Nonlinear Analysis: Modelling and Control, Vol. 22, No. 5, 679–701 ISSN 1392-5113
https://doi.org/10.15388/NA.2017.5.7

Existence of solutions of multi-term fractional differential equations with impulse effects on a half line

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Received: November 22, 2016 / **Revised:** April 17, 2017 / **Published online:** September 24, 2017

Abstract. A class of boundary value problem for impulsive fractional differential equation on a half line is proposed. Some results on existence of solutions of this kind of boundary value problem for impulsive multi-term fractional differential equation on a half line are established by constructing a weighted Banach space, a completely continuous operator and using a fixed point theorem in the Banach space. Some unsuitable lemmas in recent published papers are pointed out. An example is given to illustrate the efficiency of the main theorems.

Keywords: multi-term fractional differential equation, impulsive effect, fractional-order Abel differential equation, fixed point theorem.

1 Introduction

Fractional differential equation is a generalization of ordinary differential equation to arbitrary noninteger orders. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [10, 13]. There have been many excellent books and monographs available on this field [7, 11, 12].

Many authors have studied the existence and uniqueness of solutions of the impulsive fractional differential equations involving the Caputo fractional derivatives. Impulsive fractional differential equations is an important area of study [1]. There have been many questions needed be studied. For example, authors in papers [2, 3, 6] studied the existence of solutions of the different initial value problems for the impulsive fractional differential equations.

In the literature, $D_{0^+}^{\alpha}u(t) + f(t, u(t)) = 0$ is known as a *single term* equation. In certain cases, we find equations containing more than one differential terms. A classical example is the so-called *Bagley–Torvik equation*

$$AD_{0^+}^{2}y(x) + BD_{0^+}^{3/2}y(x) + Cy(x) = f(x),$$

where A, B, C are constants, and f is a given function. This equation arises from example for the modelling of motion of a rigid plate immersed in a Newtonian fluid. It was originally proposed in [14]. Another example for an application of equations with more than one fractional derivatives is the *Basset equation*

$$AD^1_{0+}y(x) + bD^n_{0+}y(x) + cy(x) = f(x), \qquad y(0) = y_0,$$

where $0 < n < 1$. This equation is most frequently, but not exclusively, used with $n = 1/2$. It describes the forces that occur when a spherical object sinks in a (relatively dense) incompressible viscous fluid; see [5, 10].

In recent paper [8], Liu studied existence of positive solutions for the following boundary value problems (BVP) for fractional impulsive differential equations:

$$\begin{aligned}
&D^\alpha_{0+}u(t) = -f\bigl(t, u(t)\bigr), \quad t \in (0,1),\ t \neq t_k,\ k = 1,2,\dots,m,\\
&u\bigl(t_k^+\bigr) = (1-c_k)u\bigl(t_k^-\bigr), \quad k = 1,2,\dots,m, \qquad u(0) = u(1) = 0,
\end{aligned} \tag{1}$$

where D^α_{0+} is the Riemann–Liouville fractional derivative of order $\alpha \in (1,2)$ with the base point 0, m is a positive integer, $c_k \in (0,1/2)$, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is a given continuous function, $u(t_k^+)$ and $u(t_k^-)$ denote the right limit and left limit of u at t_k and $u(t_k^+) = u(t_k)$, i.e., u is right continuous at t_k.

In [15], Zhao and Ge studied the following fractional impulsive boundary value problem on infinite intervals:

$$\begin{aligned}
&D^\alpha_{0+}u(t) + f\bigl(t, u(t)\bigr) = 0, \quad t \in (0,+\infty),\ t \neq t_k,\ k = 1,2,\dots,m,\\
&u\bigl(t_k^+\bigr) - u\bigl(t_k^-\bigr) = -I_k\bigl(u(t_k)\bigr), \quad k = 1,2,\dots,m,\\
&u(0) = 0, \qquad D^{\alpha-1}_{0+}u(+\infty) = 0,
\end{aligned} \tag{2}$$

where $\alpha \in (1,2]$, D^*_{0+} is the Riemann–Liouville fractional derivatives of orders $* > 0$, $t_0 = 0$, $1 < t_1 < \cdots < t_m < +\infty$, $u(t_k^+) = \lim_{t\to t_k^+} u(t)$, and $u(t_k^-) = \lim_{t\to t_k^-} u(t)$. $D^{\alpha-1}_{0+}u(+\infty) = \lim_{t\to+\infty} D^{\alpha-1}_{0+}u(t)$, $(t,u) \to f(t,(1+t^\alpha)u$ is nonnegative, continuous on $[0,+\infty) \times [0,+\infty)$, and $u \to I_k(u)$ is nonnegative, continuous, and bounded. Existence, uniqueness, and computational method of unbounded positive solutions were established.

We note that Lemma 3.1 in [8] and Lemma 3.1 in [15] are unsuitable; see Remarks 2 and 3 in Section 3. This motivates us to establish results on solutions of impulsive fractional differential equations with order $\alpha \in (1,2)$. In this paper, we discuss the following boundary value problems for nonlinear impulsive fractional differential equations:

$$\begin{aligned}
&D^\alpha_{0+}u(t) = m_1(t)f_1\bigl(t, u(t), D^\mu_{0+}u(t)\bigr), \quad \text{a.e. } t \in (t_s, t_{s+1}],\ s \in \mathbb{N}_0,\\
&I^{1-\alpha}_{0+}u(0) = k_1 \lim_{t\to\infty} I^{1-\alpha}_{0+}u(t),\\
&\Delta I^{1-\alpha}_{0+}u(t_s) = c_i I^{1-\alpha}_{0+}\bigl(u(t_s)\bigr), \quad s \in \mathbb{N},
\end{aligned} \tag{3}$$

where

(a) $0 < \alpha < 1$, $\mu \in (0, \alpha)$, D^b_{a+} is the Riemann–Liouville fractional derivative with order $b > 0$ and starting point a, $k_1 \in \mathbb{R}$, $c_s \in \mathbb{R}$ $(s \in \mathbb{N}_0)$ with $c_0 = 0$, c_i satisfies that $\sum_{i=1}^{\infty} |c_i|$ is convergent, and there exist constants $M_0, M_c > 0$ such that $\prod_{\tau=\omega}^{i-1} |1 + c_\tau| \leqslant M_0$ and $|c_i| \leqslant M_c$ for all ω, $i \in \mathbb{N}$, set

$$A_i = (-1)^{i+1} c_i \sum_{\omega=2}^{i-1} (-1)^{\omega} c_{\omega-1} \prod_{\tau=\omega}^{i-1} (1 + c_\tau) + c_i c_{i-1} + c_i, \quad i \in \mathbb{N},$$

$$\Pi = (1 - k_1)\Gamma(\alpha) - \Gamma(\alpha) k_1 \sum_{i=1}^{\infty} A_i \neq 0;$$

(b) $0 = t_0 < t_1 < \cdots < t_s < \cdots$ with $\lim_{s\to\infty} t_s = \infty$, $\mathbb{N} = \{1, 2, \dots\}$, and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$;

(c) $m_1|_{(t_i, t_{i+1}]} \in L^1(t_i, t_{i+1}]$ satisfies that there exist constants $k > -1$ such that $|m_1(t)| \leqslant t^k \mathrm{e}^{-t}$ for almost all $t \in (0, \infty)$;

(d) f_1 is a Carathéodory function.

A functions x with $x : (0, \infty) \to \mathbb{R}$ is said to be a solution of (3) if $x|_{(t_s, t_{s+1}]}$, $D^{\mu}_{0+} x|_{(t_s, t_{s+1}]} \in C^0(t_s, t_{s+1}]$ $(s \in \mathbb{N}_0)$, the limits $\lim_{t\to t_i^+} (t - t_i)^{1-\alpha} x(t)$ and $\lim_{t\to t_i^+} (t - t_i)^{1+\mu-\alpha} x(t)$ $(i \in \mathbb{N}_0)$ are finite, and x satisfies all equations in (3). We obtain the results on existence of solutions for BVP (3). An example is given to illustrate the efficiency of the main theorem.

The remainder of this paper is as follows: in Section 2, we present preliminary results. In Section 3, the main theorem on the existence of solutions of (3) are presented. In Section 4, an example is given to illustrate the main results.

2 Preliminary results

For the convenience of the readers, we present the necessary definitions from the fractional calculus theory. These definitions and results can be found in the monograph [11]. For $\phi \in L^1(0, \infty)$, denote $\|\phi\|_1 = \int_0^\infty |\phi(s)|\,\mathrm{d}s$. In this paper, we define $\sum_{s=a}^{b} A_s = 0$, $\prod_{s=a}^{b} B_s = 1$, $a > b$. Let the gamma and beta functions $\Gamma(\alpha)$, $\mathrm{B}(p, q)$ be defined by

$$\Gamma(\alpha) = \int_0^{+\infty} x^{\alpha-1} \mathrm{e}^{-x}\,\mathrm{d}x, \qquad \mathrm{B}(p, q) = \int_0^1 x^{p-1} (1 - x)^{q-1}\,\mathrm{d}x.$$

Definition 1. Let $a, b \in \mathbb{R}$ with $b > a$. The left Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $g : (a, b) \to \mathbb{R}$ is given by $I^{\alpha}_{a+} g(t) = (1/\Gamma(\alpha)) \times \int_a^t (t - s)^{\alpha-1} g(s)\,\mathrm{d}s$, provided that the right-hand side exists [11].

Definition 2. Let $a, b \in \mathbb{R}$ with $b > a$. The left Riemann–Liouville fractional derivative of order $\alpha > 0$ of a function $g : (a, b) \to \mathbb{R}$ is given by $D^{\alpha}_{a+} g(t) = (1/\Gamma(n - \alpha)) \times$

$(\mathrm{d}^n/\mathrm{d}t^n)\int_a^t g(s)/(t-s)^{\alpha-n+1})\,\mathrm{d}s$, where $n-1<\alpha<n$, provided that the right-hand side exists [11].

For ease of expression, denote

$$\delta_1(t)=(t-t_s)^{1-\alpha},\quad \delta_2(t)=(t-t_s)^{1+\mu-\alpha},\quad t\in(t_s,t_{s+1}],\ s\in\mathbb{N}_0,$$
$$f_x(t)=f_1\bigl(t,x(t),D^{\mu}_{0^+}x(t)\bigr),\quad t\in(t_s,t_{s+1}],\ s\in\mathbb{N}_0.$$

Definition 3. Let $\sigma>1+k$. We call K is called a Carathéodory function if it satisfies the followings:

(i) $t\to K(t,(1+t^\sigma)x/\delta_1(t),(1+t^\sigma)y/\delta_2(t))$ is measurable on $(t_s,t_{s+1}]$ $(s\in\mathbb{N}_0)$;
(ii) $(x,y)\to K(t,(1+t^\sigma)x/\delta_1(t),(1+t^\sigma)y/\delta_2(t))$ is continuous on $\mathbb{R}^2$ for almost all $t\in(0,\infty)$;
(iii) for each $r>0$, there exists a constant $A_r>0$ such that

$$\left|K\left(t,\frac{(1+t^\sigma)x}{\delta_1(t)},\frac{(1+t^\sigma)y}{\delta_2(t)}\right)\right|\leqslant A_r,\quad t\in(t_s,t_{s+1}],\ s\in\mathbb{N}_0,\ |x|,|y|\leqslant r.$$

Set

$$\begin{aligned}X=\Bigl\{x\colon\ &x|_{(t_s,t_{s+1}]},D^{\mu}_{0^+}x\big|_{(t_s,t_{s+1}]}\in C^0(t_s,t_{s+1}],\ s\in\mathbb{N}_0,\\ &\lim_{t\to t_s^+}(t-t_s)^{1-\alpha}x(t),\ \lim_{t\to t_s^+}(t-t_s)^{1+\mu-\alpha}D^{\mu}_{0^+}x(t),\ s\in\mathbb{N}_0,\\ &\lim_{t\to\infty}\frac{\delta_1(t)}{1+t^\sigma}x(t),\ \lim_{t\to\infty}\frac{\delta_2(t)}{1+t^\sigma}D^{\mu}_{0^+}x(t)\text{ are finite}\Bigr\}.\end{aligned}$$

For $x\in X$, define the norm by

$$\|x\|=\|x\|_X=\max\left\{\sup_{t\in(0,\infty)}\frac{\delta_1(t)}{1+t^\sigma}\bigl|x(t)\bigr|,\ \sup_{t\in(0,\infty)}\frac{\delta_2(t)}{1+t^\sigma}\bigl|D^{\mu}_{0^+}x(t)\bigr|\right\}.$$

By standard method, we can show that X is a real Banach space.

Lemma 1. *Suppose that $x\in X$,* (a)–(d) *hold, and*

$$\begin{aligned}\varPi&=(1-k_1)\Gamma(\alpha)\\ &\quad-\Gamma(\alpha)k_1\sum_{i=1}^{\infty}\left[(-1)^{i+1}c_i\sum_{\omega=2}^{i-1}(-1)^{\omega}c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau)+c_ic_{i-1}+c_i\right]\neq 0.\end{aligned}$$

Then $u\in X$ is a solution of

$$\begin{aligned}&D^{\alpha}_{0^+}u(t)=m_1(t)f_x(t)\ \textit{a.e.},\quad t\in(t_s,t_{s+1}],\ s\in\mathbb{N}_0,\\ &I^{1-\alpha}_{0^+}u(0)=k_1\lim_{t\to\infty}I^{1-\alpha}_{0^+}u(t),\qquad \varDelta I^{1-\alpha}_{0^+}u(t_s)=c_iI^{1-\alpha}_{0^+}\bigl(u(t_s)\bigr),\quad s\in\mathbb{N}\end{aligned}\tag{4}$$

if and only if

$$
\begin{aligned}
u(t) = {} & \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m_1(s) f_x(s)\,\mathrm{d}s + \Biggl[\frac{k_1}{\Pi} \int_0^\infty m_1(s) f_x(s)\,\mathrm{d}s \\
& + \frac{k_1}{\Pi} \sum_{i=1}^\infty c_i (-1)^{i+1} \sum_{\omega=2}^{i-1} (-1)^\omega c_{\omega-1} \prod_{\tau=\omega}^{i-1} (1+c_\tau) \int_0^{t_{\omega-1}} m_1(s) f_x(s)\,\mathrm{d}s \\
& + \frac{k_1}{\Pi} \sum_{i=1}^\infty c_i c_{i-1} \int_0^{t_{i-1}} m_1(s) f_x(s)\,\mathrm{d}s + \frac{k_1}{\Pi} \sum_{i=1}^\infty c_i \int_0^{t_i} m_1(s) f_x(s)\,\mathrm{d}s \Biggr] t^{\alpha-1} \\
& + \sum_{i=1}^k \Biggl[\frac{(-1)^{i+1} c_i}{\Gamma(\alpha)} \sum_{\omega=2}^{i-1} (-1)^\omega c_{\omega-1} \prod_{\tau=\omega}^{i-1} (1+c_\tau) \int_0^{t_{\omega-1}} m_1(s) f_x(s)\,\mathrm{d}s \\
& + \frac{c_i c_{i-1}}{\Gamma(\alpha)} \int_0^{t_{i-1}} m_1(s) f_x(s)\,\mathrm{d}s + \frac{c_i}{\Gamma(\alpha)} \int_0^{t_i} m_1(s) f_x(s)\,\mathrm{d}s + \frac{k_1 A_i}{\Pi} \int_0^\infty m_1(s) f_x(s)\,\mathrm{d}s \\
& + \frac{k_1 A_i}{\Pi} \sum_{i=1}^\infty c_i (-1)^{i+1} \sum_{\omega=2}^{i-1} (-1)^\omega c_{\omega-1} \prod_{\tau=\omega}^{i-1} (1+c_\tau) \int_0^{t_{\omega-1}} m_1(s) f_x(s)\,\mathrm{d}s \\
& + \frac{k_1 A_i}{\Pi} \sum_{i=1}^\infty c_i c_{i-1} \int_0^{t_{i-1}} m_1(s) f_x(s)\,\mathrm{d}s + \frac{k_1 A_i}{\Pi} \sum_{i=1}^\infty c_i \int_0^{t_i} m_1(s) f_x(s)\,\mathrm{d}s \Biggr] \\
& \times (t-t_i)^{\alpha-1}, \quad t \in (t_k, t_{k+1}],\ k \in \mathbb{N}_0.
\end{aligned} \tag{5}
$$

Proof. From $x \in X$, there exists $r > 0$ such that

$$\|x\| = \max\biggl\{\sup_{t\in(0,\infty)} \frac{\delta_1(t)}{1+t^\sigma}\bigl|x(t)\bigr|, \sup_{t\in(0,\infty)} \frac{\delta_2(t)}{1+t^\sigma}\bigl|D_{0^+}^\mu x(t)\bigr|\biggr\} = r < \infty.$$

Since f_1 is a Carathéodory function, by Definition 3, there exists $A_r > 0$ such that

$$\bigl|f_1\bigl(t, x(t), D_{0^+}^\mu x(t)\bigr)\bigr| \leqslant A_r, \quad t \in (t_s, t_{s+1}],\ s \in \mathbb{N}_0.$$

We divide the whole proof into two steps.

Step 1. We will prove that u satisfies (5) if $u \in X$ is a solution of (4).

Suppose that u is a solution of (4). By Theorem 3.2 ($n = 1$, $\alpha \in (0,1)$, and $g(t)$ is replaced by $m_1(t) f_x(t)$) in [9], we know that there exist constants $C_s \in \mathbb{R}$ ($s \in \mathbb{N}$) such that, for $t \in (t_i, t_{i+1}]$, $i \in \mathbb{N}_0$,

$$u(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} m_1(s) f_x(s)\,\mathrm{d}s + \sum_{\tau=0}^i C_\tau (t-t_\tau)^{\alpha-1}. \tag{6}$$

Then, by Definitions 1 and 2, we know for $t \in (t_k, t_{k+1}]$ that

$$
\begin{aligned}
D_{0^+}^{\mu} u(t) &= \frac{1}{\Gamma(1-\mu)} \left[\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t-s)^{-\mu} u(s)\,\mathrm{d}s + \int_{t_{k+1}}^{t} (t-s)^{-\mu} u(s)\,\mathrm{d}s \right]' \\
&= \frac{1}{\Gamma(1-\mu)} \left[\sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} (t-s)^{-\mu} \left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} m_1(u) f_x(u)\,\mathrm{d}u \right.\right. \\
&\qquad \left.\left. + \sum_{\tau=0}^{j} C_\tau (s-t_\tau)^{\alpha-1} \right) \mathrm{d}s \right]' \\
&\quad + \frac{1}{\Gamma(1-\mu)} \left[\int_{t_k}^{t} (t-s)^{-\mu} \left(\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} m_1(u) f_x(u)\,\mathrm{d}u \right.\right. \\
&\qquad \left.\left. + \sum_{\tau=0}^{k} C_\tau (s-t_\tau)^{\alpha-1} \right) \mathrm{d}s \right]' \\
&= \frac{1}{\Gamma(1-\mu)} \left[\sum_{\tau=0}^{k-1} \sum_{j=\tau}^{k-1} C_\tau (t-t_\tau)^{\alpha-\mu} \int_{(t_j-t_\tau)/(t-t_\tau)}^{(t_{j+1}-t_\tau)/(t-t_\tau)} (1-w)^{-\mu} w^{\alpha-1}\,\mathrm{d}w \right]' \\
&\quad + \frac{1}{\Gamma(1-\mu)} \left[\int_0^t (t-u)^{\alpha-\mu} \int_0^1 (1-w)^{-\mu} \frac{w^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}w\, m_1(u) f_x(u)\,\mathrm{d}u \right. \\
&\quad \left. + \sum_{\tau=0}^{k} C_\tau (t-t_\tau)^{\alpha-\mu} \int_{(t_{k+1}-t_\tau)/(t-t_\tau)}^{1} (1-w)^{-\mu} w^{\alpha-1}\,\mathrm{d}w \right]'.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
D_{0^+}^{\mu} u(t) &= \int_0^t \frac{(t-s)^{\alpha-\mu-1}}{\Gamma(\alpha-\mu)} m_1(s) f_x(s)\,\mathrm{d}s \\
&\quad + \sum_{\tau=0}^{i} \frac{\Gamma(\alpha) C_\tau}{\Gamma(\alpha-\mu)} (t-t_\tau)^{\alpha-\mu-1}, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}_0. \qquad (7)
\end{aligned}
$$

Similarly, we have

$$
I_{0^+}^{1-\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \left[\int_0^t (t-s)^{-\alpha} u(s)\,\mathrm{d}s \right]'
$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(1-\alpha)}\Biggl[\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}u(s)\,\mathrm{d}s + \int_{t_{k+1}}^{t}(t-s)^{-\alpha}u(s)\,\mathrm{d}s\Biggr]' \\
&= \int_0^t m_1(s)f_x(s)\,\mathrm{d}s + \sum_{\tau=0}^{i}\Gamma(\alpha)C_\tau, \quad t\in(t_i,t_{i+1}],\ i\in\mathbb{N}_0.
\end{aligned} \tag{8}
$$

From $I_{0^+}^{1-\alpha}u(0) = k_1 \lim_{t\to\infty} I_{0^+}^{1-\alpha}u(t)$, we have

$$
\Gamma(\alpha)C_0 = k_1\Biggl[\int_0^\infty m_1(s)f_x(s)\,\mathrm{d}s + \sum_{\tau=0}^{\infty}\Gamma(\alpha)C_\tau\Biggr]. \tag{9}
$$

From $\Delta I_{0^+}^{1-\alpha}u(t_s) = c_i I_{0^+}^{1-\alpha}u(t_s))$, $s\in\mathbb{N}$, we have

$$
\Gamma(\alpha)C_i = c_i\Biggl[\int_0^{t_i} m_1(s)f_x(s)\,\mathrm{d}s + \sum_{\tau=0}^{i-1}\Gamma(\alpha)C_\tau\Biggr], \quad i\in\mathbb{N}. \tag{10}
$$

By (10), we get

$$
\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -c_2 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -c_i & -c_i & \cdots & -c_i & 1 \end{pmatrix}
\begin{pmatrix} C_1 \\ C_2 \\ \cdots \\ C_{i-1} \\ C_i \end{pmatrix}
=
\begin{pmatrix}
\frac{c_1}{\Gamma(\alpha)}[\int_0^{t_1} m_1(s)f_x(s)\,\mathrm{d}s + \Gamma(\alpha)C_0] \\
\frac{c_2}{\Gamma(\alpha)}[\int_0^{t_2} m_1(s)f_x(s)\,\mathrm{d}s + \Gamma(\alpha)C_0] \\
\cdots \\
\frac{c_{i-1}}{\Gamma(\alpha)}[\int_0^{t_{i-1}} m_1(s)f_x(s)\,\mathrm{d}s + \Gamma(\alpha)C_0] \\
\frac{c_i}{\Gamma(\alpha)}[\int_0^{t_i} m_1(s)f_x(s)\,\mathrm{d}s + \Gamma(\alpha)C_0]
\end{pmatrix}.
$$

We get

$$
\begin{aligned}
C_i = {}&\Biggl[(-1)^{i+1}c_i\sum_{\omega=2}^{i-1}(-1)^\omega c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau) + c_ic_{i-1} + c_i\Biggr]C_0 \\
&+ \frac{1}{\Gamma(\alpha)}c_i\Biggl[(-1)^{i+1}\sum_{\omega=2}^{i-1}(-1)^\omega c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau)\int_0^{t_{\omega-1}} f_x(s)\,\mathrm{d}s \\
&+ c_{i-1}\int_0^{t_{i-1}} f_x(s)\,\mathrm{d}s + \int_0^{t_i} f_x(s)\,\mathrm{d}s\Biggr], \quad i\in\mathbb{N}.
\end{aligned} \tag{11}
$$

Substituting (11) into (9), we have

$$\begin{aligned}
&(1-k_1)\Gamma(\alpha)C_0\\
&\quad = k_1\int\limits_0^\infty m_1(s)f_x(s)\,\mathrm{d}s + k_1\sum_{i=1}^\infty c_i\Biggl[(-1)^{i+1}\sum_{\omega=2}^{i-1}(-1)^\omega c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau)\\
&\qquad \times \int\limits_0^{t_{\omega-1}} m_1(s)f_x(s)\,\mathrm{d}s + c_{i-1}\int\limits_0^{t_{i-1}} m_1(s)f_x(s)\,\mathrm{d}s + \int\limits_0^{t_i} m_1(s)f_x(s)\,\mathrm{d}s\Biggr]\\
&\qquad + \Gamma(\alpha)k_1\sum_{i=1}^\infty\Biggl[(-1)^{i+1}c_i\sum_{\omega=2}^{i-1}(-1)^\omega c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau) + c_ic_{i-1} + c_i\Biggr]C_0.
\end{aligned}$$

It follows that

$$\begin{aligned}
C_0 &= \Biggl(k_1\int\limits_0^\infty m_1(s)f_x(s)\,\mathrm{d}s + k_1\sum_{i=1}^\infty c_i\Biggl[\sum_{\omega=2}^{i-1}(-1)^{i+1+\omega}c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau)\\
&\qquad \times \int\limits_0^{t_{\omega-1}} m_1(s)f_x(s)\,\mathrm{d}s + c_{i-1}\int\limits_0^{t_{i-1}} m_1(s)f_x(s)\,\mathrm{d}s + \int\limits_0^{t_i} m_1(s)f_x(s)\,\mathrm{d}s\Biggr]\Biggr)\\
&\qquad \times \Biggl((1-k_1)\Gamma(\alpha) - \Gamma(\alpha)k_1\sum_{i=1}^\infty\Biggl[(-1)^{i+1}c_i\sum_{\omega=2}^{i-1}(-1)^\omega c_{\omega-1}\\
&\qquad \times \prod_{\tau=\omega}^{i-1}(1+c_\tau) + c_ic_{i-1} + c_i\Biggr]\Biggr)^{-1}. \qquad (12)
\end{aligned}$$

So, for $i \in \mathbb{N}$, we have

$$\begin{aligned}
C_i &= \frac{(-1)^{i+1}c_i}{\Gamma(\alpha)}\sum_{\omega=2}^{i-1}(-1)^\omega c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau)\int\limits_0^{t_{\omega-1}} m_1(s)f_x(s)\,\mathrm{d}s\\
&\quad + \frac{c_ic_{i-1}}{\Gamma(\alpha)}\int\limits_0^{t_{i-1}} m_1(s)f_x(s)\,\mathrm{d}s + \frac{c_i}{\Gamma(\alpha)}\int\limits_0^{t_i} m_1(s)f_x(s)\,\mathrm{d}s + \frac{k_1A_i}{\Pi}\int\limits_0^\infty m_1(s)f_x(s)\,\mathrm{d}s\\
&\quad + \frac{k_1A_i}{\Pi}\sum_{i=1}^\infty c_i(-1)^{i+1}\sum_{\omega=2}^{i-1}(-1)^\omega c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau)\int\limits_0^{t_{\omega-1}} m_1(s)f_x(s)\,\mathrm{d}s\\
&\quad + \frac{k_1A_i}{\Pi}\sum_{i=1}^\infty c_ic_{i-1}\int\limits_0^{t_{i-1}} m_1(s)f_x(s)\,\mathrm{d}s + \frac{k_1A_i}{\Pi}\sum_{i=1}^\infty c_i\int\limits_0^{t_i} m_1(s)f_x(s)\,\mathrm{d}s. \qquad (13)
\end{aligned}$$

Substituting (12) and (13) into (6), we get (5).

Step 2. We prove that $u \in X$ and u is a solution of (4) if u is defined by (5).

Remember C_i ($i \in \mathbb{N}_0$) defined by (12) and (13), then (5) is (6). Hence, we have (7) and (8). From (5) and (7), it is easy to see that $u|_{(t_s,t_{s+1}]}, D_{0+}^{\mu}|_{(t_s,t_{s+1}]} \in C(t_s,t_{s+1}]$ ($s \in \mathbb{N}_0$). Furthermore, the limits $\lim_{t\to t_s^+}(t-t_s)^{1-\alpha}u(t)$ and $\lim_{t\to t_s^+}(t-t_s)^{1+\mu-\alpha}D_{0+}^{\mu}u(t)$ exist for all $s \in \mathbb{N}_0$. Now we will prove that both

$$\lim_{t\to\infty}\frac{\delta_1(t)}{1+t^{\sigma}}u(t) \quad \text{and} \quad \lim_{t\to\infty}\frac{\delta_2(t)}{1+t^{\sigma}}D_{0+}^{\mu}u(t)$$

exist, and $D_{0+}^{\alpha}u(t) = f_x(t)$.

In fact, for $t \in (t_i, t_{i+1}]$, we have by a similar method used in the proof of Theorem 3.2 in [9] that

$$\begin{aligned}
D_{0+}^{\alpha}u(t) &= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\left(\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}m_1(u)f_x(u)\,\mathrm{d}u\right.\right.\\
&\qquad\left.\left.+\sum_{\tau=0}^{j}C_\tau(s-t_\tau)^{\alpha-1}\right)\mathrm{d}s\right]'\\
&\quad+\frac{1}{\Gamma(1-\alpha)}\left[\int_{t_i}^{t}(t-s)^{-\alpha}\left(\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}m_1(u)f_x(u)\,\mathrm{d}u\right.\right.\\
&\qquad\left.\left.+\sum_{\tau=0}^{i}C_\tau(s-t_\tau)^{\alpha-1}\right)\mathrm{d}s\right]'\\
&= \frac{1}{\Gamma(1-\alpha)}\left[\sum_{j=0}^{i-1}\int_{t_j}^{t_{j+1}}(t-s)^{-\alpha}\sum_{\tau=0}^{j}C_\tau(s-t_\tau)^{\alpha-1}\,\mathrm{d}s\right]'\\
&\quad+\frac{1}{\Gamma(1-\alpha)}\left[\int_0^t(t-s)^{-\alpha}\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}m_1(u)f_x(u)\,\mathrm{d}u\,\mathrm{d}s\right.\\
&\quad\left.+\int_{t_i}^{t}(t-s)^{-\alpha}\sum_{\tau=0}^{i}C_\tau(s-t_\tau)^{\alpha-1}\,\mathrm{d}s\right]'\\
&= m_1(t)f_x(t), \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}_0.
\end{aligned}$$

Since $\sum_{i=1}^{\infty}|c_i| < \infty$, there exists $M_c > 0$ such that $|c_i| \leqslant M_c$. By (12) and (13), we know

$$\begin{aligned}
|A_i| &\leqslant |c_i|\sum_{\omega=2}^{i-1}|c_{\omega-1}|\prod_{\tau=\omega}^{i-1}|1+c_\tau| + |c_i||c_{i-1}| + |c_i|\\
&\leqslant |c_i|\left[M_0\sum_{\omega=2}^{i-1}|c_{\omega-1}| + M_c + 1\right], \quad i \in \mathbb{N},
\end{aligned}$$

$$
\begin{aligned}
|C_0| &\leqslant \frac{1}{|\Pi|}\Biggl[|k_1| + |k_1|\sum_{i=1}^{\infty}|c_i|\sum_{\omega=2}^{i-1}|c_{\omega-1}|\prod_{\tau=\omega}^{i-1}|1+c_\tau| + |k_1|\sum_{i=1}^{\infty}|c_i||c_{i-1}| \\
&\quad + |k_1|\sum_{i=1}^{\infty}|c_i|\Biggr]\int_0^{\infty}\bigl|m_1(s)f_x(s)\bigr|\,\mathrm{d}s \\
&\leqslant \frac{1}{|\Pi|}\Biggl[|k_1| + M_0|k_1|\Biggl(\sum_{i=1}^{\infty}|c_i|\Biggr)^2 + M_c|k_1|\sum_{i=1}^{\infty}|c_i| + |k_1|\sum_{i=1}^{\infty}|c_i|\Biggr] \\
&\quad \times \int_0^{\infty}\bigl|m_1(s)f_x(s)\bigr|\,\mathrm{d}s, \\
|C_i| &\leqslant \Biggl[\frac{|c_i|}{\Gamma(\alpha)}\sum_{\omega=2}^{i-1}|c_{\omega-1}|\prod_{\tau=\omega}^{i-1}|1+c_\tau| + \frac{|c_i||c_{i-1}|}{\Gamma(\alpha)} + \frac{|c_i|}{\Gamma(\alpha)} + \frac{|k_1||A_i|}{|\Pi|} \\
&\quad + \frac{|k_1||A_i|}{|\Pi|}\sum_{i=1}^{\infty}|c_i|\sum_{\omega=2}^{i-1}|c_{\omega-1}|\prod_{\tau=\omega}^{i-1}|1+c_\tau| + \frac{|k_1||A_i|}{|\Pi|}\sum_{i=1}^{\infty}|c_i||c_{i-1}| \\
&\quad + \frac{|k_1||A_i|}{|\Pi|}\sum_{i=1}^{\infty}|c_i|\Biggr]\int_0^{\infty}\bigl|m_1(s)f_x(s)\bigr|\,\mathrm{d}s \\
&\leqslant \Biggl[\frac{|c_i|}{\Gamma(\alpha)}\sum_{\omega=2}^{i-1}|c_{\omega-1}|\prod_{\tau=\omega}^{i-1}|1+c_\tau| + \frac{|c_i||c_{i-1}|}{\Gamma(\alpha)} + \frac{|c_i|}{\Gamma(\alpha)} + \frac{|k_1||A_i|}{|\Pi|} \\
&\quad + \frac{M_0|k_1||A_i|}{|\Pi|}\Biggl(\sum_{i=1}^{\infty}|c_i|\Biggr)^2 + \frac{M_c|k_1||A_i|}{|\Pi|}\sum_{i=1}^{\infty}|c_i| + \frac{|k_1||A_i|}{|\Pi|}\sum_{i=1}^{\infty}|c_i|\Biggr] \\
&\quad \times \int_0^{\infty}\bigl|m_1(s)f_x(s)\bigr|\,\mathrm{d}s.
\end{aligned}
$$

It follows for $t \in (t_i, t_{i+1}]$ that

$$
\begin{aligned}
\frac{\delta_1(t)}{1+t^\sigma}\bigl|u(t)\bigr| &\leqslant \frac{(t-t_i)^{1-\alpha}}{1+t^\sigma}\Biggl[\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\bigl|m_1(s)f_x(s)\bigr|\,\mathrm{d}s + \sum_{\tau=0}^{i}|C_\tau|(t-t_\tau)^{\alpha-1}\Biggr] \\
&\leqslant \frac{t^{1+k_1}}{1+t^\sigma}\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}A_r + \frac{\Gamma(k+1)A_r}{|\Pi|(1+t^\sigma)}\Biggl[|k_1| + M_0|k_1|\Biggl(\sum_{i=1}^{\infty}|c_i|\Biggr)^2 \\
&\quad + M_c|k_1|\sum_{i=1}^{\infty}|c_i| + |k_1|\sum_{i=1}^{\infty}|c_i|\Biggr] \\
&\quad + \frac{\Gamma(k+1)A_r}{1+t^\sigma}\Biggl[\sum_{i=1}^{\infty}|c_i|\Biggl(\frac{M_0}{\Gamma(\alpha)}\sum_{\omega=2}^{\infty}|c_{\omega-1}| + \frac{M_c}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\Biggr)\Biggr]
\end{aligned}
$$

$$+ \frac{A_r}{1+t^\sigma}\Bigg[\Bigg(M_0\sum_{\omega=2}^{\infty}|c_{\omega-1}| + M_c + 1\Bigg)\Bigg(\frac{|k_1|}{|\Pi|} + \frac{M_0|k_1|}{|\Pi|}\Bigg(\sum_{i=1}^{\infty}|c_i|\Bigg)^2 + \frac{M_c|k_1|}{|\Pi|}\sum_{i=1}^{\infty}|c_i| + \frac{|k_1|}{|\Pi|}\sum_{i=1}^{\infty}|c_i|\Bigg)\sum_{i=1}^{\infty}|c_i|\Gamma(k+1)\Bigg].$$

Hence, $\delta_1(t)|u(t)|/(1+t^\sigma) \to 0$ as $t \to \infty$. Similarly, from (7), for $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned}
&\frac{\delta_2(t)}{1+t^\sigma}\big|D^{\mu}_{0^+}u(t)\big| \\
&\quad\leqslant \frac{t^{1+\sigma}}{1+t^\sigma}\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}A_r + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\frac{\Gamma(k+1)}{|\Pi|(1+t^\sigma)}A_r \\
&\qquad\times\Bigg[|k_1| + M_0|k_1|\Bigg(\sum_{i=1}^{\infty}|c_i|\Bigg)^2 + M_c|k_1|\sum_{i=1}^{\infty}|c_i| + |k_1|\sum_{i=1}^{\infty}|c_i|\Bigg] \\
&\qquad+ \frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\frac{\Gamma(k+1)}{1+t^\sigma}A_r\Bigg[\frac{M_0}{\Gamma(\alpha)}\sum_{\omega=2}^{i-1}|c_{\omega-1}| + \frac{M_c}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\Bigg]\sum_{i=1}^{\infty}|c_i| \\
&\qquad+ \frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\frac{\Gamma(k+1)}{1+t^\sigma}A_r\Bigg[M_0\sum_{\omega=2}^{\infty}|c_{\omega-1}| + M_c + 1\Bigg] \\
&\qquad\times\Bigg[\frac{|k_1|}{|\Pi|} + \frac{M_0|k_1|}{|\Pi|}\Bigg(\sum_{i=1}^{\infty}|c_i|\Bigg)^2 + \frac{M_c|k_1|}{|\Pi|}\sum_{i=1}^{\infty}|c_i| + \frac{|k_1|}{|\Pi|}\sum_{i=1}^{\infty}|c_i|\Bigg]\sum_{i=1}^{\infty}|c_i|.
\end{aligned}$$

We have $\delta_2(t)|D^{\mu}_{0^+}u(t)|/(1+t^\sigma) \to 0$ as $t \to \infty$. It follows that $u \in X$ satisfies (4). The proof is completed. □

Now, we define for $t \in (t_k, t_{k+1}]$, $k \in \mathbb{N}_0$, the operator T on X by

$$\begin{aligned}
(Tx)(t) = {}& \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m_1(s)f_x(s)\,\mathrm{d}s + \Bigg[\frac{k_1}{\Pi}\int_0^{\infty}m_1(s)f_x(s)\,\mathrm{d}s \\
&+ \frac{k_1}{\Pi}\sum_{i=1}^{\infty}c_i(-1)^{i+1}\sum_{\omega=2}^{i-1}(-1)^{\omega}c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau)\int_0^{t_{\omega-1}}m_1(s)f_x(s)\,\mathrm{d}s \\
&+ \frac{k_1}{\Pi}\sum_{i=1}^{\infty}c_ic_{i-1}\int_0^{t_{i-1}}m_1(s)f_x(s)\,\mathrm{d}s + \frac{k_1}{\Pi}\sum_{i=1}^{\infty}c_i\int_0^{t_i}m_1(s)f_x(s)\,\mathrm{d}s\Bigg]t^{\alpha-1} \\
&+ \sum_{i=1}^{k}\Bigg[\frac{(-1)^{i+1}c_i}{\Gamma(\alpha)}\sum_{\omega=2}^{i-1}(-1)^{\omega}c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau)\int_0^{t_{\omega-1}}m_1(s)f_x(s)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\frac{c_i c_{i-1}}{\Gamma(\alpha)}\int\limits_0^{t_{i-1}} m_1(s)f_x(s)\,\mathrm{d}s+\frac{c_i}{\Gamma(\alpha)}\int\limits_0^{t_i} m_1(s)f_x(s)\,\mathrm{d}s\\
&+\frac{k_1A_i}{\Pi}\int\limits_0^{\infty} m_1(s)f_x(s)\,\mathrm{d}s+\frac{k_1A_i}{\Pi}\sum_{i=1}^{\infty}c_i(-1)^{i+1}\sum_{\omega=2}^{i-1}(-1)^{\omega}c_{\omega-1}\\
&\times\prod_{\tau=\omega}^{i-1}(1+c_\tau)\int\limits_0^{t_{\omega-1}} m_1(s)f_x(s)\,\mathrm{d}s+\frac{k_1A_i}{\Pi}\sum_{i=1}^{\infty}c_i c_{i-1}\int\limits_0^{t_{i-1}} m_1(s)f_x(s)\,\mathrm{d}s\\
&+\frac{k_1A_i}{\Pi}\sum_{i=1}^{\infty}c_i\int\limits_0^{t_i} m_1(s)f_x(s)\,\mathrm{d}s\Bigg](t-t_i)^{\alpha-1}. \qquad (14)
\end{aligned}
$$

Remark 1. By Lemma 1, $x\in X$ is a solution of (3) if and only if $x\in X$ is a fixed point of the operator T.

Lemma 2. *Suppose that* (a)–(d) *hold, and* $\Pi\neq 0$ *defined in Lemma* 1. *Then* $T: X\to X$ *is well defined and completely continuous.*

Proof. The proof is standard and is omitted. □

3 Existence of solutions of BVP (3)

In this section, we shall establish existence result of at least one solution of (3). For easy referencing, we list the necessary conditions as follows:

(A_σ) There exist positive numbers $\sigma_1>\sigma_2>\cdots>\sigma_n>0$, bounded functions $\psi:(0,\infty)\to\mathbb{R}$, and nonnegative numbers a_j, b_j ($j=1,2,\dots,n$) such that

$$
\left|f\left(t,\frac{(1+t^\sigma)x}{\delta_1(t)},\frac{(1+t^\sigma)y}{\delta_2(t)}\right)-\psi(t)\right|\leqslant\sum_{j=1}^{n}a_j|x|^{\sigma_j}+\sum_{j=1}^{n}b_j|y|^{\sigma_j}
$$

holds for all $t\in(t_i,t_{i+1}]$, $i\in\mathbb{N}_0$, $x,y\in\mathbb{R}$.

Set

$$
\begin{aligned}
\Psi(t)=&\int\limits_0^{t}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}m_1(s)\psi(s)\,\mathrm{d}s+\Bigg[\frac{k_1}{\Pi}\int\limits_0^{\infty} m_1(s)\psi(s)\,\mathrm{d}s\\
&+\frac{k_1}{\Pi}\sum_{i=1}^{\infty}c_i(-1)^{i+1}\sum_{\omega=2}^{i-1}(-1)^{\omega}c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau)\int\limits_0^{t_{\omega-1}} m_1(s)\psi(s)\,\mathrm{d}s\\
&+\frac{k_1}{\Pi}\sum_{i=1}^{\infty}c_i c_{i-1}\int\limits_0^{t_{i-1}} m_1(s)\psi(s)\,\mathrm{d}s+\frac{k_1}{\Pi}\sum_{i=1}^{\infty}c_i\int\limits_0^{t_i} m_1(s)\psi(s)\,\mathrm{d}s\Bigg]t^{\alpha-1}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{k}\Bigg[\frac{(-1)^{i+1}c_i}{\Gamma(\alpha)}\sum_{\omega=2}^{i-1}(-1)^{\omega}c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau)\int\limits_0^{t_{\omega-1}}m_1(s)\psi(s)\,\mathrm{d}s\\
&+\frac{c_ic_{i-1}}{\Gamma(\alpha)}\int\limits_0^{t_{i-1}}m_1(s)\psi(s)\,\mathrm{d}s+\frac{c_i}{\Gamma(\alpha)}\int\limits_0^{t_i}\psi(s)\,\mathrm{d}s+\frac{k_1A_i}{\Pi}\int\limits_0^{\infty}m_1(s)\psi(s)\,\mathrm{d}s\\
&+\frac{k_1A_i}{\Pi}\sum_{i=1}^{\infty}c_i(-1)^{i+1}\sum_{\omega=2}^{i-1}(-1)^{\omega}c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau)\int\limits_0^{t_{\omega-1}}m_1(s)\psi(s)\,\mathrm{d}s\\
&+\frac{k_1A_i}{\Pi}\sum_{i=1}^{\infty}c_ic_{i-1}\int\limits_0^{t_{i-1}}m_1(s)\psi(s)\,\mathrm{d}s+\frac{k_1A_i}{\Pi}\sum_{i=1}^{\infty}c_i\int\limits_0^{t_i}m_1(s)\psi(s)\,\mathrm{d}s\Bigg]\\
&\times(t-t_i)^{\alpha-1},\quad t\in(t_k,t_{k+1}],\ k\in\mathbb{N}_0.
\end{aligned}
$$

Denote

$$
\begin{aligned}
\overline{M}=&\Bigg[|k_1|+M_0|k_1|\Bigg(\sum_{i=1}^{\infty}|c_i|\Bigg)^2+M_c|k_1|\sum_{i=1}^{\infty}|c_i|+|k_1|\sum_{i=1}^{\infty}|c_i|\Bigg]\frac{\Gamma(k+1)}{|\Pi|}\\
&+\sum_{i=1}^{\infty}|c_i|\Bigg(\frac{M_0}{\Gamma(\alpha)}\sum_{\omega=2}^{\infty}|c_{\omega-1}|+\frac{M_c}{\Gamma(\alpha)}+\frac{1}{\Gamma(\alpha)}\Bigg)\Gamma(k+1)\\
&+\Bigg(M_0\sum_{\omega=2}^{\infty}|c_{\omega-1}|+M_c+1\Bigg)\Bigg[\frac{|k_1|}{|\Pi|}+\frac{M_0|k_1|}{|\Pi|}\Bigg(\sum_{i=1}^{\infty}|c_i|\Bigg)^2\\
&+\frac{(1+M_c)|k_1|}{|\Pi|}\sum_{i=1}^{\infty}|c_i|\Bigg]\sum_{i=1}^{\infty}|c_i|\Gamma(k+1).
\end{aligned}
$$

Theorem 1. *Suppose that* (a)–(d) *and* (A_σ) *hold,* $\Pi\neq 0$ *defined in Lemma* 1. *Then system* (3) *has at least one solution if one of the following items is satisfied*:

(i) $\sigma_1>1$ *with*

$$
\Bigg[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}\Bigg]\sum_{j=1}^{n}[a_j+b_j]\|\Psi\|^{\sigma_j-1}\leqslant\frac{(\sigma_1-1)^{\sigma_1-1}}{\sigma_1^{\sigma_1}};
$$

(ii) $\sigma_1\in(0,1)$;

(iii) $\sigma_1=1$ *with*

$$
\Bigg[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}\Bigg]\sum_{j=1}^{n}[a_j+b_j]\|\Psi\|^{\sigma_j-\sigma_1}<1.
$$

Proof. Let the Banach space X and its norm be defined as in Section 2. Define the nonlinear operator T by (14). Then

(i) $T: X \to X$ is well defined;
(ii) x is a bounded positive solution of BVP (3) if and only if x is a solution of the operator equation $x = Tx$ in X;
(iii) $T: X \to X$ is completely continuous.

It is easy to show that $\Psi \in X$. Let $r > 0$, and define $M_r = \{x \in X\colon \|x - \Psi\| \leqslant r\}$. For $x \in M_r$, we find $\|x\| \leqslant \|x - \Psi\| + \|\Psi\|$ and

$$\begin{aligned}\left|f_x(t) - \psi(t)\right| &= \left|f_1\left(t,\ \frac{1+t^\sigma}{\delta_1(t)}\frac{\delta_1(t)x(t)}{1+t^\sigma},\ \frac{1+t^\sigma}{\delta_2(t)}\frac{\delta_2(t)D_{0^+}^\mu x(t)}{1+t^\sigma}\right) - \psi(t)\right| \\ &\leqslant \sum_{j=1}^n a_j\left(\frac{\delta_1(t)|x(t)|}{1+t^\sigma}\right)^{\sigma_j} + \sum_{j=1}^n b_j\left(\frac{\delta_2(t)|D_{0^+}^\mu x(t)|}{1+t^\sigma}\right)^{\sigma_j} \\ &\leqslant \sum_{j=1}^n [a_j + b_j]\|x\|^{\sigma_j}, \quad t \in (t_i, t_{i+1}],\ i \in \mathbb{N}_0. \end{aligned} \tag{15}$$

By definition of T and Ψ and (15), we have for $t \in (t_i, t_{i+1}]$ that

$$\begin{aligned}&\frac{\delta_1(t)}{1+t^\sigma}\left|(Tx)(t) - \Psi(t)\right| \\ &\leqslant \left[\frac{\mathrm{B}(\alpha, k+1)}{\Gamma(\alpha)} + \frac{\Gamma(k+1)}{|\Pi|}\left(|k_1| + M_0|k_1|\left(\sum_{i=1}^\infty |c_i|\right)^2 + M_c|k_1|\sum_{i=1}^\infty |c_i|\right.\right. \\ &\quad \left. + |k_1|\sum_{i=1}^\infty |c_i|\right) + \sum_{i=1}^\infty |c_i|\left(\frac{M_0}{\Gamma(\alpha)}\sum_{\omega=2}^\infty |c_{\omega-1}| + \frac{M_c}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\right)\Gamma(k+1) \\ &\quad + \left(M_0\sum_{\omega=2}^\infty |c_{\omega-1}| + M_c + 1\right)\left(\frac{|k_1|}{|\Pi|} + \frac{M_0|k_1|}{|\Pi|}\left(\sum_{i=1}^\infty |c_i|\right)^2\right. \\ &\quad \left.\left. + \frac{(1+M_c)|k_1|}{|\Pi|}\sum_{i=1}^\infty |c_i|\right)\sum_{i=1}^\infty |c_i|\Gamma(k+1)\right]\sum_{j=1}^n [a_j + b_j]\|x\|^{\sigma_j} \\ &= \left[\frac{\mathrm{B}(\alpha, k+1)}{\Gamma(\alpha)} + \overline{M}\right]\sum_{j=1}^n [a_j + b_j]\|x\|^{\sigma_j}. \end{aligned}$$

Similarly, we get

$$\begin{aligned}&\frac{\delta_2(t)}{1+t^\sigma}\left|D_{0^+}^\mu (Tx)(t) - D_{0^+}^\mu \Psi\right| \\ &\quad \leqslant \left[\frac{\mathrm{B}(\alpha-\mu, k+1)}{\Gamma(\alpha-\mu)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}\right]\sum_{j=1}^n [a_j + b_j]\|x\|^{\sigma_j}. \end{aligned}$$

It follows that

$$\|Tx-\Psi\| \leqslant \left[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}\right]\sum_{j=1}^{n}[a_j+b_j]\|x\|^{\sigma_j}$$
$$\leqslant \left[r+\|\Psi\|\right]^{\sigma_1}\left[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}\right]$$
$$\times\sum_{j=1}^{n}[a_j+b_j]\|\Psi\|^{\sigma_j-\sigma_1}.$$

Then

$$\|Tx-\Psi\| \leqslant \left[r+\|\Psi\|\right]^{\sigma_1}\left[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}\right]$$
$$\times\sum_{j=1}^{n}[a_j+b_j]\|\Psi\|^{\sigma_j-\sigma_1}.$$

(i) $\sigma_1>1$. Let $r=r_0=\|\Psi\|/(\sigma_1-1)$. By assumption

$$\left[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}\right]\sum_{j=1}^{n}[a_j+b_j]\|\Psi\|^{\sigma_j-1}$$
$$\leqslant \frac{(\sigma_1-1)^{\sigma_1-1}}{\sigma_1^{\sigma_1}}$$

for $x\in M_{r_0}$, we have

$$\|Tx-\Psi\| \leqslant \left[r_0+\|\Psi\|\right]^{\sigma_1}\left[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}\right]$$
$$\times\sum_{j=1}^{n}[a_j+b_j]\|\Psi\|^{\sigma_j-\sigma_1}$$
$$\leqslant r_0.$$

Hence, we have a bounded subset $M_{r_0}\subseteq X$ such that $T(M_{r_0})\subseteq M_{r_0}$. Then T has a fixed point $x\in M_{r_0}$. Hence, x is a solution of BVP (3).

(ii) $\sigma_1\in(0,1)$. Choose $r>0$ sufficiently large such that

$$\left[r+\|\Psi\|\right]^{\sigma_1}\left[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}\right]$$
$$\times\sum_{j=1}^{n}[a_j+b_j]\|\Psi\|^{\sigma_j-\sigma_1}\leqslant r.$$

Then, for $x \in M_r$, we have

$$\begin{aligned}\|Tx-\Psi\| &\leqslant \big[r+\|\Psi\|\big]^{\sigma_1}\left[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}\right]\\ &\quad\times\sum_{j=1}^{n}[a_j+b_j]\|\Psi\|^{\sigma_j-\sigma_1}\\ &\leqslant r.\end{aligned}$$

So, $T(M_r) \subseteq M_r$. Then T has a fixed point $x \in M_r$. This x is a solution of BVP (3).

(iii) $\sigma_1 = 1$. We choose

$$r \geqslant \frac{[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}]\sum_{j=1}^{n}[a_j+b_j]\|\Psi\|^{\sigma_j-\sigma_1}\|\Psi\|}{1-[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}]\sum_{j=1}^{n}[a_j+b_j]\|\Psi\|^{\sigma_j-\sigma_1}}.$$

Then, for $x \in M_r$, we have

$$\begin{aligned}\|Tx-\psi_0\| &\leqslant \big[r+\|\Psi\|\big]^{\sigma_1}\left[\frac{\mathrm{B}(\alpha-\mu,k+1)}{\Gamma(\alpha-\mu)}+\frac{\mathrm{B}(\alpha,k+1)}{\Gamma(\alpha)}+\frac{\Gamma(\alpha)}{\Gamma(\alpha-\mu)}\overline{M}+\overline{M}\right]\\ &\quad\times\sum_{j=1}^{n}[a_j+b_j]\|\Psi\|^{\sigma_j-\sigma_1}\\ &\leqslant r.\end{aligned}$$

Hence, as in earlier cases, we conclude that T has a fixed point $x \in M_r$, which is a solution of BVP (3).

From above discussion, the proof is completed. □

Theorem 2. *Suppose that* (a)–(d)*, and there exists a constant* $M_f \geqslant 0$ *such that*

$$\left|f_1\left(t,\frac{1+t^\sigma}{\delta_1(t)}x,\frac{1+t^\sigma}{\delta_2(t)}y\right)\right| \leqslant M_f, \quad t \in (t_i,t_{i+1}],\ i \in \mathbb{N}_0,\ x,y \in \mathbb{R}.$$

Then system (3) *has at least one solution.*

Proof. In Theorem 1, choose $\psi(t) \equiv 0$, $n = 1$, $\sigma_1 = 0$, $a_1 = M_f$, and $b_1 = 0$. The result follows. □

Remark 2. (See [15, Lemma 3.1].) Let $y \in C^0[0,\infty)$, $\int_0^\infty y(t)\,\mathrm{d}t$ be convergent, and $\alpha \in (1,2)$. In order to consider the solvability of BVP (2), it was proved that if u is a solution of

$$u(t) = \int_0^\infty G(t,s)y(s)\,\mathrm{d}s + \sum_{i=1}^{m} W_i\big(t,u(t_i)\big), \tag{16}$$

where

$$G(t,s) = \begin{cases} (t^{\alpha-1} - (t-s)^{\alpha-1})/\Gamma(\alpha), & 0 \leqslant s \leqslant t < \infty, \\ t^{\alpha-1}/\Gamma(\alpha), & 0 \leqslant t \leqslant s < \infty, \end{cases}$$

$$W_i\bigl(t, u(t_i)\bigr) = \begin{cases} I_i(u(t_i))t^{\alpha-1}/(t_i^{\alpha-1} - t_i^{\alpha-2}), & 0 \leqslant t \leqslant t_i, \\ I_i(u(t_i))t^{\alpha-2}/(t_i^{\alpha-1} - t_i^{\alpha-2}), & t_i < t, < \infty, \end{cases}$$

then u is a solution of

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + y(t) = 0, \quad t \in (0, +\infty),\ t \neq t_k,\ k = 1, 2, \dots, m, \\ &u(t_k^+) - u(t_k^-) = -I_k\bigl(u(t_k)\bigr), \quad k = 1, 2, \dots, m, \qquad u(0) = 0, \\ &D_{0+}^{\alpha-1} u(+\infty) = 0. \end{aligned}$$

We find that this result is wrong. In fact, (16) can be rewritten by

$$\begin{aligned} u(t) = &-\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\,\mathrm{d}s + \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty y(s)\,\mathrm{d}s + \sum_{i=k+1}^{m} \frac{I_i(u(t_i))t^{\alpha-1}}{t_i^{\alpha-1} - t_i^{\alpha-2}} \\ &+ \sum_{i=1}^{k} \frac{I_i(u(t_i))t^{\alpha-2}}{t_i^{\alpha-1} - t_i^{\alpha-2}}, \quad t \in (t_k, t_{k+1}],\ k = 0, 1, 2, \dots. \end{aligned}$$

Hence, we have for $t \in (t_k, t_{k+1}]$ $(k = 1, 2, \dots, m)$ that

$$\begin{aligned} D_{0+}^{\alpha} u(t) &= \frac{1}{\Gamma(2-\alpha)} \left[\int_0^t (t-s)^{1-\alpha} u(s)\,\mathrm{d}s \right]'' \\ &= \frac{1}{\Gamma(2-\alpha)} \left[\sum_{\nu=0}^{k-1} \int_{t_\nu}^{t_{\nu+1}} (t-s)^{1-\alpha} u(s)\,\mathrm{d}s + \int_{t_k}^{t} (t-s)^{1-\alpha} u(s)\,\mathrm{d}s \right]'' \\ &= \frac{1}{\Gamma(2-\alpha)} \left[\sum_{\nu=0}^{k-1} \int_{t_\nu}^{t_{\nu+1}} (t-s)^{1-\alpha} \left(-\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,\mathrm{d}u + \frac{s^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty y(u)\,\mathrm{d}u \right.\right. \\ &\qquad \left.\left. + \sum_{i=\nu+1}^{m} \frac{I_i(u(t_i))s^{\alpha-1}}{t_i^{\alpha-1} - t_i^{\alpha-2}} + \sum_{i=1}^{\nu} \frac{I_i(u(t_i))s^{\alpha-2}}{t_i^{\alpha-1} - t_i^{\alpha-2}} \right) \mathrm{d}s \right]'' \\ &\quad + \frac{1}{\Gamma(2-\alpha)} \left[\int_{t_k}^{t} (t-s)^{1-\alpha} \left(-\int_0^s \frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)} y(u)\,\mathrm{d}u + \frac{s^{\alpha-1}}{\Gamma(\alpha)} \int_0^\infty y(u)\,\mathrm{d}u \right.\right. \\ &\qquad \left.\left. + \sum_{i=k+1}^{m} \frac{I_i(u(t_i))s^{\alpha-1}}{t_i^{\alpha-1} - t_i^{\alpha-2}} + \sum_{i=1}^{k} \frac{I_i(u(t_i))s^{\alpha-2}}{t_i^{\alpha-1} - t_i^{\alpha-2}} \right) \mathrm{d}s \right]'' \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(2-\alpha)}\Bigg[\sum_{\nu=0}^{k-1}\sum_{i=\nu+1}^{m}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}s^{\alpha-1}\,\mathrm{d}s \\
&\quad+\sum_{\nu=0}^{k-1}\sum_{i=1}^{\nu}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}s^{\alpha-2}\,\mathrm{d}s\Bigg]'' \\
&\quad+\frac{1}{\Gamma(2-\alpha)}\Bigg[\sum_{i=k+1}^{m}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_k}^{t}(t-s)^{1-\alpha}s^{\alpha-1}\,\mathrm{d}s \\
&\quad+\sum_{i=1}^{k}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_k}^{t}(t-s)^{1-\alpha}s^{\alpha-2}\,\mathrm{d}s\Bigg]'' \\
&\quad+\frac{1}{\Gamma(2-\alpha)}\Bigg[-\int_0^t(t-s)^{1-\alpha}\int_0^s\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}y(u)\,\mathrm{d}u\,\mathrm{d}s \\
&\quad+\int_0^t(t-s)^{1-\alpha}\frac{s^{\alpha-1}}{\Gamma(\alpha)}\int_0^\infty y(u)\,\mathrm{d}u\,\mathrm{d}s\Bigg]''.
\end{aligned}
$$

By changing the order of sum and integral, we get

$$
\begin{aligned}
D_{0^+}^{\alpha}u(t) &= \frac{1}{\Gamma(2-\alpha)}\Bigg[\sum_{i=1}^{k}\sum_{\nu=0}^{i-1}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}s^{\alpha-1}\,\mathrm{d}s\Bigg]'' \\
&\quad+\frac{1}{\Gamma(2-\alpha)}\Bigg[\sum_{i=k+1}^{m}\sum_{\nu=0}^{k-1}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}s^{\alpha-1}\,\mathrm{d}s \\
&\quad+\sum_{i=1}^{k-1}\sum_{\nu=i}^{k-1}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_\nu}^{t_{\nu+1}}(t-s)^{1-\alpha}s^{\alpha-2}\,\mathrm{d}s\Bigg]'' \\
&\quad+\frac{1}{\Gamma(2-\alpha)}\Bigg[\sum_{i=k+1}^{m}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_k}^{t}(t-s)^{1-\alpha}s^{\alpha-1}\,\mathrm{d}s \\
&\quad+\sum_{i=1}^{k}\frac{I_i(u(t_i))}{t_i^{\alpha-1}-t_i^{\alpha-2}}\int_{t_k}^{t}(t-s)^{1-\alpha}s^{\alpha-2}\,\mathrm{d}s\Bigg]'' \\
&\quad+\frac{1}{\Gamma(2-\alpha)}\Bigg[-\int_0^t\int_u^t(t-s)^{1-\alpha}\frac{(s-u)^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s\,y(u)\,\mathrm{d}u \\
&\quad+\int_0^\infty\int_0^t(t-s)^{1-\alpha}\frac{s^{\alpha-1}}{\Gamma(\alpha)}\,\mathrm{d}s\,y(u)\,\mathrm{d}u\Bigg]''.
\end{aligned}
$$

Use $(s-u)/(t-u) = w$, $s/t = w$.

$$D_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(2-\alpha)} \left[\sum_{i=1}^{k} \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} \int_0^{t_i} (t-s)^{1-\alpha} s^{\alpha-1} \, \mathrm{d}s \right]''$$

$$+ \frac{1}{\Gamma(2-\alpha)} \left[\sum_{i=k+1}^{m} \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} \int_0^{t} (t-s)^{1-\alpha} s^{\alpha-1} \, \mathrm{d}s \right.$$

$$\left. + \sum_{i=1}^{k} \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} \int_{t_i}^{t} (t-s)^{1-\alpha} s^{\alpha-2} \, \mathrm{d}s \right]''$$

$$+ \frac{1}{\Gamma(2-\alpha)} \left[- \int_0^{t} (t-u) \int_0^{1} (1-w)^{1-\alpha} \frac{w^{\alpha-1}}{\Gamma(\alpha)} \, \mathrm{d}w \, y(u) \, \mathrm{d}u \right.$$

$$\left. + \int_0^{\infty} t \int_0^{1} (1-w)^{1-\alpha} \frac{w^{\alpha-1}}{\Gamma(\alpha)} \, \mathrm{d}w \, y(u) \, \mathrm{d}u \right]''.$$

Use $\mathrm{B}(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} \, \mathrm{d}x = \Gamma(p)\Gamma(q)/\Gamma(p+q)$.

$$D_{0^+}^{\alpha} u(t) = y(t) + \frac{1}{\Gamma(2-\alpha)} \left[\sum_{i=1}^{k} \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_0^{t_i/t} (1-w)^{1-\alpha} w^{\alpha-1} \, \mathrm{d}w \right]''$$

$$+ \frac{1}{\Gamma(2-\alpha)} \left[\sum_{i=k+1}^{m} \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_0^{1} (1-w)^{1-\alpha} w^{\alpha-1} \, \mathrm{d}w \right.$$

$$\left. + \sum_{i=1}^{k} \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_{t_i/t}^{1} (1-w)^{1-\alpha} w^{\alpha-2} \, \mathrm{d}w \right]''$$

$$= y(t) + \frac{1}{\Gamma(2-\alpha)} \left[\sum_{i=1}^{k} \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_0^{t_i/t} (1-w)^{1-\alpha} w^{\alpha-1} \, \mathrm{d}w \right]''$$

$$- \frac{1}{\Gamma(2-\alpha)} \left[\sum_{i=1}^{k} \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_0^{t_i/t} (1-w)^{1-\alpha} w^{\alpha-2} \, \mathrm{d}w \right]''$$

$$= y(t) + \frac{1}{\Gamma(2-\alpha)} \left[\sum_{i=1}^{k} \frac{I_i(u(t_i))}{t_i^{\alpha-1} - t_i^{\alpha-2}} t \int_0^{t_i/t} (1-w)^{1-\alpha} \left[w^{\alpha-1} - w^{\alpha-2} \right] \mathrm{d}w \right]''$$

$$\neq y(t), \quad t \in (t_k, t_{k+1}],\ k = 1, 2, \dots, m.$$

Hence, Lemma 3.1 of [15] is wrong.

Remark 3. (See [8, Lemma 3.1].) Let u is a fixed point of the operator A defined for all $x \in PC([0,1]$ by

$$Ax(t) = \int_0^1 G(t,s)f\big(s,x(s)\big)\,\mathrm{d}s + t^{\alpha-1}\sum_{t<t_k<1}\frac{c_k}{1-c_k}t_k^{1-\alpha}x(t_k).$$

It was proved that u is a solution of BVP (1) (x is continuous at each point $t \neq t_i$, right continuous at t_i, the left limit $\lim_{t\to t_i^-} x(t)$ is finite and satisfies (1)). Here

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases}[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1,\\ [t(1-s)]^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1.\end{cases}$$

We find that Lemma of [8] is wrong. In fact, if u is a fixed point of A, then we get

$$x(t) = \int_0^1 G(t,s)f\big(s,x(s)\big)\,\mathrm{d}s + t^{\alpha-1}\sum_{t<t_k<1}\frac{c_k}{1-c_k}t_k^{1-\alpha}x(t_k).$$

It is rewritten by

$$\begin{aligned}x(t) = {}& -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}f\big(s,x(s)\big)\,\mathrm{d}s\\ &+ t^{\alpha-1}\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}f\big(s,x(s)\big)\,\mathrm{d}s + \sum_{i=1}^m \frac{c_i}{1-c_i}t_i^{1-\alpha}x(t_i)\right]\\ &- t^{\alpha-1}\sum_{i=1}^{k-1}\frac{c_i}{1-c_i}t_i^{1-\alpha}x(t_i), \quad t \in [t_{k-1},t_k),\ k = 1,\dots,m+1.\end{aligned}$$

One can easily verifies that $D_{0^+}^{\alpha}x(t) \neq f(t,x(t)), t \in (t_i,t_{i+1}]$, $i = 1,\dots,m$, similarly to above discussion in Remark 2.

4 An example

To illustrate the usefulness of our main results, we present an example that Theorem 1 can be easily applied.

Example 1. Consider the following impulsive fractional differential equation:

$$\begin{aligned}&D_{0^+}^{2/5}x(t) = t^{-1/2}\mathrm{e}^{-t}\left[c_0 + b_0\left(\frac{(t-s)^{3/5}}{1+t^2}x(t)\right)^{\rho} + a_0\left(\frac{(t-s)^{4/5}}{1+t^2}D_{0^+}^{1/5}x(t)\right)^{\rho}\right],\\ &\quad t \in (s,s+1],\ s \in \mathbb{N}_0,\\ &I_{0^+}^{3/5}x(0) = -I_{0^+}^{3/5}x(\infty), \qquad \Delta I_{0^+}^{3/5}x(s) = 0, \quad s \in \mathbb{N},\end{aligned} \tag{17}$$

where $c_0 \geqslant 0$, $b_0 \geqslant 0$, $a_0 \geqslant 0$, $\rho \geqslant 0$ are constants. Then system (17) has at least one solution if one of the following items holds:

(i) $\rho > 1$;

$$a_0 + b_0 \leqslant (\rho-1)^{\rho-1}\left(\rho^\rho\left[\frac{\mathrm{B}(\frac{1}{5},\frac{1}{2})}{\Gamma(\frac{1}{5})} + \frac{\mathrm{B}(\frac{2}{5},\frac{1}{2})}{\Gamma(\frac{2}{5})} + \frac{\Gamma(\frac{2}{5})}{\Gamma(\frac{1}{5})}\frac{\Gamma(\frac{1}{2})}{2\Gamma(\frac{2}{5})} + \frac{\Gamma(\frac{1}{2})}{2\Gamma(\frac{2}{5})}\right]\right)^{-1};$$

(ii) $\rho \in (0,1)$;

(iii) $\rho = 1$,

$$a_0 + b_0 < \left(\frac{\mathrm{B}(\frac{1}{5},\frac{1}{2})}{\Gamma(\frac{1}{5})} + \frac{\mathrm{B}(\frac{2}{5},\frac{1}{5})}{\Gamma(\frac{2}{5})} + \frac{\Gamma(\frac{2}{5})}{\Gamma(\frac{1}{5})}\frac{\Gamma(\frac{1}{2})}{2\Gamma(\frac{2}{5})} + \frac{\Gamma(\frac{1}{2})}{2\Gamma(\frac{2}{5})}\right)^{-1}.$$

Proof. According to (3), we have

(a) $\alpha = 2/5$, $\mu = 1/5 \in (0,\alpha)$, $k_1 = -1$, $c_s = 0$ ($s \in \mathbb{N}_0$) with $c_0 = 0$, c_i satisfies that $\sum_{i=1}^{\infty}|c_i| = 0$, and there exist constants $M_0, M_c > 0$ such that $\prod_{\tau=\omega}^{i-1}|1+c_\tau| \leqslant M_0 = 1$ and $|c_i| \leqslant M_c = 1$ for all ω, $i \in \mathbb{N}$, set

$$A_i = (-1)^{i+1}c_i\sum_{\omega=2}^{i-1}(-1)^\omega c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau) + c_ic_{i-1} + c_i = 0, \quad i \in \mathbb{N};$$

(b) $0 = t_0 < t_1 = 1 < \cdots < t_s = s < \cdots$ with $\lim_{s\to\infty} t_s = \infty$;

(c) $m_1(t) = \mathrm{e}^{-1/2}\mathrm{e}^{-t} \in L^1(0,\infty)$ satisfies $|m_1(t)| \leqslant t^k\mathrm{e}^{-t}$ for almost all $t \in (0,\infty)$ with $k = -1/2$;

(d) Choose $\sigma = 2$ and

$$\delta_1(t) = (t-s)^{3/5}, \quad \delta_2(t) = (t-s)^{4/5}, \quad t \in (s, s+1],\ s \in \mathbb{N}_0,$$

then $f_1(t,x,y) = c_0 + b_0x^\rho + a_0y^\rho$ is a Carathéodory function.

One sees that

$$\begin{aligned}\mathit{\Pi} &= -\Gamma(\alpha)k_1\sum_{i=1}^{\infty}\left[(-1)^{i+1}c_i\sum_{\omega=2}^{i-1}(-1)^\omega c_{\omega-1}\prod_{\tau=\omega}^{i-1}(1+c_\tau) + c_ic_{i-1} + c_i\right] \\ &\quad + (1-k_1)\Gamma(\alpha) \\ &= 2\Gamma\left(\frac{2}{5}\right) \neq 0.\end{aligned}$$

One sees that

(A_σ) $n = 1$, $\sigma_1 = \rho \geqslant 0$, $\psi(t) = c_0$, $a_1 = a_0$, $b_1 = b_0$, and for $t \in (s, s+1]$, $s \in \mathbb{N}$, $x, y \in \mathbb{R}$,

$$\left|f\left(t, \frac{1+t^\sigma}{\delta_1(t)}x, \frac{1+t^\sigma}{\delta_2(t)}y\right) - \psi(t)\right| \leqslant a_1|x|^{\sigma_1} + b_1|y|^{\sigma_1}.$$

We have from $c_i = A_i = 0$ that

$$\Psi(t) = c_0 \int\limits_0^t \frac{(t-s)^{-3/5}}{\Gamma(\frac{2}{5})} s^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s - \frac{c_0}{2\Gamma(\frac{2}{5})} \int\limits_0^\infty s^{-1/2} \mathrm{e}^{-s} \, \mathrm{d}s t^{\alpha-1}, \quad t \in (0, \infty),$$

and $\overline{M} = \Gamma(k+1)|k_1|/|\Pi| = \Gamma(1/2)/(2\Gamma(2/5))$. Then from Theorem 1, BVP (17) has at least one solution if (i) or (ii) or (iii) holds. □

Remark 4. It is easy to see that for sufficiently large $c_0 > 0$ and sufficiently small $a_0, b_0 \geqslant 0$, (17) has at least one solution if $\rho > 1$, (17) has at least one solution if $\rho \in (0,1)$, and (17) has at least one solution for sufficiently small $a_0, b_0 \geqslant 0$.

Remark 5. It is well known that the differential equation $x'(t) = \sum_{j=0}^n f_i(t)[x(t)]^j$ is called Abel differential equation [4]. Hence, the equation in (17) is a fractional-order Abel differential equation. So, (3) contains the following impulsive ecological models as special cases.

Model 1. Fractional order logistic differential equation

$$\begin{aligned}
&D_{0^+}^{\alpha} x(t) = t^{-4/5} \mathrm{e}^{-t} \left(b_0 + b_1 \frac{t^{1/3}}{1+t^{2/3}} x(t) \right), \quad t \in (0, \infty),\ \alpha \in (0,1), \\
&I_{0^+}^{1-\alpha} x(0) = k_1 \lim_{t \to \infty} I_{0^+}^{1-\alpha} x(t), \\
&\Delta I_{0^+}^{1-\alpha} u(t_s) = c_i I_{0^+}^{1-\alpha} \bigl(u(t_s)\bigr), \quad s \in \mathbb{N}.
\end{aligned}$$

Model 2. Forced fractional-order powered logistic equation or fractional-order Riccati equation

$$\begin{aligned}
&D_{0^+}^{2/3} x(t) = \frac{t^{-4/5}}{\mathrm{e}^t} \left(b_0 + b_1 \frac{t^{1/3}}{1+t^{2/3}} x(t) + b_2 \frac{t^{4/3}}{(1+t^{2/3})^2} \bigl[D_{0^+}^{1/3} x(t)\bigr]^2 \right), \quad t \in (0, \infty), \\
&I_{0^+}^{1/3} x(0) = k_1 \lim_{t \to \infty} I_{0^+}^{1/3} x(t), \\
&\Delta I_{0^+}^{1-\alpha} u(t_s) = c_i I_{0^+}^{1-\alpha} \bigl(u(t_s)\bigr), \quad s \in \mathbb{N}.
\end{aligned}$$

Model 3. Forced fractional-order Bernoulli equation

$$\begin{aligned}
&D_{0^+}^{2/3} x(t) = \frac{t^{-4/5}}{\mathrm{e}^t} \left(b_0 + b_1 \frac{t^{1/3}}{1+t^{2/3}} x(t) + b_2 \frac{t^{2n/3}}{(1+t^{2/3})^n} \bigl[D_{0^+}^{1/3} x(t)\bigr]^n \right), \quad t \in (0, \infty), \\
&I_{0^+}^{1/3} x(0) = k_1 \lim_{t \to \infty} I_{0^+}^{1/3} x(t), \\
&\Delta I_{0^+}^{1-\alpha} u(t_s) = c_i I_{0^+}^{1-\alpha} \bigl(u(t_s)\bigr), \quad s \in \mathbb{N}.
\end{aligned}$$

By applying Theorem 1, we can establish the global existence results for solutions of these kinds of models. We omit the details.

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