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Controllability of nonlinear stochastic neutral fractional dynamical systems*

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Abstract. In this paper, we obtain an equivalent nonlinear integral equation to the stochastic neutral fractional system with bounded operator. Using the integral equation, the sufficient conditions for ensuring the complete controllability of the stochastic fractional neutral systems with Wiener and Lévy noise are obtained. Banach's fixed point theorem is used to obtain the results. Examples are provided to illustrate the theory.

Keywords: stochastic fractional differential equation, controllability, neutral systems, Wiener process, Lévy process.

1 Introduction

Controllability is a qualitative property of dynamical systems and is of particular importance in control theory. Theory of controllability originates from the famous work of Kalman in 1960, where the concept of controllability was defined for finite dimensional deterministic linear systems. The natural extension of the concept of controllability to infinite dimensional systems is studied by many authors. A discussion on the concepts of controllability of infinite dimensional systems can be found in [2, 6, 8].

In recent years, fractional differential equations (FDEs) have attracted considerable interest due to its ability to model complex phenomena by capturing nonlocal relations in space and time. At the same time, the fluctuations in nature can be captured only by adding random elements into the differential equations, which are called stochastic differential equations (SDEs). Also, in many applications, one assumes that the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past states and is determined solely by the present. However, under closer scrutiny, it becomes apparent that the principle of causality is

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often only a first approximation to the true situation and that a more realistic model would include some of the past states of the system. There are also a number of applications in which the delayed argument occurs in the derivative of the state variable as well as in the independent variable, the so-called neutral differential difference equations. Such problems are more difficult to motivate but often arise in the study of two or more simple oscillatory systems with some interconnections between them. In some cases, the connection can be replaced by differential equations involving delays in the highest order derivatives. Neutral differential equations are encountered in the description of various physical scenarios like the lossless transmission connection, stunted transmission connection [7], vibrating masses attached to an elastic bar [11], and collision problem in electrodynamics [10]. The controllability of such systems are studied in [12] and the references therein. Therefore, the investigation of fractional neutral differential equations with stochastic nature attracts great attention, especially as regards to controllability.

The controllability of fractional and stochastic dynamical systems have been studied by many authors separately. The controllability of linear and nonlinear fractional dynamical systems is studied in [4] and the references therein. The natural extension of the controllability concepts from deterministic to stochastic control systems has no meaning. Therefore, there is a need in further weakening of these concepts in order to extend them to stochastic control systems. For the controllability of SDEs, one can refer to [3,5,18,19, 23]. It is worth pointing out that most of the works on controllability of stochastic systems only focused on the case of SDEs driven by a Brownian motion [9]. Unfortunately, the fluctuations in financial markets, sudden changes in the environment, and many other real systems cannot be described by Brownian motion, and this leads to the use of Lévy noise to model such discontinuous systems. Lévy processes have stationary and independent increments, their sample paths are right continuous having number of discontinuities at random times, and they are special classes of semi martingales and Markov processes. Along with these advantages, Lévy processes have applications in diverse fields like mathematical finance, financial economics, stochastic control, and quantum field theory. These form the reason for making the study of SDEs with Lévy noise important inspite of its increased mathematical complexities. A detailed study of Lévy process in finite and infinite dimensions can be found in [1, 20] and the references therein.

In [4], controllability of linear system of the form

$$\begin{aligned} {}^{C}D^{\alpha}x(t) &= Ax(t) + Bu(t), \quad t \in [0, T], \\ x(0) &= x_0, \end{aligned} \tag{1}$$

where $1/2 < \alpha \leqslant 1$ and A and B are bounded linear operators, is investigated. The controllability of stochastic counterpart of the above fractional dynamical integro-differential systems is studied in [15]. In this paper, our aim is to extend the results to stochastic neutral fractional dynamical system driven by Wiener and Lévy noise. The Lévy–Itô decomposition of an arbitrary Lévy process into Brownian and Poisson parts is used to study the stochastic fractional system with Lévy noise. Examples are provided to support the developed theory.

2 Preliminaries

Let X, U, and K be separable Hilbert spaces, and for convenience, we will use the same notation $\|\cdot\|$ to represent their norms. $\mathbb{L}(X,U)$ is the space of all bounded linear operators from X to U, $\mathbb{L}_p(X)$ is the Lebesgue space of p-integrable functions on X, $\mathcal{B}(X)$ is the Borel σ-algebra of subsets of X, and J denotes the interval $[0,T]$.

We assume that a filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geqslant 0},\mathbf{P})$ with the probability measure $\mathbf{P}$ on Ω satisfies the *usual hypothesis*:

(i) $\mathcal{F}_0$ contains all $A\in\mathcal{F}$ such that $\mathbf{P}(A)=0$,
(ii) $\mathcal{F}_t=\mathcal{F}_{t^+}$ for all $t\in J$, where $\mathcal{F}_{t^+}$ is the intersection of all $\mathcal{F}_s$, $s>t$, i.e., the filtration is right continuous.

Let us consider the following space settings:

- $Y:=\mathbb{L}_2(\Omega,\mathcal{F}_T,X)$ is a closed subspace of $\mathbb{L}_2(\Omega,X)$ consisting of all $\mathcal{F}_T$-measurable square integrable random variables with values in X.
- $\mathcal{H}_2$ is a closed subspace of $C(J,\mathbb{L}_2(\Omega,X))$ consisting of all $\mathcal{F}_t$-measurable processes with values in X, identifying processes, which are modification of each other and endowed with the norm,
$$\|\phi\|^2_{\mathcal{H}_2}=\sup_{t\in J}\mathbf{E}\big\|\phi(t)\big\|^2,$$
where $\mathbf{E}$ denotes expectation with respect to $\mathbf{P}$.
- $U_{\mathrm{ad}}:=\mathbb{L}_2^{\mathcal{F}}(J,U)$ is a Hilbert space of all square integrable and $\mathcal{F}_t$-measurable processes with values in U.
- $\mathcal{H}_2^0:=\mathbb{L}_2(\Omega,\mathcal{F}_0,X)$ is the Hilbert space of all $\mathcal{F}_0$-measurable square integrable random variables with values in X.

Let us recall some basic definitions from fractional calculus. Let $\alpha,\beta>0$ with $n-1<\alpha,\beta<n$ and $n\in\mathbb{N}$. Suppose $f\in\mathbb{L}_1(\mathbb{R}_+)$, $\mathbb{R}_+=[0,\infty)$.

Definition 1. (See [13].) The Riemann–Liouville fractional integral of a function f is defined as
$$I^\alpha f(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s)\,\mathrm{d}s,$$
and the Caputo derivative of f is ${}^CD^\alpha f=I^{n-\alpha}D^nf$, that is,
$${}^CD^\alpha f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t(t-s)^{n-\alpha-1}f^{(n)}(s)\,\mathrm{d}s,$$
where the function $f(t)$ has absolutely continuous derivative up to order $n-1$.

Definition 2. (See [13].) Let A be a bounded linear operator, the Mittag–Leffler operator function is given by

$$E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha + \beta)}.$$

In particular, for $\beta = 1$,

$$E_{\alpha,1}(A) = E_\alpha(z) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(k\alpha + 1)}.$$

3 Controllability results for systems with Wiener noise

In this section, we obtain sufficient conditions for the controllability of nonlinear stochastic fractional neutral differential system

$$\begin{aligned}
&{}^C D^\alpha \big(x(t) - g\big(t, x(t)\big)\big) \\
&\quad = Ax(t) + Bu(t) + f\big(t, x(t)\big) + \sigma\big(t, x(t)\big)\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t \in J, \\
&x(0) = x_0,
\end{aligned} \tag{2}$$

where $0 < \alpha \leqslant 1$, $\alpha \neq 1/2$, $A : X \to X$ is a bounded linear operator, $W(t)$ is a K-valued Wiener process with positive symmetric trace class covariance operator, $\sigma : J \times X \to \mathbb{L}_2^0(K, X)$ (where $\mathbb{L}_2^0$ is the space of Hilbert–Schmidt operators [22]), functions $f, g : J \times X \to X$ are continuous, and g is continuously differentiable, $u \in U_{\mathrm{ad}}$, a Hilbert space of admissible control functions, and $B : U \to X$ is a bounded linear operator.

Lemma 1. (*See* [14].) *Suppose that A is a linear bounded operator defined on a Banach space, and assume that $\|A\| < 1$. Then $(I - A)^{-1}$ is linear, bounded, and*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

The convergence of the above series is in the operator norm, and $\|(I - A)^{-1}\| \leqslant (1 - \|A\|)^{-1}$.

Let us assume the following hypothesis:

(H1) The operator $A \in \mathbb{L}(X)$ and $\|A\|^2 < (2\alpha - 1)\Gamma^2(\alpha)/T^{2\alpha}$.

Let $x \in \mathcal{H}_2$, then by (H1) we have

$$\begin{aligned}
\big\|(I^\alpha A)x\big\|_{\mathcal{H}_2} &\leqslant \frac{T}{\Gamma^2(\alpha)} \sup_{t\in J} \int_0^t (t-s)^{2\alpha-2}\mathbf{E}\big\|Ax(s)\big\|_X^2 \,\mathrm{d}s \\
&\leqslant \frac{T^{2\alpha}}{(2\alpha - 1)\Gamma^2(\alpha)} \sup_{t\in J} \mathbf{E}\|Ax\|_X^2 < \|x\|_{\mathcal{H}_2},
\end{aligned}$$

which implies that $\|I^\alpha A\| < 1$. Hence, by Lemma 1 we conclude that $(I - I^\alpha A)^{-1}$ is a bounded linear operator satisfying $(I - I^\alpha A)^{-1} = \sum_{k=0}^{\infty}(I^\alpha A)^k$ and $\|(I - I^\alpha A)^{-1}\| \leqslant 1/(1 - \|I^\alpha A\|)$. On the other hand, taking I^α on both sides of (2), we have

$$\begin{aligned} x(t) &= x_0 - g(0, x_0) + g\big(t, x(t)\big) + I^\alpha A x(t) + I^\alpha B u(t) + I^\alpha f\big(t, x(t)\big) \\ &\quad + I^\alpha \sigma\big(t, x(t)\big)\frac{\mathrm{d}W(t)}{\mathrm{d}t} \\ &= \big(I - I^\alpha A\big)^{-1}\bigg(x_0 - g(0, x_0) + g(t, x(t)) + I^\alpha B u(t) + I^\alpha f\big(t, x(t)\big) \\ &\quad + I^\alpha \sigma\big(t, x(t)\big)\frac{\mathrm{d}W(t)}{\mathrm{d}t}\bigg). \end{aligned}$$

Therefore, using Lemma 1 and the fact that I^α commutes with A, we obtain

$$\begin{aligned} x(t) &= \sum_{k=0}^{\infty}\big(I^\alpha A\big)^k\bigg(x_0 - g(0, x_0) + g\big(t, x(t)\big) + I^\alpha B u(t) + I^\alpha f\big(t, x(t)\big) \\ &\quad + I^\alpha \sigma\big(t, x(t)\big)\frac{\mathrm{d}W(t)}{\mathrm{d}t}\bigg) \\ &= \sum_{k=0}^{\infty} I^{k\alpha} A^k\big[x_0 - g(0, x_0)\big] + g\big(t, x(t)\big) + \sum_{k=1}^{\infty} I^{k\alpha} A^k g\big(t, x(t)\big) \\ &\quad + \sum_{k=0}^{\infty} I^{k\alpha+\alpha} A^k\bigg[Bu(t) + f\big(t, x(t)\big) + \sigma\big(t, x(t)\big)\frac{\mathrm{d}W(t)}{\mathrm{d}t}\bigg] \\ &= \sum_{k=0}^{\infty}\frac{A^k t^{\alpha k}}{\Gamma(k\alpha + 1)}\big[x_0 - g(0, x_0)\big] \\ &\quad + \int_0^t A(t-s)^{\alpha-1}\sum_{k=0}^{\infty}\frac{A^k (t-s)^{\alpha k}}{\Gamma(k\alpha + \alpha)} g\big(s, x(s)\big)\,\mathrm{d}s \\ &\quad + g\big(t, x(t)\big) + \int_0^t (t-s)^{\alpha-1}\sum_{k=0}^{\infty}\frac{A^k (t-s)^{\alpha k}}{\Gamma(k\alpha + \alpha)}\big(f\big(s, x(s)\big) + Bu(s)\big)\,\mathrm{d}s \\ &\quad + \int_0^t (t-s)^{\alpha-1}\sum_{k=0}^{\infty}\frac{A^k (t-s)^{\alpha k}}{\Gamma(k\alpha + \alpha)}\sigma\big(s, x(s)\big)\,\mathrm{d}W(s) \\ &= E_\alpha\big(At^\alpha\big)\big[x_0 - g(0, x_0)\big] + g\big(t, x(t)\big) \\ &\quad + \int_0^t A(t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) g\big(s, x(s)\big)\,\mathrm{d}s \end{aligned}$$

$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)f\big(s,x(s)\big)\,\mathrm{d}s$$
$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)Bu(s)\,\mathrm{d}s$$
$$+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\sigma\big(s,x(s)\big)\,\mathrm{d}W(s). \tag{3}$$

Thus, the solution of (2) is the solution of the above nonlinear integral equation (3).

Similarly to the conventional controllability concept, the controllability of the stochastic fractional dynamical system is defined as follows: the set of all states attainable from x_0 in time $t > 0$ is given by the set

$$\mathcal{R}_t(x_0) = \big\{x(t)\colon\ u \in U_{\mathrm{ad}}\big\},$$

where $x(t)$ is given in (3).

Definition 3. (See [18].) The stochastic fractional system (2) is said to be completely controllable on the interval J if for every $x_1 \in Y$, there exists a control $u \in U_{\mathrm{ad}}$ such that the solution $x(t)$ given in (3) satisfies $x(T) = x_1$.

In other words,

$$\mathcal{R}_T(x_0) = Y.$$

Define the operator $L_T : U_{\mathrm{ad}} \to X$ as (see [18])

$$L_T u = \int_0^T E_{\alpha,\alpha}\big(A(T-s)^\alpha\big)Bu(s)\,\mathrm{d}s.$$

Clearly, the adjoint operator L_T^* of L_T satisfying $L_T^* \in \mathbb{L}(X, U_{\mathrm{ad}})$ is obtained as

$$(L_T^* x)(t) = B^* E_{\alpha,\alpha}\big(A^*(T-t)^\alpha\big)\mathbf{E}\{x \mid \mathcal{F}_t\}.$$

Definition 4. (See [21].) The controllability Grammian operator $\mathcal{W}_T : X \to X$ is defined as

$$\mathcal{W}_T z = \int_0^T E_{\alpha,\alpha}\big(A(T-s)^\alpha\big)BB^* E_{\alpha,\alpha}\big(A^*(T-s)^\alpha\big)\mathbf{E}\{z \mid \mathcal{F}_s\}\,\mathrm{d}s,$$

where $*$ denotes adjoint operator.

The corresponding deterministic operator $\varGamma_T : X \to X$ is given by

$$\varGamma_s^T x = \int_s^T E_{\alpha,\alpha}\big(A(T-s)^\alpha\big)BB^* E_{\alpha,\alpha}\big(A^*(T-s)^\alpha\big)x\,\mathrm{d}s.$$

The linear system corresponding to (2) is

$$\begin{aligned} &{}^{C}D^{\alpha}\big(x(t)-g(t)\big)=Ax(t)+Bu(t)+f(t)+\sigma(t)\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t\in J,\\ &x(0)=x_0, \end{aligned} \tag{4}$$

where the functions $f, g : J \to X$ are continuous and g is continuously differentiable.

Theorem 1. *The fractional system* (4) *is controllable on* J *if and only if for some* $\gamma > 0$*,*

$$\langle \mathcal{W}_T x, x\rangle_X \geqslant \gamma\|x\|_X^2 \quad \forall x\in X.$$

The proof is similar to that of the integer order case given in [17], provided that the relation between $\mathcal{W}_T$ and Γ_s^T [17, Lemma 5] remains the same for the fractional order case. The following lemma asserts that the relation between $\mathcal{W}_T$ and Γ_s^T remains the same even for the fractional order systems.

Lemma 2. *For every* $z \in Y$*, there exists a process* $\phi(\cdot) \in \mathbb{L}_2^{\mathcal{F}}(J, L(K, X))$ *such that*:

(i) $z = \mathbf{E}z + \int_0^T \phi(s)\,\mathrm{d}W(s)$;

(ii) $\mathcal{W}_T z = \Gamma_T \mathbf{E}z + \int_0^T \Gamma_{T-s}\phi(s)\,\mathrm{d}W(s)$.

Proof. (i) can be obtained as in [17].

Now, we prove (ii). Let $z \in \mathbb{L}_2(\Omega, \mathcal{F}_T, X)$, then from the first equality we have

$$\mathbf{E}\{z \mid \mathcal{F}_t\} = \mathbf{E}z + \int_0^t \phi(s)\,\mathrm{d}W(s).$$

Now, the definition of the operator and stochastic Fubini's theorem lead to the desired representation:

$$\begin{aligned} \mathcal{W}_T z &= \int_0^T E_{\alpha,\alpha}\big(A(T-t)^\alpha\big)BB^*E_{\alpha,\alpha}\big(A^*(T-t)^\alpha\big)\mathbf{E}\{z\mid\mathcal{F}_t\}\,\mathrm{d}t\\ &= \int_0^T E_{\alpha,\alpha}\big(A(T-t)^\alpha\big)BB^*E_{\alpha,\alpha}\big(A^*(T-t)^\alpha\big)\left[\mathbf{E}z+\int_0^t\phi(s)\,\mathrm{d}W(s)\right]\mathrm{d}t\\ &= \Gamma_0^T\mathbf{E}z+\int_0^T\int_s^T E_{\alpha,\alpha}\big(A(T-t)^\alpha\big)BB^*E_{\alpha,\alpha}\big(A^*(T-t)^\alpha\big)\phi(s)\,\mathrm{d}t\,\mathrm{d}W(s)\\ &= \Gamma_T\mathbf{E}z+\int_0^T \Gamma_s^T\phi(s)\,\mathrm{d}W(s). \end{aligned}$$

This completes the proof of the lemma. □

For simplicity, take

$$M_1 = \max_{0\leqslant s\leqslant t\leqslant T} \bigl\|E_\alpha\bigl(At^\alpha\bigr)\bigr\|^2, \qquad M_2 = \max_{0\leqslant s\leqslant t\leqslant T} \bigl\|E_{\alpha,\alpha}\bigl(At^\alpha\bigr)\bigr\|^2,$$

$$N_2 = \max_{t\in J} \bigl\|g(t,0)\bigr\|^2, \qquad N_4 = \max_{t\in J} \bigl\|f(t,0)\bigr\|^2, \qquad N_6 = \max_{t\in J} \bigl\|\sigma(t,0)\bigr\|^2.$$

Let us further assume the following conditions:

(H2) $g : J \times X \to X$ is continuous, and there exists a constant $N_1 > 0$ such that

$$\bigl\|g(t,x_1) - g(t,x_2)\bigr\|_X^2 \leqslant N_1\|x_1 - x_2\|_X^2 \quad \forall x_1, x_2 \in X.$$

(H3) $f : J \times X \to X$ is continuous, and there exists a constant $N_3 > 0$ such that

$$\bigl\|f(t,x_1) - f(t,x_2)\bigr\|_X^2 \leqslant N_3\|x_1 - x_2\|_X^2 \quad \forall x_1, x_2 \in X.$$

(H4) $\sigma : J \times X \to \mathbb{L}_2^0$ is continuous, and there exists a constant $N_5 > 0$ such that

$$\bigl\|\sigma(t,x_1) - \sigma(t,x_2)\bigr\|_{\mathbb{L}_2^0}^2 \leqslant N_5\|x_1 - x_2\|_X^2 \quad \forall x_1, x_2 \in X.$$

(H5) Let $\rho_1 = 16(N_1 + (2\alpha-1)\Gamma^2(\alpha)N_1M_2 + T^{2\alpha}N_3M_2 + T^{2\alpha-1}N_5M_2)$ be such that $0 \leqslant \rho_1 < 1$.

Theorem 2. *If hypothesis* (H1)–(H5) *are satisfied and if the linear stochastic fractional neutral system corresponding to* (2) *is completely controllable, then the nonlinear stochastic fractional neutral system* (2) *is completely controllable.*

Proof. Let x_1 be an arbitrary random variable in Y. Define the operator Φ on $\mathcal{H}_2$ by

$$\begin{aligned}
\Phi x(t) &= E_\alpha\bigl(At^\alpha\bigr)\bigl[x_0 - g(0,x_0)\bigr] + g\bigl(t,x(t)\bigr) \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(A(t-s)^\alpha\bigr)Bu(s)\,\mathrm{d}s \\
&\quad + \int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(A(t-s)^\alpha\bigr)g\bigl(s,x(s)\bigr)\,\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(A(t-s)^\alpha\bigr)f\bigl(s,x(s)\bigr)\,\mathrm{d}s \\
&\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\bigl(A(t-s)^\alpha\bigr)\sigma\bigl(s,x(s)\bigr)\,\mathrm{d}W(s).
\end{aligned}$$

Since the linear system corresponding to the nonlinear system (2) is controllable, we have that $\mathcal{W}_T$ is invertible (see [15]). Define the control variable u as

$$\begin{aligned}
u(t) &= (T-t)^{1-\alpha} B^* E_{\alpha,\alpha}\big(A^*(T-t)^\alpha\big) \\
&\quad \times \mathbf{E}\Bigg\{\mathcal{W}_T^{-1}\Bigg(x_1 - E_\alpha\big(AT^\alpha\big)\big(x_0 - g(0,x_0)\big) \\
&\qquad - g\big(T,x(T)\big) - \int_0^T A(T-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(T-s)^\alpha\big)g\big(s,x(s)\big)\,\mathrm{d}s \\
&\qquad - \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(T-s)^\alpha\big)f\big(s,x(s)\big)\,\mathrm{d}s \\
&\qquad - \int_0^T (T-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(T-s)^\alpha\big)\sigma\big(s,x(s)\big)\,\mathrm{d}W(s)\Bigg) \;\Big|\; \mathcal{F}_t\Bigg\}.
\end{aligned}$$

We now show that Φ has a fixed point. This fixed point is then a solution of the control problem. Clearly, $\Phi(x(T)) = x_1$, which means that the control u steers the nonlinear system from the initial state x_0 to x_1 in the time T, provided we can obtain a fixed point of the nonlinear operator Φ. First, we show that Φ maps $\mathcal{H}_2$ into itself. From the assumptions we have

$$\begin{aligned}
\sup_{t\in J}\mathbf{E}\big\|\Phi x(t)\big\|^2 &\leqslant 7M_1\mathbf{E}\big(\|x_0\|^2 + \big\|g(0,x_0)\big\|^2\big) \\
&\quad + 7\Big(N_1\sup_{t\in J}\mathbf{E}\big\|x(t)\big\|^2 + N_2\Big)\big(1 + M_2\Gamma^2(\alpha)\big) \\
&\quad + 49M_2\|B\|^2\|L_T^*\|^2\big\|\mathcal{W}_T^{-1}\big\|^2\Big[\mathbf{E}\|x_1\|^2 + \mathbf{E}\big\|g\big(T,x(T)\big)\big\|^2 \\
&\quad + M_1\mathbf{E}\big(\|x_0\|^2 + \big\|g(0,x_0)\big\|^2\big) \\
&\quad + M_2\frac{T^{2\alpha-1}}{2\alpha-1}\Big((TN_3+N_5)\sup_{t\in J}\mathbf{E}\big\|x(t)\big\|^2 + TN_4 + N_6\Big) \\
&\quad + M_2\Gamma^2(\alpha)\Big(N_1\sup_{t\in J}\mathbf{E}\big\|x(t)\big\|^2 + N_2\Big)\Big] \\
&\quad + 7M_2\frac{T^{2\alpha-1}}{2\alpha-1}\Big((N_3T+N_5)\sup_{t\in J}\mathbf{E}\big\|x(t)\big\|^2 + N_4T + N_6\Big). \qquad (5)
\end{aligned}$$

From (5) it follows that there exists a constant $C_1 > 0$ such that

$$\sup_{t\in J}\mathbf{E}\big\|\Phi x(t)\big\|^2 \leqslant C_1\Big(1 + \sup_{t\in J}\mathbf{E}\big\|x(t)\big\|^2\Big).$$

Thus, Φ maps $\mathcal{H}_2$ into itself. Now, for $x_1, x_2 \in \mathcal{H}_2$, we have

$$\begin{aligned}
&\sup_{t\in J}\mathbf{E}\big\|\Phi x_1(t) - \Phi x_2(t)\big\|_X^2 \\
&\quad = \sup_{t\in J}\mathbf{E}\Bigg\| g\big(t, x_1(t)\big) - g\big(t, x_2(t)\big) \\
&\qquad + \int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big(g\big(s, x_1(s)\big) - g\big(s, x_2(s)\big)\big)\,\mathrm{d}s \\
&\qquad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)BL_T^*\mathcal{W}_T^{-1} \\
&\qquad\quad \times \Bigg[g\big(T, x_1(T)\big) - g\big(T, x_2(T)\big) \\
&\qquad\quad + \int_0^T A(T-\theta)^{\alpha-1}E_{\alpha,\alpha}\big(A(T-\theta)^\alpha\big)\big(g\big(\theta, x_1(\theta)\big) - g\big(\theta, x_2(\theta)\big)\big)\,\mathrm{d}\theta \\
&\qquad\quad + \int_0^T (T-\theta)^{\alpha-1}E_{\alpha,\alpha}\big(A(T-\theta)^\alpha\big)\big(\sigma\big(\theta, x_1(\theta)\big) - \sigma\big(\theta, x_2(\theta)\big)\big)\,\mathrm{d}W(\theta) \\
&\qquad\quad + \int_0^T (T-\theta)^{\alpha-1}E_{\alpha,\alpha}\big(A(T-\theta)^\alpha\big)\big(f\big(\theta, x_1(\theta)\big) - f\big(\theta, x_2(\theta)\big)\big)\,\mathrm{d}\theta \Bigg]\,\mathrm{d}s \\
&\qquad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big(f\big(s, x_1(s)\big) - f\big(s, x_2(s)\big)\big)\,\mathrm{d}s \\
&\qquad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(T-s)^\alpha\big)\big[\sigma\big(s, x_1(s)\big) - \sigma\big(s, x_2(s)\big)\big]\,\mathrm{d}W(s)\Bigg\|^2 \\
&\quad \leqslant 16\sup_{t\in J}\mathbf{E}\big\|x_1(t) - x_2(t)\big\|_X^2 \\
&\qquad \times \big(N_1 + (2\alpha-1)\Gamma^2(\alpha)2N_1M_2 + T^{2\alpha}N_3M_2 + T^{2\alpha-1}N_5M_2\big) \\
&\quad \leqslant \rho_1\sup_{t\in J}\mathbf{E}\big\|x_1(t) - x_2(t)\big\|_X^2.
\end{aligned}$$

Using (H5), we conclude that Φ is a contraction mapping, and hence, there exists a unique fixed point $x \in \mathcal{H}_2$ for Φ. This fixed point of Φ satisfies $x(T) = x_1$ for any arbitrary $x_1 \in Y$. Therefore, system (2) is completely controllable on J. □

4 Controllability results for systems with Lévy noise

Consider the nonlinear stochastic neutral fractional differential system driven by Lévy noise of the form

$$\begin{aligned}
&{}^{C}D^{\alpha}\big(x(t) - g\big(t, x(t)\big)\big) \\
&\quad = Ax(t) + Bu(t) + f\big(t, x(t)\big) + \sigma\big(t, x(t)\big)\frac{\mathrm{d}W(t)}{\mathrm{d}t} + \int_Z h\big(t, x(t), z\big)\frac{\mathrm{d}\widetilde{N}(t, z)}{\mathrm{d}t}, \quad (6)\\
&x(0) = x_0.
\end{aligned}$$

Here $\mathrm{d}\widetilde{N}(t,z) = \widetilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - \nu(\mathrm{d}z)\mathrm{d}t$ is a compensated Poisson random measure, where $N(\mathrm{d}t, \mathrm{d}z)$ denotes the Poisson random measure associated to Poisson point process on $Z \in \mathcal{B}(X)$, and $\nu(\mathrm{d}z)$ is a σ-finite Lévy measure on $(Z, \mathcal{B}(Z))$, $h : J \times Z \times X \to X$ is a continuous function satisfying $\int_0^T \int_Z \mathbf{E}\|h(s, x(s), z)\|^2 \,\nu(\mathrm{d}z)\,\mathrm{d}t < \infty$. If hypothesis (H1) is satisfied, then by the Lemma 1 the solution of system (2) is the same as the solution of the following nonlinear integral equation:

$$\begin{aligned}
x(t) = {}& E_\alpha\big(At^\alpha\big)\big[x_0 - g(0, x_0)\big] + g\big(t, x(t)\big) \\
&+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) Bu(s)\,\mathrm{d}s \\
&+ \int_0^t A(t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) g\big(s, x(s)\big)\,\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) f\big(s, x(s)\big)\,\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) \sigma\big(s, x(s)\big)\,\mathrm{d}W(s) \\
&+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) \int_Z h\big(s, x(s), z\big)\,\widetilde{N}(\mathrm{d}s, \mathrm{d}z), \qquad (7)
\end{aligned}$$

which can be obtained similar to (3). We assume the following conditions:

(H6) $h : J \times X \times Z \to X$ is continuous, and there exists a constant $N_7 > 0$ such that

$$\int_Z \big\|h(t, x_1, z) - h(t, x_2, z)\big\|_X^2 \,\nu(\mathrm{d}z) \leqslant N_7 \|x_1 - x_2\|_X^2 \quad \forall x_1, x_2 \in X.$$

(H7) Let $\rho_2 = 16(N_1 + (2\alpha-1)\Gamma^2(\alpha)N_1M_2 + T^{2\alpha}N_3M_2 + T^{2\alpha-1}(N_5 + N_7)M_2)$ be such that $0 \leqslant \rho_2 < 1$.

Also, we denote $N_8 = \max_{t \in J} \int_Z \|h(t, 0, z)\|_X^2 \,\nu(\mathrm{d}z)$.

Theorem 3. *If hypothesis* (H1)–(H4), (H6), *and* (H7) *are satisfied and if the linear neutral fractional system corresponding to* (6) *is completely controllable, then the nonlinear neutral fractional system driven by Lévy noise* (6) *is completely controllable.*

Proof. Let x_1 be an arbitrary random variable in Y. Define the operator Φ on $\mathcal{H}_2$ by

$$\begin{aligned}\Phi x(t) = {} & E_\alpha\big(At^\alpha\big)\big[x_0 - g(0,x_0)\big] + g\big(t,x(t)\big) \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) Bu(s)\,\mathrm{d}s \\ & + \int_0^t A(t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) g\big(s,x(s)\big)\,\mathrm{d}s \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) f\big(s,x(s)\big)\,\mathrm{d}s \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) \sigma\big(s,x(s)\big)\,\mathrm{d}W(s) \\ & + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) \int_Z h\big(s,x(s),z\big)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}z).\end{aligned}$$

Since the linear system corresponding to the nonlinear system (6) is completely controllable, we have that $\mathcal{W}_T$ is invertible [16]. Define the control variable u as

$$\begin{aligned}u(t) = {} & (T-t)^{1-\alpha} B^* E_{\alpha,\alpha}\big(A^*(T-t)^\alpha\big) \\ & \times \mathbf{E}\Bigg\{ \mathcal{W}_T^{-1}\Bigg(x_1 - E_\alpha\big(AT^\alpha\big)\big(x_0 - g\big(0,x_0\big)\big) \\ & \quad - g\big(T,x(T)\big) - \int_0^T A(T-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(T-s)^\alpha\big) g\big(s,x(s)\big)\,\mathrm{d}s \\ & \quad - \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(T-s)^\alpha\big) f\big(s,x(s)\big)\,\mathrm{d}s \\ & \quad - \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(T-s)^\alpha\big) \sigma\big(s,x(s)\big)\,\mathrm{d}W(s) \\ & \quad - \int_0^T (T-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(T-s)^\alpha\big) \int_Z h\big(s,x(s),z\big)\,\widetilde{N}(\mathrm{d}s,\mathrm{d}z) \Bigg) \;\Big|\; \mathcal{F}_t \Bigg\}.\end{aligned}$$

We now show that Φ has a fixed point. This fixed point is then a solution of the control problem. Clearly, $\Phi(x(T)) = x_1$, which means that the control u steers the nonlinear system from the initial state x_0 to x_1 in the time T, provided we can obtain a fixed point of the nonlinear operator Φ. First, we show that Φ maps $\mathcal{H}_2$ into itself. From the assumptions we have

$$\begin{aligned}
&\sup_{t\in J}\mathbf{E}\big\|\Phi x(t)\big\|^2 \\
&\quad\leqslant 8M_1\mathbf{E}\big(\|x_0\|^2+\big\|g(0,x_0)\big\|^2\big)+8\Big(N_1\sup_{t\in J}\mathbf{E}\big\|x(t)\big\|^2+N_2\Big)\big(1+M_2\Gamma^2(\alpha)\big) \\
&\qquad+64M_2\|B\|^2\|L_T^*\|^2\big\|\mathcal{W}_T^{-1}\big\|^2\Big[\mathbf{E}\|x_1\|^2+\mathbf{E}\big\|g\big(T,x(T)\big)\big\|^2 \\
&\qquad+M_1\mathbf{E}\big(\|x_0\|^2+\big\|g(0,x_0)\big\|^2\big) \\
&\qquad+M_2\frac{T^{2\alpha-1}}{2\alpha-1}\Big((TN_3+N_5+N_7)\sup_{t\in J}\mathbf{E}\big\|x(t)\big\|^2+TN_4+N_6+N_8\Big) \\
&\qquad+M_2\Gamma^2(\alpha)2\Big(N_1\sup_{t\in J}\mathbf{E}\big\|x(t)\big\|^2+N_2\Big)\Big] \\
&\qquad+8M_2\frac{T^{2\alpha-1}}{2\alpha-1}\Big((N_3T+N_5+N_7)\sup_{t\in J}\mathbf{E}\big\|x(t)\big\|^2+N_4T+N_6+N_8\Big).
\end{aligned}\tag{8}$$

From (8) it follows that there exists a constant $C_2 > 0$ such that

$$\sup_{t\in J}\mathbf{E}\big\|\Phi x(t)\big\|^2\leqslant C_2\Big(1+\sup_{t\in J}\mathbf{E}\big\|x(t)\big\|^2\Big).$$

Thus, Φ maps $\mathcal{H}_2$ into itself. Now, for $x_1, x_2 \in \mathcal{H}_2$, we have

$$\begin{aligned}
&\sup_{t\in J}\mathbf{E}\big\|\Phi x_1(t)-\Phi x_2(t)\big\|_X^2 \\
&\quad=\sup_{t\in J}\mathbf{E}\bigg\|g\big(t,x_1(t)\big)-g\big(t,x_2(t)\big) \\
&\qquad+\int_0^t A(t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big(g\big(s,x_1(s)\big)-g\big(s,x_2(s)\big)\big)\,\mathrm{d}s \\
&\qquad+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)BL_T^*\mathcal{W}_T^{-1}\Big[g\big(T,x_1(T)\big)-g\big(T,x_2(T)\big) \\
&\qquad\quad+\int_0^T A(T-\theta)^{\alpha-1}E_{\alpha,\alpha}\big(A(T-\theta)^\alpha\big)\big(g\big(\theta,x_1(\theta)\big)-g\big(\theta,x_2(\theta)\big)\big)\,\mathrm{d}\theta
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^T (T-\theta)^{\alpha-1} E_{\alpha,\alpha}\big(A(T-\theta)^\alpha\big)\big(f\big(\theta, x_1(\theta)\big) - f\big(\theta, x_2(\theta)\big)\big)\,\mathrm{d}\theta \\
&\quad + \int_0^T (T-\theta)^{\alpha-1} E_{\alpha,\alpha}\big(A(T-\theta)^\alpha\big)\big(\sigma\big(\theta, x_1(\theta)\big) - \sigma\big(\theta, x_2(\theta)\big)\big)\,\mathrm{d}W(\theta) \\
&\quad + \int_0^T (T-\theta)^{\alpha-1} E_{\alpha,\alpha}\big(A(T-\theta)^\alpha\big) \int_Z \big[h\big(\theta, x_1(\theta), z\big) - h\big(\theta, x_2(\theta), z\big)\big]\,\widetilde{N}(\mathrm{d}\theta, \mathrm{d}z)\Bigg]\,\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big)\big(f\big(s, x_1(s)\big) - f\big(s, x_2(s)\big)\big)\,\mathrm{d}s \\
&+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(T-s)^\alpha\big)\big(\sigma\big(s, x_1(s)\big) - \sigma\big(s, x_2(s)\big)\big)\,\mathrm{d}W(s) \\
&+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\big(A(t-s)^\alpha\big) \int_Z \big[h\big(s, x_1(s), z\big) - h\big(s, x_2(s), z\big)\big]\,\widetilde{N}(\mathrm{d}s, \mathrm{d}z)\Bigg\|^2 \\
&\leqslant 20 \sup_{t\in J} \mathbf{E}\big\|x_1(t) - x_2(t)\big\|_X^2 \\
&\quad \times \big(N_1 + (2\alpha-1)\Gamma^2(\alpha) 2N_1M_2 + T^{2\alpha}N_3M_2 + T^{2\alpha-1}(N_5+N_7)M_2\big) \\
&\leqslant \rho_2 \sup_{t\in J} \mathbf{E}\big\|x_1(t) - x_2(t)\big\|_X^2.
\end{aligned}
$$

Using (H7), we conclude that Φ is a contraction mapping, and hence, there exists a unique fixed point $x \in \mathcal{H}_2$ for Φ. This fixed point of Φ satisfies $x(T) = x_1$ for any arbitrary $x_1 \in Y$. Therefore, system (6) is completely controllable on J. □

5 Example

In this section, we provide examples to support the theory developed in the previous sections.

Example 1. Consider the nonlinear stochastic fractional neutral system

$$
\begin{aligned}
&{}^C D^{3/4}\left(x(t) - \frac{1}{10\sqrt{2}}\begin{pmatrix}\sqrt{x_1^2+5} \\ \cos x_2\end{pmatrix}\right) \\
&\quad = \begin{pmatrix}0.2 & 0.3 \\ 0.4 & 0.3\end{pmatrix} x(t) + \begin{pmatrix}0 \\ 1\end{pmatrix} u(t) \\
&\qquad + \frac{1}{10\sqrt{2}}\begin{pmatrix}\sin x_1 \\ \sqrt{x_2^2+1}\end{pmatrix} + \frac{1}{10\sqrt{2}}\begin{pmatrix}\ln(\cosh x_1) \\ \tan^{-1} x_2\end{pmatrix}\frac{\mathrm{d}W(t)}{\mathrm{d}t}, \quad t \in [0,1], \qquad (9) \\
&x(0) = \begin{pmatrix}6 \\ 90\end{pmatrix},
\end{aligned}
$$

where $x(t) = (x_1(t),\, x_2(t))^{\mathrm{T}}$ and $W(t)$ is a Wiener process on $\mathbb{R}$. Comparing with (2), we have

$$A = \begin{pmatrix} 0.2 & 0.3 \\ 0.4 & 0.3 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$
$$g\big(t, x(t)\big) = \frac{1}{10\sqrt{2}} \begin{pmatrix} \sqrt{x_1^2 + 5} \\ \cos x_2 \end{pmatrix}, \qquad f\big(t, x(t)\big) = \frac{1}{10\sqrt{2}} \begin{pmatrix} \sin x_1 \\ \sqrt{x_2^2 + 1} \end{pmatrix},$$
$$\sigma\big(t, x(t)\big) = \frac{1}{10\sqrt{2}} \begin{pmatrix} \ln(\cosh x_1) \\ \tan^{-1} x_2 \end{pmatrix}, \quad \text{and} \quad \alpha = \frac{3}{4}.$$

We wish to steer the system from the initial point $x(0)$ to $x(1) = (100,\, 1)^{\mathrm{T}}$. We see that, $g(t, x(t))$, $f(t, x(t))$, and $\sigma(t, x(t))$ are Lipschitz continuous with Lipschitz constant $1/200$. By the calculation we obtain $\rho = 0.9270 < 1$. To prove that the linear system corresponding to the above system is controllable in $[0, 1]$, it is enough to show Γ_1 is invertible. Since Γ_1 is self adjoint, it is enough to prove it is coercive.

$$\Gamma_1 x = \int_0^1 E_{3/4,3/4}\big(A(1-s)^{3/4}\big) BB^* E_{3/4,3/4}\big(A^*(1-s)^{3/4}\big) x \,\mathrm{d}s,$$
$$\langle \Gamma_1 x, x \rangle = 0.0918 x_1^2 + 0.6358 x_1 x_2 + 1.2553 x_2^2 \geqslant \gamma\big(x_1^2 + x_2^2\big),$$

where $0 < \gamma \leqslant 0.0018$. All the hypothesis of Theorem 2 are thus verified, and hence, system (9) is controllable.

Example 2. Consider the nonlinear stochastic fractional system driven by Lévy noise

$$\begin{aligned} {}^{C}D^{0.8}&\left(x(t) - \frac{1}{15} \begin{pmatrix} \mathrm{e}^{-\sin(x_1)} \\ \mathrm{e}^{-\cos(x_2)} \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 & -0.5 \\ 0.5 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \frac{1}{15} \begin{pmatrix} (1+t)^{-1} \\ \mathrm{e}^{-sin(x_2)/(1+t)} \end{pmatrix} \frac{\mathrm{d}W(t)}{\mathrm{d}t} \\ &\quad + \int_{\mathbb{R}^2 \setminus (0,0)} \frac{1}{15} \begin{pmatrix} t z_1 \\ \cos(x_2) z_2 \end{pmatrix} \frac{\mathrm{d}\widetilde{N}(t,z)}{\mathrm{d}t}, \quad t \in [0, 2], \qquad (10) \\ x(0) &= \begin{pmatrix} 1000 \\ 0 \end{pmatrix}, \end{aligned}$$

where $x(t) = (x_1(t),\, x_2(t))^{\mathrm{T}}$, $\mathrm{d}\widetilde{N}(t, z) = \mathrm{d}N(t)$ is a Poisson process with jump intensity $\lambda = 3$, $\nu(\mathrm{d}z) = \lambda f(z)\mathrm{d}z$, $f(z)$ is log-normal density function, and $\mathbf{E}z = \mathrm{e}^{\mu + \sigma^2/2}$, where μ is the mean and σ is the standard deviation of z. We wish to steer the solution from the initial point $x(0)$ to the final point $x(2) = (500,\, 10)^{\mathrm{T}}$. Comparing with (6), we have

$$A = \begin{pmatrix} 0 & -0.5 \\ 0.5 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$g\bigl(t, x(t)\bigr) = \frac{1}{15}\begin{pmatrix} \mathrm{e}^{-\sin x_1} \\ \mathrm{e}^{-\cos x_2} \end{pmatrix}, \qquad \sigma\bigl(t, x(t)\bigr) = \frac{1}{15}\begin{pmatrix} (1+t)^{-1} \\ \mathrm{e}^{-\sin x_2/(1+t)} \end{pmatrix},$$

$$h\bigl(t, x(t), z\bigr) = \frac{1}{15}\begin{pmatrix} tz_1 \\ \cos x_2 \cdot z_2 \end{pmatrix}, \quad f\bigl(t, x(t)\bigr) = 0, \quad \text{and} \quad \alpha = 0.8.$$

To show that the nonlinear system (10) is controllable, it is enough to check if the hypotheses of Theorem 3 are satisfied. We first check if the linear system corresponding to (10) is controllable by showing the operator is invertible.

$$\Gamma_2 = \begin{pmatrix} 0.4754 & -0.5709 \\ -0.5709 & 1.3249 \end{pmatrix}.$$

Now, we consider

$$\langle \Gamma_2 x, x\rangle = 0.5181x_1^2 - 1.0872x_1x_2 + 0.8871x_2^2 \geqslant \gamma\bigl(x_1^2 + x_2^2\bigr),$$

where $0 < \gamma \leqslant 0.7869$. We see that $\sigma(t, x(t))$, $g(t, x(t), z)$, and $h(t, x(t))$ are Lipschitz continuous with $1/225$ as Lipschitz constant. We also obtain the value of ρ in hypothesis (H7) to be $\rho = 0.6295 < 1$. All the hypothesis of Theorem 3 are thus verified, and hence, system (10) is controllable.

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