{m^*}$, we have

$$\int_{\mathbb{R}^N} \left|(-\Delta)^{s/2} u\right|^2 \mathrm{d}x \leqslant C \sum_{i=1}^{m^*} |c_i|^2 \int_{\mathbb{R}^N} \left|(-\Delta)^{s/2} f_\lambda^i\right|^2 \mathrm{d}x$$

for some constant $C > 0$,

$$\int_{\mathbb{R}^N} V(x)|u|^2 \,\mathrm{d}x = \sum_{i=1}^{m^*} |c_i|^2 \int_{\mathbb{R}^N} V(x)\left|f_\lambda^i\right|^2 \mathrm{d}x,$$

and

$$\begin{aligned} &\frac{1}{2^*(s)} \int_{\mathbb{R}^N} |u|^{2^*(s)} \,\mathrm{d}x + \int_{\mathbb{R}^N} H(x,u) \,\mathrm{d}x \\ &\quad = \sum_{i=1}^{m^*} \left(\frac{1}{2^*(s)} \int_{\mathbb{R}^N} \left|c_i f_\lambda^i\right|^{2^*(s)} \mathrm{d}x + \int_{\mathbb{R}^N} H\left(x, c_i f_\lambda^i\right) \mathrm{d}x \right). \end{aligned}$$

Therefore,

$$J_\lambda(u) \leqslant C \sum_{i=1}^{m^*} J_\lambda\left(c_i f_\lambda^i\right)$$

for some constant $C > 0$. By a similar argument as the one before, we know that

$$J_\lambda\left(c_i f_\lambda^i\right) \leqslant \lambda^{1-N/(2s)} \Psi\left(|c_i| f_\lambda^i\right).$$

Set

$$\beta_\zeta := \max\left\{\left|\phi_\zeta^i\right|_2^2 \colon j = 1, 2, \ldots, m^*\right\}$$

and choose $\Lambda_{m^*\delta} > 0$ so that

$$V\left(\lambda^{-1/(2s)} x\right) \leqslant \frac{\zeta}{\beta_\zeta} \quad \text{for all } |x| \leqslant r_\zeta^{m^*} \text{ and } \lambda \geqslant \Lambda_{m^*\zeta}.$$

As before, we can obtain the following:

$$\max_{u \in H_{\lambda\delta}^{m^*}} J_\lambda(u) \leqslant C m^* T^* \zeta \lambda^{1-N/(2s)} \tag{28}$$

for all $\lambda \geqslant \Lambda_{m^*\zeta}$ and some constant $C > 0$.

Using this estimate we have the following.

Lemma 8. *Under the assumptions of Lemma* 5*, for any* $m^* \in \mathbb{N}$ *and* $\kappa > 0$*, there exists* $\Lambda_{m^*\kappa} > 0$ *such that for each* $\lambda \geqslant \Lambda_{m^*\kappa}$*, there exists an* m^**-dimensional subspace* $F_{\lambda m^*}$ *satisfying*

$$\max_{u \in F_{\lambda m^*}} J_\lambda(u) \leqslant \kappa \lambda^{1-N/(2s)}.$$

Proof. Choose $\zeta > 0$ so small that $Cm^*T^*\zeta \leqslant \kappa$. Taking $F_{\lambda m^*} = H_{\lambda\zeta}^{m^*} = \operatorname{span}\{f_\lambda^1, f_\lambda^2, \dots, f_\lambda^{m^*}\}$, where $f_\lambda^i(x) = \phi_\delta^i(\lambda^{1/(2s)}x)$ for $i = 1, 2, \dots, m^*$ are given by (27). From (28) we know that the conclusion of Lemma 8 holds. □

We now establish the existence and multiplicity results.

Proof of Theorem 3. (i) For any $0 < \kappa < \Sigma_0$, by Lemma 4, we choose $\Lambda_\Sigma > 0$ and, for $\lambda \geqslant \Lambda_\Sigma$, define the minimax value

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} J_\lambda(t\widehat{f}_\lambda),$$

where

$$\Gamma_\lambda := \bigl\{\gamma \in C\bigl([0,1], E\bigr) \colon \gamma(0) = 0 \text{ and } \gamma(1) = \widehat{f}_\lambda\bigr\}.$$

By Lemma 5, we have $\alpha_\lambda \leqslant c_\lambda \leqslant \kappa\lambda^{1-N/(2s)}$. In virtue of Lemma 4, we get that $(\mathrm{PS}_{c_\lambda})$ condition holds for J_λ at c_λ. Thus, there is u_λ such that $J_\lambda'(u_\lambda) = 0$ and $J_\lambda(u_\lambda) = c_\lambda$. Then u_λ is a nontrivial solution of (7). Moreover, it is well known that a mountain pass solution is a state solution of (7).

Because u_λ is a critical point of J_λ, for $\rho \in [2, 2_s^*]$,

$$\begin{aligned}
\kappa\lambda^{1-N/(2s)} &\geqslant J_\lambda(u_\lambda) = J_\lambda(u_\lambda) - \frac{1}{\rho}J_\lambda'(u_\lambda)u_\lambda \\
&= \frac{1}{2}G\biggl(\int_{\mathbb{R}^N} \bigl|(-\Delta)^{s/2}u_\lambda\bigr|^2\,\mathrm{d}x\biggr) \\
&\quad - \frac{1}{\rho}g\biggl(\int_{\mathbb{R}^N} \bigl|(-\Delta)^{s/2}u_\lambda\bigr|^2\,\mathrm{d}x\biggr)\int_{\mathbb{R}^N} \bigl|(-\Delta)^{s/2}u_\lambda\bigr|^2\,\mathrm{d}x \\
&\quad + \biggl(\frac{1}{2} - \frac{1}{\rho}\biggr)\lambda\int_{\mathbb{R}^N} V(x)|u_\lambda|^2\,\mathrm{d}x + \biggl(\frac{1}{\rho} - \frac{1}{2_s^*}\biggr)\lambda\int_{\mathbb{R}^N} |u_\lambda|^{2_s^*}\,\mathrm{d}x \\
&\quad + \lambda\int_{\mathbb{R}^N} \biggl[\frac{1}{\rho}h(x,u_\lambda)u_\lambda - H(x,u_\lambda)\biggr]\,\mathrm{d}x \\
&\geqslant \biggl(\frac{\Sigma}{2} - \frac{1}{\rho}\biggr)\alpha_0\int_{\mathbb{R}^N} \bigl|(-\Delta)^{s/2}u_\lambda\bigr|^2\,\mathrm{d}x + \biggl(\frac{1}{2} - \frac{1}{\rho}\biggr)\lambda\int_{\mathbb{R}^N} V(x)|u_\lambda|^2\,\mathrm{d}x \\
&\quad + \biggl(\frac{1}{\rho} - \frac{1}{2_s^*}\biggr)\lambda\int_{\mathbb{R}^N} |u_\lambda|^{2_s^*}\,\mathrm{d}x + \biggl(\frac{\mu}{\rho} - 1\biggr)\lambda\int_{\mathbb{R}^N} H(x,u_\lambda)\,\mathrm{d}x.
\end{aligned} \tag{29}$$

Taking $\rho = 2/\Sigma$, we obtain estimate (9), and taking $\rho = \mu$, we obtain estimate (10). This completes the proof of Theorem 3(i).

(ii) Denote the set of all symmetric (in the sense that $-Z = Z$) and closed subsets of E by Σ for each $Z \in \Sigma$. Let $\operatorname{gen}(Z)$ be the Krasnoselkski genus and

$$j(Z) := \min_{\iota \in \Gamma_{m^*}} \operatorname{gen}\bigl(\iota(Z) \cap \partial B_{\rho_\lambda}\bigr),$$

where Γ_{m^*} is the set of all odd homeomorphisms $\iota \in C(E, E)$, and ρ_λ is the number from Lemma 5. Then j is a version of Benci's pseudoindex [5]. Let

$$c_{\lambda i} := \inf_{j(Z) \geqslant i} \sup_{u \in Z} J_\lambda(u), \quad 1 \leqslant i \leqslant m^*.$$

Since $J_\lambda(u) \geqslant \alpha_\lambda$ for all $u \in \partial B^+_{\rho\lambda}$ and since $j(F_{\lambda m^*}) = \dim F_{\lambda m^*} = m^*$,

$$\alpha_\lambda \leqslant c_{\lambda 1} \leqslant \cdots \leqslant c_{\lambda m^*} \leqslant \sup_{u \in H_{\lambda m^*}} J_\lambda(u) \leqslant \kappa \lambda^{1-N/(2s)}.$$

It follows from Lemma 4 that J_λ satisfies (PS$_{c_\lambda}$) condition at all levels $c < \Sigma_0 \lambda^{1-N/2s}$. By the usual critical point theory, all $c_{\lambda i}$ are critical levels, and J_λ has at least m^* pairs of nontrivial critical points satisfying

$$\alpha_\lambda \leqslant J_\lambda(u_\lambda) \leqslant \kappa \lambda^{1-N/(2s)}.$$

Hence, problem (7) has at least m^* pairs of solutions. In the end, as in the proof of Theorem 3(i), we see that these solutions satisfy estimates (9) and (10). □

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