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Common fixed point of a power graphic (F, ψ)-contraction pair on partial b-metric spaces with application

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Abstract. The aim of this paper is to inaugurate power graphic (F, ψ)-contraction pair and to establish fixed point results for such mappings defined on partial b-metric spaces endowed with a graph. It is mentioning that, first time, we launch a class of fixed point results in the frame of partial b-metric spaces involving a graph. Results of this paper extend and generalize known results from metric, partial metric, and partial b-metric spaces in partial b-metric spaces with a graph. Further, appropriate examples are presented to emphasize the utility of the obtained results. At the end, an attempt to correlate the given work with application is turned out as solution for an integral equation.

Keywords: partial b-metric space, directed graph, common fixed point, power graphic (F, ψ)-contraction pair.

1 Introduction

In 2014, Shukla [11] introduced the notion of partial b-metric space and extended the famous Banach contraction principle in the setting of partial b-metric space. The work of Shukla has been extended by many authors; (see [8, 9]). Recently, a very interesting generalization of Banach contraction principle was obtained by Jachymski and Jóźwik [7].

They introduced Banach G-contractions. Here G stands for a directed graph in a metric space whose vertex set coincidences with the metric space. Also, Gu and He [6] presented some common fixed point results for self-maps with twice power type Φ-contractive condition. In recent years, Abbas et al. [2] carried out some common fixed point theorems for a power graphic contraction pair in partial metric spaces equipped with a graph.

On the other hand, in 2012, Wardowski [14] inaugurated the notion of F-contraction. This kind of contractions generalizes the Banach contraction. Newly, Piri and Kumam [10] enhanced the results of Wardowski [14] by launching the concept of an F-Suzuki contraction and obtained some curious fixed point results. (See some recent results about these direction in [5].)

In the current paper, our aim is to introduce the concept of power graphic (F, ψ)-contraction pair defined on partial b-metric space involving a directed graph and set up some common fixed point results regarding such contractions. Some related results are also derived besides furnishing illustrative examples. Finally, we utilize our results to prove the existences of solution of integral equation.

2 Basic facts and definitions

In this section, we list some basic definitions and fundamental results that are useful tool in subsequent analysis.

Definition 1. (See [11].) Let X be a nonempty set, and $s \geqslant 1$ be a given real number. A function $p_b : X \times X \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in X$, the following conditions are satisfied:

(p_{b1}) $x = y$ iff $p_b(x, x) = p_b(x, y) = p_b(y, y)$;
(p_{b2}) $p_b(x, x) \leqslant p_b(x, y)$;
(p_{b3}) $p_b(x, y) = p_b(y, x)$.

The pair (X, p_b) is called a partial b-metric space. The number $s \geqslant 1$ is called the coefficient of (X, p_b).

In the following definition, Mustafa et al. [9] modified Definition 1 in order to find that each partial b-metric p_b generates a b-metric d_{p_b}.

Definition 2. (See [9].) Let X be a nonempty set and $s \geqslant 1$ be a given real number. A function $p_b : X \times X \to [0, \infty)$ is called a partial b-metric if for all $x, y, z \in X$, the following conditions are satisfied:

(p_{b1}) $x = y$ iff $p_b(x, x) = p_b(x, y) = p_b(y, y)$;
(p_{b2}) $p_b(x, x) \leqslant p_b(x, y)$;
(p_{b3}) $p_b(x, y) = p_b(y, x)$;
(p_{b4}) $p_b(x, y) \leqslant s(p_b(x, z) + p_b(z, y) - p_b(z, z)) + (1 - s)(p_b(x, x) + p_b(y, y))/2$.

The pair (X, p_b) is called a partial b-metric space. The number $s \geqslant 1$ is called the coefficient of (X, p_b).

Proposition 1. (*See* [9].) *Every partial b-metric p_b defines a b-metric d_{p_b}, where*

$$d_{p_b}(x,y) = 2p_b(x,y) - p_b(x,x) - p_b(y,y)$$

for all $x, y \in X$.

Example 1. (See [11].) Let $X = \mathbb{R}^+$, $q > 1$ be a constant, and $p_b : X \times X \to \mathbb{R}^+$ be defined by

$$p_b(x,y) = \left[\max\{x,y\}\right]^q + |x-y|^q$$

for all $x, y \in X$. Then (X, p_b) is a partial b-metric space with the coefficient $s = 2^{q-1} > 1$, but it is neither a b-metric nor a partial metric space.

Remark 1. The class of partial b-metric space (X, p_b) is effectively larger than the class of partial metric space since a partial metric space is a special case of a partial b-metric space (X, p_b) when $s = 1$. Also, the class of partial b-metric space (X, p_b) is effectively larger than the class of b-metric space since a b-metric space is a special case of a partial b-metric space (X, p_b) when the self distance $p(x,x) = 0$.

Definition 3. (See [9].) A sequence $\{x_n\}$ in a partial b-metric space (X, p_b) is said to be:

(a) p_b convergent to a point $x \in X$ if $p_b(x,x) = \lim_{n\to\infty} p_b(x, x_n)$;
(b) a p_b-Cauchy sequence if $\lim_{n,m\to\infty} p_b(x_n, x_m)$ exists (and is finite);
(c) a partial b-metric space (X, p_b) is said to be p_b-complete if every p_b-Cauchy sequence $\{x_n\}$ in X p_b-converges to a point $x \in X$ such that

$$p_b(x,x) = \lim_{n,m\to\infty} p_b(x_n, x_m) = \lim_{n\to\infty} p_b(x, x_n).$$

Lemma 1. (*See* [6].) *Let (X, p_b) be a partial b-metric space. Then*:

(i) *a sequence $\{x_n\}$ is a p_b-Cauchy sequence in (X, p_b) if and only if it is a b-Cauchy sequence in the b-metric space (X, d_{p_b});*
(ii) *(X, p_b) is p_b-complete if and only if the b-metric space (X, d_{p_b}) is complete, moreover,* $\lim_{n\to\infty} d_{p_b}(x_n, x) = 0$ *if and only if*

$$p_b(x,x) = \lim_{n\to\infty} p_b(x_n, x) = \lim_{n,m\to\infty} p_b(x_n, x_m).$$

Definition 4. (See [14].) Let $F : \mathbb{R}^+ \to \mathbb{R}$ be a mapping satisfying:

(F_1) F is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}^+$, $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;
(F_2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F_3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

We denote the set of all functions satisfying (F1)–(F3) by $\mathcal{F}$.

Definition 5. (See [10].) Let us denote by Δ_F the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

(Δ_{F1}) F is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}^+$, $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;
(Δ_{F2}) there is a sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive real numbers such that $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(Δ_{F3}) F is continuous on $(0, \infty)$.

Definition 6. Let Δ_D be the set of all continuous functions $D(t_1, t_2, t_3, t_4) : {\mathbb{R}^+}^4 \to \mathbb{R}^+$ satisfying: for all $t_1, t_2, t_3, t_4 \in \mathbb{R}^+$, if $t_i = t_j$ for $i, j = 1, 2, 3, 4$, where $i \neq j$, then there exists $\tau > 0$ such that $D(t_1, t_2, t_3, t_4) = \tau$.

Take $\Psi = \{\psi : [0,\infty) \to [0,\infty)$: ψ is upper semi continuous and nondecreasing with $\psi(t) < t$ for each $t > 0\}$. Furthermore, let (X, p_b) be a partial metric space, and Δ denotes the diagonal of $X \times X$. Let G be a directed graph, which has no parallel edges such that the set $V(G)$ of its vertices coincides with X, and $E(G) \subseteq X \times X$ contains all loops (i.e. $\Delta \subseteq E(G)$). Hence, G is identify by the pair $(V(G), E(G))$. Denote by G^{-1} the graph obtained from G by reversing the direction of its edges. That is,

$$E\bigl(G^{-1}\bigr) = \bigl\{(x, y) \in X \times X\colon (y, x) \in E(G)\bigr\}.$$

It is more adaptable to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention, we have that

$$E(\widetilde{G}) = E(G) \cup E\bigl(G^{-1}\bigr).$$

In $V(G)$, we define the relation R in the following way: for $x, y \in V(G)$, we have xRy if and only if there is a path in G from x to y. If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, the equivalence class $[x]_{\widetilde{G}}$ in $V(G)$ defined by the relation R is $V(G_x)$. Recall that if $f : X \to X$ is an operator, then by $\mathrm{Fix}(f)$ we denote the set of all fixed points of f. Let

$$X_f := \bigl\{x \in X\colon (x, fx) \in E(G)\bigr\}.$$

Abbas et al. [1] used the following property: a graph is said to satisfy property (P*) if for any sequence $\{x_n\}$ in $V(G)$ with $x_n \to x$ as $n \to \infty$, $(x_n, x_{n+1}) \in E(G)$ for $n \in \mathbb{N}$ implies that there is a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ with an edge between $x_{n(k)}$ and x for $k \in \mathbb{N}$.

Throughout this paper, G is a weighted graph such that the weight of each vertex x is $p_b(x, x)$, and the weight of each edge (x, y) is $p_b(x, y)$. Since (X, p_b) is a partial b-metric space, the weight assigned to each vertex x need not to be zero, and whenever a zero weight is assigned to some edge (x, y), it reduces to a loop (x, x).

3 Common fixed point theorems for power graphic (F, ψ)-contraction pair

We begin this section by introducing the following definition.

Definition 7. Let (X, p_b) be a partial b-metric space endowed with a directed graph G. Let $f, g : X \to X$ be two self-mappings on X. We say that (f, g) is a power graphic (F, ψ)-contraction pair on a partial b-metric space X if:

(a) for every vertex $v \in G$, we have $(v, fv), (v, gv) \in E(G)$;
(b) there exist $F \in \Delta_F$, $D \in \Delta_D$, and $\psi \in \Psi$ such that, for all $x, y \in X$ and $s > 1$ with $fx \neq gy$,

$$\begin{aligned} F\bigl(sp_b^\lambda(fx, gy)\bigr) \leqslant{} & F\bigl(\psi\bigl(M_s^\eta(x,y)\, p_b^\mu(x,fx)\, p_b^\nu(y,gy)\bigr)\bigr) \\ & - D\bigl(p_b(x,y), p_b(x,fx), p_b(y,fy), p_b(fx,fy)\bigr), \end{aligned} \tag{1}$$

where

$$M_s(x,y) = \max\left\{p_b(x,y), p_b(x,fx), p_b(y,gy), \frac{p_b(x,gy) + p_b(y,fx)}{2s}\right\} \tag{2}$$

for all $(x, y) \in E(G)$, where $\eta, \mu, \nu \geqslant 0$ with $\lambda = \eta + \mu + \nu \in (0, \infty)$.

Remark 2. If $f = g$ in Definition 7, then we say that f is a power graphic (F, ψ)-contraction on X.

Our main result run as follows.

Theorem 1. *Let (X, p_b) be a complete partial b-metric space endowed with a directed graph G and the mappings $f, g : X \to X$ such that (f, g) is a power graphic (F, ψ)-contraction pair on X. Then the following assertions are true*:

(i) $\mathrm{Fix}(f) \neq \phi$ *or* $\mathrm{Fix}(g) \neq \phi$ *if and only if* $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) \neq \phi$;
(ii) *if $x^* \in \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$, then the weight assigned to the vertex x^* is 0*;
(iii) $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) \neq \phi$, *provided that G satisfies property* (P*);
(iv) $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ *is complete if and only if* $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ *is a singleton.*

Proof. To prove (i), let $\mathrm{Fix}(f) \neq \phi$, and there exists $x^* \in \mathrm{Fix}(f)$. By the hypothesis, there exists an edge between x^* and fx^*, so $(x^*, fx^*) \in E(G)$. Now, we will show that $x^* \in \mathrm{Fix}(g)$, i.e., the weight assigned to the edge (x^*, gx^*) is zero. Suppose to the contrary that a non zero weight assigned to the edge (x^*, gx^*). As $(x^*, gx^*) \in E(G)$ and (f, g) is a power graphic (F, ψ)-contraction pair, so we deduce that

$$\begin{aligned} F\bigl(p_b^\lambda(x^*, gx^*)\bigr) &\leqslant F\bigl(sp_b^\lambda(fx^*, gx^*)\bigr) \\ &\leqslant F\bigl(\psi\bigl(M_s^\eta(x^*, x^*)\, p_b^\mu(x^*, fx^*)\, p_b^\nu(x^*, gx^*)\bigr)\bigr) \\ &\quad - D\bigl(p_b(x^*, x^*), p_b(x^*, fx^*), p_b(x^*, gx^*), p_b(fx^*, gx^*)\bigr), \end{aligned} \tag{3}$$

where

$$\begin{aligned} M_s(x^*, x^*) &= \max\left\{p_b(x^*, x^*), p_b(x^*, fx^*), p_b(x^*, gx^*), \frac{p_b(x^*, fx^*) + p_b(x^*, gx^*)}{2s}\right\} \\ &= p_b(x^*, gx^*). \end{aligned}$$

Thus, by the definition of function D and ψ, inequality (3) turns into the following:

$$\begin{aligned} F\big(p_b^\lambda(x^*,gx^*)\big) &\leqslant F\big(\psi\big(p_b^\eta(x^*,gx^*)\,p_b^\mu(x^*,fx^*)\,p_b^\nu(x^*,gx^*)\big)\big)-\tau \\ &\leqslant F\big(\psi\big(p_b^{\eta+\mu+\nu}(x^*,gx^*)\big)\big)-\tau \\ &< F\big(p_b^\lambda(x^*,gx^*)\big)-\tau, \end{aligned}$$

which is not possible since $\tau>0$. Thus, the weight assigned to the edge (x^*,gx^*) is zero, i.e., $x^*=gx^*$. So that $x^*\in\mathrm{Fix}(f)\cap\mathrm{Fix}(g)\neq\phi$. Analogously, one can show that if $x^*\in\mathrm{Fix}(g)$, then $x^*\in\mathrm{Fix}(f)$.

Conversely, let $\mathrm{Fix}(f)\cap\mathrm{Fix}(g)\neq\phi$ and there exists x^* such that $x^*\in\mathrm{Fix}(f)\cap\mathrm{Fix}(g)$, then $x^*\in\mathrm{Fix}(f)$ and $x^*\in\mathrm{Fix}(g)$. The proof of (i) is completed.

To prove (ii), let $x^*\in\mathrm{Fix}(f)\cap\mathrm{Fix}(g)$ and on the contrary, we assign a non zero weight to the vertex x^*. As $(x^*,x^*)\in E(G)$ and (f,g) is a power graphic (F,ψ)-contraction pair, so we have

$$\begin{aligned} F\big(p_b^\lambda(x^*,x^*)\big) &\leqslant F\big(sp_b^\lambda\big(fx^*,gx^*\big)\big) \\ &\leqslant F\big(\psi\big(M_s^\eta(x^*,x^*)\,p_b^\mu(x^*,fx^*)\,p_b^\nu(x^*,gx^*)\big)\big) \\ &\quad - D\big(p_b(x^*,x^*),p_b(x^*,fx^*),p_b(x^*,gx^*),p_b(fx^*,gx^*)\big). \end{aligned}$$

By the routine calculation, one can find that

$$\begin{aligned} F\big(p_b^\lambda(x^*,x^*)\big) &\leqslant F\big(\psi\big(p_b^\eta(x^*,x^*)\,p_b^\mu(x^*,x^*)\,p_b^\nu(x^*,x^*)\big)\big)-\tau \\ &\leqslant F\big(\psi\big(p_b^{\eta+\mu+\nu}(x^*,x^*)\big)\big)-\tau \\ &< F\big(p_b^\lambda(x^*,x^*)\big)-\tau, \end{aligned}$$

a contradiction. Hence, the weight assigned to the edge (x^*,x^*) is zero. The proof of (2) is finished.

To prove (iii), let x_0 be an arbitrary point in X. If $x_0\in\mathrm{Fix}(f)$ or $x_0\in\mathrm{Fix}(g)$, then from (i) the proof is completed. Assume that $x_0\notin\mathrm{Fix}(f)$, so $x_0\neq fx_0$. As there is an edge between x_0 and fx_0 i.e., $(x_0,fx_0)\in E(G)$, which gives that there exists $fx_0=x_1\in X$ such that $(x_0,x_1)\in E(G)$. Also, $(x_1,gx_1)\in E(G)$ implies $(x_1,x_2)\in E(G)$. Continuing in this manner, define a sequence $\{x_n\}\in X$ such that $(x_n,x_{n+1})\in E(G)$. In the consecutive way, $fx_{2n}=x_{2n+1}$ and $gx_{2n+1}=x_{2n+2}$ for all $n\in\mathbb{N}$. If for some $m\in\mathbb{N}$, the weight assigned to the edge (x_{2m},x_{2m+1}) is zero, then $x_{2m}=x_{2m+1}=fx_{2m}$, which yields $x_{2m}\in\mathrm{Fix}(f)$, and by (i), $x_{2m}\in\mathrm{Fix}(f)\cap\mathrm{Fix}(g)$. Hence, x_{2m} is a common fixed point of f and g. Assume that the weight assigned to the edge (x_{2n},x_{2n+1}) is non zero for all $n\in\mathbb{N}$, i.e., $x_{2n}\neq x_{2n+1}$ for all $n\in\mathbb{N}$. By inequality (1), we arrive at

$$\begin{aligned} &F\big(p_b^\lambda(x_{2n+1},x_{2n})\big) \\ &\quad\leqslant F\big(sp_b^\lambda(fx_{2n},gx_{2n-1})\big) \\ &\quad\leqslant F\big(\psi\big(M_s^\eta(x_{2n},x_{2n-1})\,p_b^\mu(x_{2n},x_{2n+1})\,p_b^\nu(x_{2n-1},x_{2n})\big)\big) \\ &\qquad - D\big(p_b(x_{2n},x_{2n-1}),p_b(x_{2n},x_{2n+1}),p_b(x_{2n-1},x_{2n}),p_b(x_{2n+1},x_{2n})\big), \end{aligned} \tag{4}$$

in which

$$M_s(x_{2n},x_{2n-1}) = \max\biggl\{p_b(x_{2n},x_{2n-1}), p_b(x_{2n},x_{2n+1}), p_b(x_{2n-1},x_{2n}), \frac{p_b(x_{2n},x_{2n}) + p_b(x_{2n-1},x_{2n+1})}{2s}\biggr\} = \max\bigl\{p_b(x_{2n},x_{2n-1}), p_b(x_{2n},x_{2n+1})\bigr\}.$$

If $M_s(x_{2n},x_{2n-1}) = p_b(x_{2n},x_{2n-1})$ for all $n \in \mathbb{N}$, then from (4) and by the definition of functions D and ψ we get

$$F\bigl(p_b^{\lambda}(x_{2n+1},x_{2n})\bigr) \leqslant F\bigl(\psi\bigl(p_b^{\eta+\nu}(x_{2n},x_{2n-1})\, p_b^{\mu}(x_{2n},x_{2n+1})\bigr)\bigr) - \tau, \qquad (5)$$
$$F\bigl(p_b^{\eta+\mu+\nu}(x_{2n+1},x_{2n})\bigr) < F\bigl(p_b^{\eta+\nu}(x_{2n},x_{2n-1})\, p_b^{\mu}(x_{2n},x_{2n+1})\bigr).$$

By (Δ_{F1}), the above inequality turns into

$$p_b^{\eta+\nu}(x_{2n+1},x_{2n}) < p_b^{\eta+\nu}(x_{2n},x_{2n-1}),$$

which is not possible if $\eta+\nu = 0$. Thus, $\eta+\nu > 0$, and we get

$$p_b(x_{2n+1},x_{2n}) < p_b(x_{2n},x_{2n-1}) \qquad (6)$$

for all $n \in \mathbb{N}$. If $M_s(x_{2n},x_{2n-1}) = p_b(x_{2n},x_{2n+1})$ for all $n \in \mathbb{N}$, then by (4) and in account of the properties of functions D and ψ, we acquire

$$F\bigl(p_b^{\lambda}(x_{2n+1},x_{2n})\bigr) \leqslant F\bigl(\psi\bigl(p_b^{\eta+\mu}(x_{2n},x_{2n+1})\, p_b^{\nu}(x_{2n-1},x_{2n})\bigr)\bigr) - \tau,$$
$$F\bigl(p_b^{\eta+\mu+\nu}(x_{2n+1},x_{2n})\bigr) < F\bigl(p_b^{\eta+\mu}(x_{2n},x_{2n+1})\, p_b^{\nu}(x_{2n-1},x_{2n})\bigr).$$

Due to property (Δ_{F1}), the above inequality reduces to

$$p_b^{\nu}(x_{2n+1},x_{2n}) < p_b^{\nu}(x_{2n-1},x_{2n}). \qquad (7)$$

At this junction, we attend two cases:

Case 1. If $\nu > 0$, then from above inequality we conclude that

$$p_b(x_{2n+1},x_{2n}) < p_b(x_{2n},x_{2n-1}) \qquad (8)$$

for all $n \in \mathbb{N}$.

Case 2. If $\nu = 0$, then (7) leads to a contradiction, this forces the maximum term to be $p_b(x_{2n},x_{2n-1})$, and the same conclusion follows at once.

Analogously, one can find that

$$p_b(x_{2n+2},x_{2n+1}) < p_b(x_{2n+1},x_{2n}) \qquad (9)$$

for all $n \in \mathbb{N}$. Notice that (6), (8), and (9) yields that $\{p_b(x_{2n+1},x_{2n})\}$ is a decreasing sequence of positive real numbers. Therefore, from (5) and utilizing the property of ψ, we deduce that

$$\begin{aligned} &F\big(p_b^\lambda(x_{2n+1},x_{2n})\big) \leqslant F\big(\psi\big(p_b^{\eta+\nu}(x_{2n},x_{2n-1})\,p_b^\mu(x_{2n},x_{2n+1})\big)\big) - \tau, \\ &F\big(p_b^\lambda(x_{2n+1},x_{2n})\big) < F\big(p_b^\lambda(x_{2n},x_{2n-1})\big) - \tau. \end{aligned} \tag{10}$$

Repeated use of (10) gives

$$\begin{aligned} &F\big(p_b^\lambda(x_{2n+1},x_{2n})\big) \\ &\quad < F\big(p_b^\lambda(x_{2n-1},x_{2n-2})\big) - 2\tau < F\big(p_b^\lambda(x_{2n-2},x_{2n-3})\big) - 3\tau \\ &\quad < \cdots < F\big(p_b(x_1,x_0)\big) - 2n\tau. \end{aligned} \tag{11}$$

Similarly, we get

$$F\big(p_b^\lambda(x_{2n+2},x_{2n+1})\big) < F\big(p_b(x_1,x_0)\big) - (2n+1)\tau. \tag{12}$$

As $F \in \Delta_F$, making the limit as $n \to \infty$ in (11) and (12), we have

$$\lim_{n\to\infty} F\big(p_b^\lambda(x_{n+1},x_n)\big) = -\infty \quad \Longleftrightarrow \quad \lim_{n\to\infty} p_b^\lambda(x_{n+1},x_n) = 0,$$

which implies that

$$\lim_{n\to\infty} p_b(x_{n+1},x_n) = 0.$$

Further, from (p_{b2}) we have the following:

$$\lim_{n\to\infty} p_b(x_n,x_n) = 0.$$

Now, we will prove that $\{x_n\}$ is a p_b-Cauchy sequence in X. Consequently, we have

$$F\big(p_b^\lambda(x_{n+1},x_n)\big) \leqslant F\big(sp_b^\lambda(x_{n+1},x_n)\big) \leqslant F\big(\psi\big(p_b^\lambda(x_n,x_{n-1})\big)\big).$$

Due to property (Δ_{F1}), the aforesaid inequality turns into

$$\begin{aligned} p_b^\lambda(x_{n+1},x_n) &\leqslant \psi\big(p_b^\lambda(x_n,x_{n-1})\big) \leqslant \psi\big(\psi\big(p_b^\lambda(x_{n-1},x_{n-2})\big)\big) \\ &= \psi^2\big(p_b^\lambda(x_{n-1},x_{n-2})\big) \leqslant \cdots \leqslant \psi^n\big(p_b^\lambda(x_1,x_0)\big). \end{aligned}$$

According to property (p_{b4}) of partial b-metric space, for all $m,n \in \mathbb{N}$ $(m > n)$, we have

$$\begin{aligned} p_b^\lambda(x_n,x_m) &\leqslant sp_b^\lambda(x_n,x_{n+1}) + sp_b^\lambda(x_{n+1},x_m) - p_b^\lambda(x_{n+1},x_{n+1}) \\ &\leqslant sp_b^\lambda(x_n,x_{n+1}) + s^2p_b^\lambda(x_{n+1},x_{n+2}) + s^2p_b^\lambda(x_{n+2},x_m) \\ &\quad - p_b^\lambda(x_{n+1},x_{n+1}) - sp_b^\lambda(x_{n+2},x_{n+2}) \\ &\leqslant sp_b^\lambda(x_n,x_{n+1}) + s^2p_b^\lambda(x_{n+1},x_{n+2}) + s^3p_b^\lambda(x_{n+2},x_{n+3}) \\ &\quad + \cdots + s^{m-n}p_b^\lambda(x_{m-1},x_m) - p_b^\lambda(x_{n+1},x_{n+1}) \\ &\quad - sp_b^\lambda(x_{n+2},x_{n+2}) - \ldots - s^{m-n-1}p_b^\lambda(x_{m-1},x_{m-1}) \end{aligned}$$

$$\begin{aligned}
&\leqslant s p_b^\lambda(x_n, x_{n+1}) + s^2 p_b^\lambda(x_{n+1}, x_{n+2}) + s^3 p_b^\lambda(x_{n+2}, x_{n+3}) \\
&\quad + \cdots + s^{m-n} p_b^\lambda(x_{m-1}, x_m) \\
&\leqslant s\psi^n\big(p_b^\lambda(x_1, x_0)\big) + s^2\psi^{n+1}\big(p_b^\lambda(x_1, x_0)\big) + s^3\psi^{n+2}\big(p_b^\lambda(x_1, x_0)\big) \\
&\quad + \cdots + s^{m-n}\psi^{m-1}\big(p_b^\lambda(x_1, x_0)\big).
\end{aligned}$$

Letting $n, m \to \infty$ in aforesaid inequality, we get $p_b^\lambda(x_n, x_m) \to 0$. Therefore, $\{x_n\}$ is a Cauchy sequence in (X, p_b) and, from Lemma 1, in the b-metric space (X, d_{p_b}). Since (X, p_b) is complete, then, by Lemma 1, (X, d_{p_b}) is also complete. Therefore, the sequence converges to some point $x^* \in X$ such that $\lim_{n\to\infty} d_{p_b}(x_n, x^*) = 0$. Because of Lemma 1, we have

$$\lim_{n\to\infty} p_b(x_n, x^*) = \lim_{n,m\to\infty} p_b(x_n, x_m) = p_b(x^*, x^*) = 0.$$

Next, we claim that x^* is a fixed point of f, i.e., $x^* \in \mathrm{Fix}(f)$, so the weight assigned to the edge (x^*, fx^*) is zero. Assume that $p_b(x^*, fx^*) > 0$ reasoning by contradiction. For $x_{2n+1} \in V(G)$, $n \in \mathbb{N}$, we get $(x_{2n+1}, x_{2n+2}) = (x_{2n+1}, gx_{2n+1}) \in E(G)$. Due to property (P*), there is a subsequence $\{x_{2n(k)+1}\}$ of $\{x_{2n+1}\}$ with an edge between $x_{2n(k)+1}$ and x^* for $k \in \mathbb{N}$. From (1), we obtain that

$$\begin{aligned}
&F\big(p_b^\lambda(fx^*, x_{2n(k)+2})\big) \\
&\quad \leqslant F\big(s p_b^\lambda(fx^*, gx_{2n(k)+1})\big) \\
&\quad \leqslant F\big(\psi\big(M_s^\eta(x^*, x_{2n(k)+1})\, p_b^\mu(x^*, fx^*)\, p_b^\nu(x_{2n(k)+1}, x_{2n(k)+2})\big)\big) \\
&\qquad - D\big(p_b(x^*, x_{2n(k)+1}), p_b(x^*, fx^*), p_b(x_{2n(k)+1}, x_{2n(k)+2}), \\
&\qquad\qquad p_b(fx^*, x_{2n(k)+2})\big).
\end{aligned} \tag{13}$$

Indeed,

$$\begin{aligned}
M_s(x^*, x_{2n(k)+1}) = \max\bigg\{&p_b(x^*, x_{2n(k)+1}), p_b(x^*, fx^*), p_b(x_{2n(k)+1}, x_{2n(k)+2}), \\
&\frac{p_b(x^*, x_{2n(k)+2}) + p_b(x_{2n(k)+1}, fx^*)}{2s}\bigg\},
\end{aligned}$$

$$\limsup_{k\to\infty} M_s(x^*, x_{2n(k)+1}) = p_b(x^*, fx^*). \tag{14}$$

On taking upper limit as $k \to \infty$ in (13) and utilizing (14), we arrive at

$$\begin{aligned}
F\big(p_b^\lambda(fx^*, x^*)\big) &\leqslant F\big(\psi\big(p_b^\eta(x^*, fx^*)\, p_b^\mu(x^*, fx^*)\, p_b^\nu(x^*, x^*)\big)\big) - \tau \\
&\leqslant F\big(\psi\big(p_b^{\eta+\mu+\nu}(x^*, fx^*)\big)\big) - \tau < F\big(p_b^\lambda(fx^*, x^*)\big),
\end{aligned}$$

which is impossible. Hence, the assigned weight of the edge (x^*, fx^*) is zero, i.e., $x^* = fx^*$ and, from (1), $x^* \in \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$. Thus, the proof of (iii) is ended.

To prove (iv), let $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is complete and we will show that $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is a singleton. On the contrary, suppose that there exist $x^*, y^* \in \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ with

$x^* \neq y^*$. Since $(x^*, y^*) \in E(G)$, then from (1) we have

$$\begin{aligned} F\big(p_b^\lambda(x^*, y^*)\big) &\leqslant F\big(sp_b^\lambda(fx^*, gy^*)\big) \\ &\leqslant F\big(\psi\big(M_s^\eta(x^*, y^*)\, p_b^\mu(x^*, fx^*)\, p_b^\nu(y^*, gy^*)\big)\big) \\ &\quad - D\big(p_b(x^*, y^*), p_b(x^*, fx^*), p_b(y^*, gy^*), p_b(fx^*, gy^*)\big). \end{aligned}$$

By a routine calculation, one can find that $M_s^\eta(x^*, y^*) = p_b(x^*, y^*)$, and the above inequality reduces to

$$\begin{aligned} F\big(p_b^\lambda(x^*, y^*)\big) &\leqslant F\big(\psi\big(p_b^\eta(x^*, y^*)\, p_b^\mu(x^*, x^*)\, p_b^\nu(y^*, y^*)\big)\big) - \tau, \\ F\big(p_b^\lambda(x^*, y^*)\big) &< F\big(p_b^\lambda(x^*, y^*)\big), \end{aligned}$$

which gives a contradiction, and so $x^* = y^*$. Conversely, let $\mathrm{Fix} \cap \mathrm{Fix}(g)$ is a singleton, which yields that $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is complete. This complete the proof. □

Remark 3. We also point out some slip up in [2], where the authors used $u = gu$ and, finally, shown u to be a fixed point of g again.

Now, we consider the following example to validate our results obtained in Theorem 1.

Example 2. Let $X = \{2, 4, 6, 8, 10\}$, and $p_b : X \times X \to [0, \infty)$ be defined by $p_b(x, y) = (\max\{x, y\})^2$ for all $x, y \in X$. Then (X, p_b) is a complete partial b-metric space. Consider

$$E(G) = \varDelta \cup \big\{(4,2), (8,2), (10,2), (6,4), (8,4), (10,4), (8,6), (10,6), (10,8)\big\}.$$

Let $f, g : X \to X$ are defined as follows:

$$f(x) = \begin{cases} 2, & x \in \{2, 4, 10\}, \\ 4, & x \in \{6, 8\}, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 2, & x \in \{2, 10\}, \\ 4, & x \in \{4, 6, 8\}. \end{cases}$$

Take $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = 123t/124$ and $F(t) = \log t$. It is easy to verify that, for every vertex $v \in G$, we have $(v, fv), (v, gv) \in E(G)$. Without loss of generality, we may consider $x \neq y$ and $\eta = \mu = \nu = 1$.

The following table demonstrates that (f, g) is a power graphic (F, ψ)-contraction.

Table 1. Verification of the contractive condition (1).

(x,y)	$F(sp_b^3(fx,gy))$	$F(\psi(M_s(x,y)p_b(x,fx)p_b(y,gy))) - \tau$	τ
$(4,2)$	2.1072	2.2067 at $\tau = 0.8$	$(0, 0.8]$
$(8,2)$	2.1072	2.2108 at $\tau = 2$	$(0, 2]$
$(10,2)$	2.1072	2.2985 at $\tau = 2.3$	$(0, 2.3]$
$(6,4)$	3.9133	3.9631 at $\tau = 0.35$	$(0, 0.35]$
$(8,4)$	3.9133	3.9629 at $\tau = 0.85$	$(0, 0.85]$
$(10,4)$	2.1072	2.2005 at $\tau = 3$	$(0, 3]$
$(8,6)$	3.9133	3.9651 at $\tau = 1.2$	$(0, 1.2]$
$(10,6)$	3.9133	3.9527 at $\tau = 1.6$	$(0, 1.6]$
$(10,8)$	3.9133	3.9526 at $\tau = 1.85$	$(0, 1.85]$

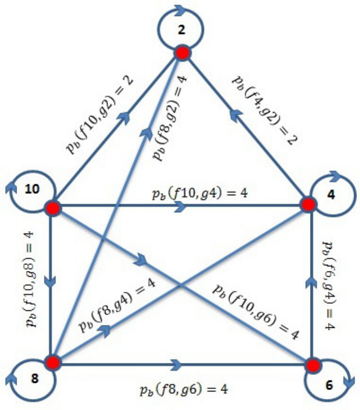

Figure 1. The graph G defined in Example 2.

Thus, all the conditions of Theorem 1 are satisfied, and $f2 = g2 = 2$ is the unique common fixed point of f and g. Figure 1 represents the graph with all the possible cases.

4 Some consequences

From Theorem 1, if $s = 1$, we deduce the following theorem.

Theorem 2. *Let (X, p) be a complete partial metric space endowed with a directed graph G and the mappings $f, g : X \to X$ such that*:

(a) *for every vertex $v \in G$, we have $(v, fv), (v, gv) \in E(G)$*;

(b) *there exist $F \in \Delta_F$, $D \in \Delta_D$, and $\psi \in \Psi$ such that, for all $x, y \in X$ with $fx \neq gy$,*

$$F\big(p^{\lambda}(fx, gy)\big) \leqslant F\big(\psi\big(M^{\eta}(x, y)\, p^{\mu}(x, fx)\, p^{\nu}(y, gy)\big)\big) - D\big(p(x, y), p(x, fx), p(y, gy), p(fx, gy)\big), \tag{15}$$

where

$$M(x, y) = \max\left\{p(x, y), p(x, fx), p(y, gy), \frac{p(x, gy) + p(y, fx)}{2}\right\}$$

for all $(x, y) \in E(G)$, where $\eta, \mu, \nu \geqslant 0$ with $\lambda = \eta + \mu + \nu \in (0, \infty)$. Then the following assertions are true:

(i) $\mathrm{Fix}(f) \neq \phi$ *or* $\mathrm{Fix}(g) \neq \phi$ *if and only if* $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) \neq \phi$;

(ii) *if $x^* \in \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$, then the weight assigned to the vertex x^* is* 0;

(iii) $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) \neq \phi$, *provided that G satisfies property* (P*);

(iv) $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ *is complete if and only if* $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ *is a singleton.*

We furnish an example in favour of Theorem 2.

Example 3. Let $X=\{1,2,3,\dots,n\}=V(G)$, $n>2$, and

$$E(G)=\big\{(1,1),(2,2),\dots,(n,n),(1,2),(1,3),\dots,(1,n),\\ (2,3),(2,4),\dots,(2,n),\dots,(n-1,n)\big\}.$$

Let $p:X\times X\to[0,\infty)$ be given by

$$p(x,y)=\begin{cases}0, & x=y,\\ \frac{1}{n+2}, & x,y\in\{1,2\} \text{ with } x\neq y,\\ \frac{n+2}{n+3}, & x \text{ or } y \text{ (or both) } \notin\{1,2\} \text{ with } x\neq y.\end{cases}$$

Easily, one can find that (X,p) is a complete partial metric space. Define $f,g\colon X\to X$ as follows:

$$fx=\begin{cases}1, & x\in\{1,2\},\\ 2, & x\notin\{1,2\},\end{cases}\quad \text{and} \quad gx=1 \quad \text{for all } x\in X.$$

Notice that, for every vertex $v\in G$, we have $(v,fv),(v,gv)\in E(G)$.

Take $\psi:[0,\infty)\to[0,\infty)$ by $\psi(t)=1023t/1024$. Further, for all $x,y\in X$, $\eta=\mu=\nu=1$ with $fx\neq gy$, we have to consider the following cases:

Case 1. If $x\notin\{1,2\}$ and $y\in\{1,2\}$, then we get

$$\text{l.h.s.}=F\big(p^3(fx,gy)\big)=F\bigg(\frac{1}{(n+2)^3}\bigg)$$

and

$$\begin{aligned}\text{r.h.s.}&=F\big(\psi\big(M(x,y)p(x,fx)p(y,gy)\big)\big)\\ &\quad-D\big(p(x,y),p(x,fx),p(y,gy),p(fx,gy)\big)\\ &=F\bigg(\frac{1023(n+2)}{1024(n+3)^2}\bigg)-\tau.\end{aligned}$$

Simple calculation shows that inequality (15) satisfied for $F(t)=\log t+t$, $t>0$, and $\tau=(0,1]$.

Case 2. If $x\notin\{1,2\}$ and $y\notin\{1,2\}$, then we arrive at

$$\text{l.h.s.}=F\big(p^3(fx,gy)\big)=F\bigg(\frac{1}{(n+2)^3}\bigg)$$

and

$$\begin{aligned}\text{r.h.s.}&=F\big(\psi\big(M(x,y)p(x,fx)p(y,gy)\big)\big)\\ &\quad-D\big(p(x,y),p(x,fx),p(y,gy),p(fx,gy)\big)\\ &=F\bigg(\frac{1023(n+2)^3}{1024(n+3)^3}\bigg)-\tau.\end{aligned}$$

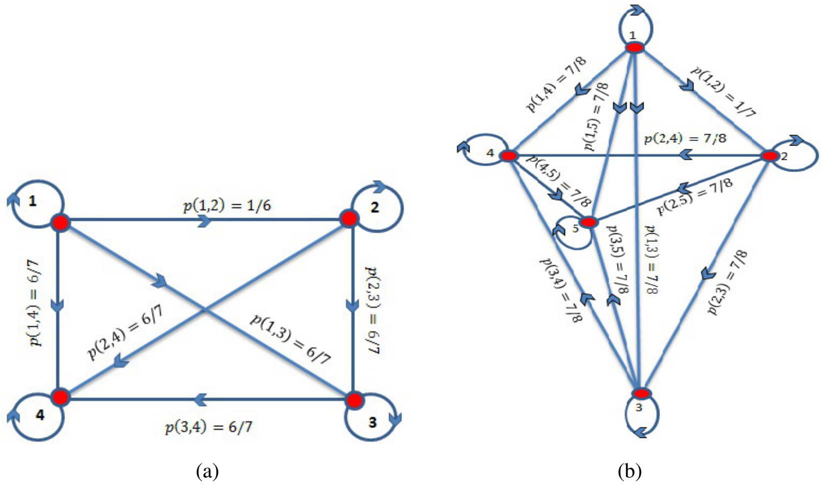

Figure 2. The graph G: (a) defined in Example 3 for $n = 4$; (b) defined in Example 3 for $n = 5$.

It is easy to verify that inequality (15) satisfied with $F(t) = \log t + t$, $t > 0$, and $\tau = (0, 0.5]$. Thus, inequality (15) holds in Cases 1, 2, and so (f, g) is a power graphic (F, ψ)-contraction pair. Hence, all the conditions of Theorem 2 are fulfilled, and we conclude that 1 is the common fixed point of f and g, thus, $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is complete.

Figure 2(a) demonstrates the weighted graph for $n = 4$.

For $n = 5$, under the same conditions, graph in Fig. 2(b) is worked out.

If $f = g$, $D(t_1, t_2, t_3, t_4) = \tau > 0$, $\mu = \nu = 0$, and $\eta \geqslant 1$ in Theorem 1, then we obtain the following corollary.

Corollary 1. *Let (X, p_b) be a complete partial b-metric space endowed with a directed graph G, and let the mapping $f : X \to X$ such that*:

(a) *for every vertex $v \in G$, we have $(v, fv) \in E(G)$*;

(b) *there exist $F \in \Delta_F$, $D \in \Delta_D$, and $\psi \in \Psi$ such that, for all $x, y \in X$ and $s > 1$ with $fx \neq fy$,*

$$F\big(sp_b^\eta(fx, fy)\big) \leqslant F\big(\psi\big(M_s^\eta(x, y)\big)\big) - \tau \tag{16}$$

for all $(x, y) \in E(G)$ and $\eta \geqslant 1$. Here $M_s(x, y)$ is obtained from (2) *by taking $f = g$.*

Then the following assertions are true:

(i) *if $x^* \in \mathrm{Fix}(f)$, then the weight assigned to the vertex x^* is 0*;

(ii) $\mathrm{Fix}(f) \neq \phi$*, provided that G satisfies property* (P*);

(iii) $\mathrm{Fix}(f)$ *is complete if and only if* $\mathrm{Fix}(f)$ *is a singleton.*

Remark 4. Theorems 1 and 2 involving the graph G generalize, improve, and extend Theorem 2.1 of Wardowski [14] for partial b-metric space and partial metric space along with power graphic contraction pair, respectively.

Remark 5. Take $\psi(t) = t$ and $\eta = 1$ in Corollary 1. Then Corollary 1 with the graph G enhances and generalizes Theorem 2.4 of Wardowski et al. [15].

Remark 6. To be specific, taking $D(t_1, t_2, t_3, t_4) = \tau > 0$, $\eta = 1$, and $\mu = \nu = 0$ in Theorem 2. So, Theorem 2 endowed with graph G upgrades Theorem 10 of Shukla along with Radenovic [12].

Remark 7. By taking $s = 1$ and $\eta = 1$ in Corollary 1, we extended variant of Theorem 2.1 of Singh et al. [13] in the sense of graphical contraction is attained.

Remark 8. Taking $s = 1$ and $\eta = 1$, in Corollary 1, we obtain Theorem 2.2 of Abbas et al. [4] in the sense of F-contraction.

Remark 9. Our Theorems 1 and 2 improve, enhance, and generalize Theorem 3.1 in Jachymski et al. [7] for F-contraction.

Remark 10. By introducing Theorem 1 we generalized the results of Abbas et al. [3] and obtained the F-contraction version of [3] in partial b-metric spaces.

Remark 11. To be specific, taking $\eta = 2$, $\mu = \nu = 0$, and $s = 1$ in Theorem 1, we reduce it to Corollary 2.5 of Zheng et al. [16] for graphical F-contraction with a note that other product terms [16] reduce to the similar terms of Theorem 1.

5 Application to nonlinear integral equation

As an application of our result, we are going to study the existence of solution for the following nonlinear integral equation:

$$u(t) = \Omega\bigl(\phi(t), t\bigr) + K\bigl(t, t, \phi(t)\bigr) + \int_a^b K\bigl(t, z, u(z)\bigr)\,\mathrm{d}z, \quad t \in [a, b], \tag{17}$$

where $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$, $\phi : [a, b] \to \mathbb{R}$, $\Omega : \mathbb{R} \times [a, b] \to \mathbb{R}$ are given continuous functions.

Let X be the set $C[a, b]$ of real continuous functions defined on $[a, b]$. Define a mapping $p_b : X \times X \to [0, \infty)$ by

$$p_b(u, v) = \max_{a \leqslant t \leqslant b} \bigl|u(t) - v(t)\bigr|^q, \quad q > 1,$$

for all $u, v \in X$.

Moreover, we define the graph G with partial ordered relation by

$$x, y \in C[a, b], \quad x \leqslant y \quad \Longleftrightarrow \quad x(t) \leqslant y(t)$$

for all $t \in [a, b]$. Let $E(G) = \{(x, y) \in X \times X\colon x \leqslant y\}$. Obviously, (X, p_b) is a complete partial b-metric space with coefficient $s = 2^{q-1} > 1$ including a directed graph G. Clearly, $\Delta(X \times X) \in E(G)$, and (X, p_b, G) has property (P*). Now, we prove the subsequent theorems to show that the existence of solution of integral equation.

Theorem 3. *Assume that the following assumptions hold for the integral equation* (17):

(i) $K : [a,b]\times[a,b]\times\mathbb{R}\to\mathbb{R}$ *is continuous and non decreasing in the third ordered*;

(ii) *for every* $x \in X$, *we have* $x \leqslant \Omega(\phi(t),t) + K(t,t,\phi(t)) + \int_a^b K(t,z,x(z))\,\mathrm{d}z$ *for all* $t,z \in [a,b]$;

(iii) *for all* $t,z\in[a,b]$, *for all* $x,y\in\mathbb{R}$ *with* $x\leqslant y$, *following inequality holds*:

$$\bigl|K\bigl(t,z,x(z)\bigr) - K\bigl(t,z,y(z)\bigr)\bigr| \leqslant G(t,z)\bigl|x(z)-y(z)\bigr|\mathrm{e}^{-\tau/(q\eta)},$$

where $G : [a,b]\times[a,b]\to[0,\infty)$ *is a continuous function such that*

$$\max_{a\leqslant t\leqslant b}\int_a^b \bigl|G(t,z)\bigr|^q\,\mathrm{d}z \leqslant \frac{3}{2^{(q+2\eta-1)/\eta}(b-a)}.$$

Then there exists at least one solution of the integral equation (17).

Proof. Let $f : X\to X$ is defined by

$$fu(t) = \Omega\bigl(\phi(t),t\bigr) + K\bigl(t,t,\phi(t)\bigr) + \int_a^b K\bigl(t,z,u(z)\bigr)\,\mathrm{d}z, \quad t\in[a,b].$$

Then system (17) can be written as $fu = u$, which yields that the solution of problem (17) is a fixed point of the mapping f.

From condition (ii) it is easy to show that, for every $v\in X$, we have $v\leqslant fv$, i.e., $(v,fv)\in E(G)$.

It follows from condition (ii) that $X_f = \{x\in X\colon\ x\leqslant fx,\ \text{i.e., } (x,fx)\in E(G)\} \neq \phi$.

Further, utilizing condition (iii) and in account of inequality (16) of Corollary 1, we have

$$\begin{aligned}
&2^{(q-1)/\eta}p_b(fu,fv)\\
&\quad= 2^{(q-1)/\eta}\max_{a\leqslant t\leqslant b}\bigl|fu(t)-fv(t)\bigr|^q\\
&\quad= 2^{(q-1)/\eta}\max_{a\leqslant t\leqslant b}\left|\int_a^b K\bigl(t,z,u(z)\bigr)\,\mathrm{d}z - \int_a^b K\bigl(t,z,v(z)\bigr)\,\mathrm{d}z\right|^q\\
&\quad= 2^{(q-1)/\eta}\max_{a\leqslant t\leqslant b}\int_a^b \bigl|K\bigl(t,z,u(z)\bigr)-K\bigl(t,z,v(z)\bigr)\bigr|^q\,\mathrm{d}z\\
&\quad\leqslant 2^{(q-1)/\eta}\max_{a\leqslant t\leqslant b}\left[\int_a^b\bigl|G(t,z)\bigr|^q\,\mathrm{d}z\right]\int_a^b\bigl|u(z)-v(z)\bigr|^q\mathrm{e}^{-\tau/\eta}\,\mathrm{d}z\\
&\quad= 2^{(q-1)/\eta}\frac{3}{2^{(q+2\eta-1)/\eta}(b-a)}\ \max_{a\leqslant t\leqslant b}\int_a^b\bigl|u(t)-v(t)\bigr|^q\mathrm{e}^{-\tau/\eta}\,\mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \frac{3}{2^2} \max_{a\leqslant t\leqslant b} \bigl|u(t)-v(t)\bigr|^q \mathrm{e}^{-\tau/\eta} \leqslant \frac{3}{4} p_b(u,v)\mathrm{e}^{-\tau/\eta} \\
&\leqslant \frac{3}{4} M_s(u,v)\mathrm{e}^{-\tau/\eta},
\end{aligned}
$$

that is,

$$
s^{1/\eta} p_b(fu,fv) \leqslant \frac{3}{4} M_s(u,v)\mathrm{e}^{-\tau/\eta},
$$

where

$$
M_(u,v) = \max\biggl\{p(u,v), p(u,fu), p(v,fv), \frac{p(u,fv)+p(v,fu)}{2}\biggr\}.
$$

The above inequality is equivalent to

$$
sp_b^\eta(fu,fv) \leqslant \biggl(\frac{3}{4}\biggr)^\eta M_s^\eta(u,v)\mathrm{e}^{-\tau}.
$$

Sequentially, by passing to logarithm, we get

$$
\log\bigl(sp_b^\eta(fu,fv)\bigr) \leqslant \log\bigl(\psi\bigl(M_s^\eta(u,v)\bigr)\bigr) - \tau,
$$

where, $F(t) = \log t$ and $\psi(t) = (3/4)^\eta$, $\eta \in (0,\infty)$. Hence, f is power graphic contraction on X. Thus, all the conditions of Corollary 1 are satisfied. Finally, there exists a fixed point $u^* \in X$ of the mapping f, which is the solution of the integral equation (17). □

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