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Solitons and other solutions to Wu–Zhang system

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Abstract. This paper addresses Wu–Zhang system to study dispersive long waves. The extended trial equation method extracts solitary waves, shock waves, and singular solitary waves solutions. Subsequently, Lie group formalism is also applied to derive symmetries of the Wu–Zhang system, and the derived ordinary differential equations are further analyzed to retrieve exact solutions are obtained. Finally, implementation of mapping method secures additional exact solutions.

Keywords: extended trial equation method, Lie symmetry analysis, mapping method, Wu–Zhang system.

1 Introduction

The study of nonlinear evolution equations (NLEEs) forms the basic fabric for various areas of mathematical physics and engineering. There are various forms of NLEEs that are studied for this purpose [1–31]. The nonlinear Schrödinger's equation, for example, is studied in nonlinear optics. In the context of plasma physics, Zakharov–Kuznetsov equation as well as complex-valued Korteweg–de Vries (KdV) equation are addressed. Also, for fluid dynamics, particularly for shallow water wave dynamics, KdV equation, Kawahara equation, Boussinesq equation, and other such models are analyzed. This paper will study one such NLEE that appears in dispersive long wave dynamics. This is the Wu-Zhang model [29]. The focus is on its integrability aspect.

Many powerful methods for obtaining the exact solutions of NLEEs have been presented in the literature [1–4, 6–8, 10, 11, 15, 23, 27, 28, 30]. The powerful and effective method for finding exact solutions of PDEs has been proposed by Liu and hence called the Liu method [17]. This method is one of the most direct and effective algebraic methods for finding exact solutions of NLEEs. The most complete description of this method is given in [19]. The successful application of this method to NLEEs has been performed in several works [9, 18, 20–22, 25]. The Liu's method [9, 17–22, 25] can be used to construct the exact solutions for fractional NLEEs. The present paper is motivated by the desire to extend the extended trial equation method to obtain generalized solutions of the Wu–Zhang system [29, 31]. An additional integration tool developed in this context is the Lie symmetry analysis. Finally, the mapping method has been employed to obtain periodic wave solutions in terms of Jacobi elliptic functions (JEFs) and their infinite period counterparts have also been deduced [12–14, 16, 26].

2 Extended trial equation method

In this section, we describe the extended trial equation method for finding traveling wave solutions of NLEEs and subsequently will apply it to solve the Wu–Zhang system. We suppose that the given nonlinear PDE for $u(x,t)$ is in the form

$$P(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \dots) = 0, \tag{1}$$

where P is a polynomial. The essence of the extended trial equation method can be presented in the following steps:

Step 1. To find the traveling wave solutions of Eq. (1), we introduce the new wave variable

$$u(x,t) = U(\xi), \quad \xi = kx - vt, \tag{2}$$

where k and v are constants to be determined later.

Substituting Eq. (2) into Eq. (1), we obtain the following ordinary differential equations (ODEs):

$$Q(U, U', U'', \dots) = 0. \tag{3}$$

Step 2. Take transformation and trial equation as follows:

$$U = \sum_{i=0}^{\varsigma} \tau_i \Psi^i, \tag{4}$$

where

$$(\Psi')^2 = \Lambda(\Psi) = \frac{\Phi(\Psi)}{\Upsilon(\Psi)} = \frac{\mu_\sigma \Psi^\sigma + \dots + \mu_1 \Psi + \mu_0}{\zeta_\rho \Psi^\rho + \dots + \zeta_1 \Psi + \zeta_0}. \tag{5}$$

Using relations (4) and (5), we can find

$$(U')^2 = \frac{\Phi(\Psi)}{\Upsilon(\Psi)} \left(\sum_{i=0}^{\varsigma} i\tau_i \Psi^{i-1} \right)^2,$$

$$U'' = \frac{\Phi'(\Psi)\Upsilon(\Psi) - \Phi(\Psi)\Upsilon'(\Psi)}{2\Upsilon^2(\Psi)} \sum_{i=0}^{\varsigma} i\tau_i \Psi^{i-1} + \frac{\Phi(\Psi)}{\Upsilon(\Psi)} \sum_{i=0}^{\varsigma} i(i-1)\tau_i \Psi^{i-2}, \tag{6}$$

where $\Phi(\Psi)$ and $\Upsilon(\Psi)$ are polynomials. Substituting these terms into Eq. (3) yields an equation of polynomial $\Omega(\Psi)$ of Ψ

$$\Omega(\Psi) = \varrho_s \Psi^s + \cdots + \varrho_1 \Psi + \varrho_0 = 0.$$

According to the balance principle, we can determine a relation of ρ, σ, and ς. We can take some values of ρ, σ, and ς.

Step 3. Let the coefficients of $\Omega(\Psi)$ all be zero will yield an algebraic equations system

$$\varrho_i = 0, \quad i = 0, \dots, s. \tag{7}$$

Solving this equations system (7), we will determine the values of $\zeta_0, \dots, \zeta_\rho$, $\mu_0, \dots, \mu_\sigma$, and $\tau_0, \dots, \tau_\varsigma$.

Step 4. Reduce Eq. (5) to the elementary integral form

$$\pm (\xi - \xi_0) = \int \frac{\mathrm{d}\Psi}{\sqrt{\Lambda(\Psi)}} = \int \sqrt{\frac{\Upsilon(\Psi)}{\Phi(\Psi)}}\, \mathrm{d}\Psi. \tag{8}$$

Using a complete discrimination system for polynomial to classify the roots of $\Phi(\Psi)$, we solve the infinite integral (8) and obtain the exact solutions to Eq. (3). Furthermore, we can write the exact traveling wave solutions to Eq. (1), respectively.

2.1 Application to the Wu–Zhang system

In this section, we apply the extended trial equation method for finding exact solutions of the Wu–Zhang system [29, 31] in the form

$$\begin{aligned} u_t &= -u\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x}, \\ v_t &= -v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} - \frac{1}{3}\frac{\partial^3 u}{\partial x^3}. \end{aligned} \tag{9}$$

For our purpose, we introduce the following transformations:

$$u(x,t) = U(\xi), \quad v(x,t) = V(\xi), \quad \xi = x - lt, \tag{10}$$

where l is a constant to be determined later. Substituting Eq. (10) into Eq. (9), integrating once the resulting equation with respect to ξ, and choosing constant of integration to zero, we obtain

$$V = lU - \frac{1}{2}U^2, \tag{11}$$

$$lV = UV + \frac{1}{3}U''. \tag{12}$$

Inserting Eq. (11) into Eq. (12), we get ODEs as follows:

$$3l^2U - \frac{9}{2}lU^2 + \frac{3}{2}U^3 - U'' = 0. \tag{13}$$

Substituting Eqs. (4) and (6) into Eq. (13) and using the balance principle, we find

$$\sigma = \rho + 2\varsigma + 2. \tag{14}$$

If we take $\sigma = 4$, $\rho = 0$, and $\varsigma = 1$ in Eq. (14), then

$$U = \tau_0 + \tau_1\Psi, \tag{15}$$

$$(U')^2 = \frac{\tau_1^2(\mu_4\Psi^4 + \mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)}{\zeta_0},$$

$$U'' = \frac{\tau_1(4\mu_4\Psi^3 + 3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\zeta_0}, \tag{16}$$

where $\mu_4 \neq 0$, $\zeta_0 \neq 0$. Substituting Eqs. (15) and (16) into Eq. (13), collecting the coefficients of Ψ, and solving the resulting algebraic equations system, we obtain

$$\mu_1 = \frac{\mu_3\tau_0(2\mu_3 - 3\zeta_0\tau_0\tau_1)}{3\zeta_0\tau_1^3}, \qquad \mu_2 = \frac{\mu_3^2}{3\zeta_0\tau_1^2} + \frac{\mu_3\tau_0}{\tau_1} - \frac{3\zeta_0\tau_0^2}{2}, \qquad \mu_4 = \frac{3\zeta_0\tau_1^2}{4},$$
$$\mu_0 = \mu_0, \qquad \mu_3 = \mu_3, \qquad \zeta_0 = \zeta_0, \qquad \tau_0 = \tau_0, \qquad \tau_1 = \tau_1, \qquad l = \tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}.$$

Substituting these results into Eqs. (5) and (8), we get

$$\pm(\xi - \xi_0) = W\int \frac{\mathrm{d}\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{17}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4}, \qquad W = \frac{2}{\sqrt{3\tau_1^2}}.$$

Integrating Eq. (17), we obtain the solutions to Eq. (13) as follows.

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$, we obtain

$$\pm(\xi - \xi_0) = -\frac{W}{\Psi - \lambda_1}. \tag{18}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, we obtain

$$\pm (\xi - \xi_0) = \frac{2W}{\lambda_1 - \lambda_2} \sqrt{\frac{\Psi - \lambda_2}{\Psi - \lambda_1}}. \tag{19}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$, we obtain

$$\pm (\xi - \xi_0) = \frac{W}{\lambda_1 - \lambda_2} \ln \left| \frac{\Psi - \lambda_1}{\Psi - \lambda_2} \right|. \tag{20}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$, we obtain

$$\begin{aligned} \pm(\xi - \xi_0) = {} & \frac{W}{\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}} \\ & \times \ln \left| \frac{\sqrt{(\Psi - \lambda_2)(\lambda_1 - \lambda_3)} - \sqrt{(\Psi - \lambda_3)(\lambda_1 - \lambda_2)}}{\sqrt{(\Psi - \lambda_2)(\lambda_1 - \lambda_3)} + \sqrt{(\Psi - \lambda_3)(\lambda_1 - \lambda_2)}} \right|. \end{aligned} \tag{21}$$

If $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$, then we obtain

$$\pm(\xi - \xi_0) = \frac{2W}{\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}} F(\varphi, k), \tag{22}$$

where

$$F(\varphi, k) = \int_0^{\varphi} \frac{d\psi}{\sqrt{1 - k^2 \sin^2 \psi}},$$

$$\varphi = \arcsin \sqrt{\frac{(\Psi - \lambda_1)(\lambda_2 - \lambda_4)}{(\Psi - \lambda_2)(\lambda_1 - \lambda_4)}}, \qquad k^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}.$$

Also λ_1, λ_2, λ_3, and λ_4 are the roots of the polynomial equation

$$\Lambda(\Psi) = 0.$$

Substituting solutions (18)–(22) into (15), we achieve the exact solutions to Eq. (13), respectively. Then we can write the traveling wave solutions to the Wu–Zhang system (9) as:

$$u(x,t) = \tau_0 + \tau_1\lambda_1 \pm \frac{\tau_1 W}{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0}, \tag{23}$$

$$\begin{aligned} v(x,t) = {} & \left(\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}\right)\left[\tau_0 + \tau_1\lambda_1 \pm \frac{\tau_1 W}{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0}\right] \\ & - \frac{1}{2}\left[\tau_0 + \tau_1\lambda_1 \pm \frac{\tau_1 W}{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0}\right]^2, \end{aligned} \tag{24}$$

$$u(x,t) = \tau_0 + \tau_1\lambda_1 + \frac{4W^2(\lambda_2 - \lambda_1)\tau_1}{4W^2 - [(\lambda_1 - \lambda_2)\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0\}]^2}, \tag{25}$$

$$\begin{aligned} v(x,t) = {} & \left(\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}\right) \\ & \times \left[\tau_0 + \tau_1\lambda_1 + \frac{4W^2(\lambda_2 - \lambda_1)\tau_1}{4W^2 - [(\lambda_1 - \lambda_2)\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0\}]^2}\right] \\ & - \frac{1}{2}\left[\tau_0 + \tau_1\lambda_1 + \frac{4W^2(\lambda_2 - \lambda_1)\tau_1}{4W^2 - [(\lambda_1 - \lambda_2)\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0\}]^2}\right]^2, \end{aligned} \tag{26}$$

$$u(x,t) = \tau_0 + \tau_1\lambda_2 + \frac{(\lambda_2 - \lambda_1)\tau_1}{\exp[\frac{\lambda_1 - \lambda_2}{W}\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0\}] - 1}, \tag{27}$$

$$\begin{aligned} v(x,t) = {} & \left(\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}\right) \\ & \times \left[\tau_0 + \tau_1\lambda_2 + \frac{(\lambda_2 - \lambda_1)\tau_1}{\exp[\frac{\lambda_1 - \lambda_2}{W}\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0\}] - 1}\right] \\ & - \frac{1}{2}\left[\tau_0 + \tau_1\lambda_2 + \frac{(\lambda_2 - \lambda_1)\tau_1}{\exp[\frac{\lambda_1 - \lambda_2}{W}\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0\}] - 1}\right]^2, \end{aligned} \tag{28}$$

$$u(x,t) = \tau_0 + \tau_1\lambda_1 + \frac{(\lambda_1 - \lambda_2)\tau_1}{\exp[\frac{\lambda_1 - \lambda_2}{W}\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0\}] - 1}, \tag{29}$$

$$\begin{aligned} v(x,t) = {} & \left(\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}\right) \\ & \times \left[\tau_0 + \tau_1\lambda_1 + \frac{(\lambda_1 - \lambda_2)\tau_1}{\exp[\frac{\lambda_1 - \lambda_2}{W}\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0\}] - 1}\right] \\ & - \frac{1}{2}\left[\tau_0 + \tau_1\lambda_1 + \frac{(\lambda_1 - \lambda_2)\tau_1}{\exp[\frac{\lambda_1 - \lambda_2}{W}\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t - \xi_0\}] - 1}\right]^2, \end{aligned} \tag{30}$$

$$u(x,t) = \tau_0 + \tau_1\lambda_1 - \frac{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\tau_1}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2)\cosh(B\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t\})}, \tag{31}$$

$$\begin{aligned} v(x,t) = {} & \left(\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}\right) \\ & \times \left[\tau_0 + \tau_1\lambda_1 - \frac{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\tau_1}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2)\cosh(B\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t\})}\right] \\ & - \frac{1}{2}\bigg[\tau_0 + \tau_1\lambda_1 \\ & - \frac{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\tau_1}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2)\cosh(B\{x - (\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1})t\})}\bigg]^2, \end{aligned} \tag{32}$$

and

$$\begin{aligned}u(x,t) &= \tau_0 + \tau_1\lambda_2 \\ &\quad + \frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\operatorname{sn}^2[B_j\{x-(\tau_0-\frac{\mu_3}{3\zeta_0\tau_1})t-\xi_0\},k]},\end{aligned} \tag{33}$$

$$\begin{aligned}v(x,t) &= \left(\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}\right) \\ &\quad\times\left[\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\operatorname{sn}^2[B_j\{x-(\tau_0-\frac{\mu_3}{3\zeta_0\tau_1})t-\xi_0\},k]}\right] \\ &\quad-\frac{1}{2}\left[\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\operatorname{sn}^2[B_j\{x-(\tau_0-\frac{\mu_3}{3\zeta_0\tau_1})t-\xi_0\},k]}\right]^2.\end{aligned} \tag{34}$$

Here, B and B_j are given by

$$B = \frac{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W}, \qquad B_j = \frac{(-1)^j\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W}, \quad j=1,2.$$

If we take $\tau_0 = -\tau_1\lambda_1$ and $\xi_0 = 0$, then solutions (23)–(32) can reduce to rational function solutions

$$\begin{aligned}u(x,t) &= \pm\frac{\tau_1 W}{x-\tilde{l}t}, \\ v(x,t) &= \left(\tau_0-\frac{\mu_3}{3\zeta_0\tau_1}\right)\left[\pm\frac{\tau_1 W}{x-\tilde{l}t}\right]-\frac{1}{2}\left[\pm\frac{\tau_1 W}{x-\tilde{l}t}\right]^2, \\ u(x,t) &= \frac{4W^2(\lambda_2-\lambda_1)\tau_1}{4W^2-[(\lambda_1-\lambda_2)(x-\tilde{l}t)]^2}, \\ v(x,t) &= \left(\tau_0-\frac{\mu_3}{3\zeta_0\tau_1}\right)\left[\frac{4W^2(\lambda_2-\lambda_1)\tau_1}{4W^2-[(\lambda_1-\lambda_2)(x-\tilde{l}t)]^2}\right] \\ &\quad-\frac{1}{2}\left[\frac{4W^2(\lambda_2-\lambda_1)\tau_1}{4W^2-[(\lambda_1-\lambda_2)(x-\tilde{l}t)]^2}\right]^2,\end{aligned}$$

singular solitary wave solutions

$$\begin{aligned}u(x,t) &= \frac{(\lambda_2-\lambda_1)\tau_1}{2}\left\{1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W}(x-\tilde{l}t)\right]\right\}, \\ v(x,t) &= \left(\tau_0-\frac{\mu_3}{3\zeta_0\tau_1}\right)\left[\frac{(\lambda_2-\lambda_1)\tau_1}{2}\left\{1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W}(x-\tilde{l}t)\right]\right\}\right] \\ &\quad-\frac{1}{2}\left[\frac{(\lambda_2-\lambda_1)\tau_1}{2}\left\{1\mp\coth\left[\frac{\lambda_1-\lambda_2}{2W}(x-\tilde{l}t)\right]\right\}\right]^2,\end{aligned}$$

and solitary wave solutions

$$u(x,t) = \frac{A}{C + \cosh[B(x - \tilde{l}t)]},$$
$$v(x,t) = \left(\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}\right)\left[\frac{A}{C + \cosh[B(x - \tilde{l}t)]}\right] - \frac{1}{2}\left[\frac{A}{C + \cosh[B(x - \tilde{l}t)]}\right]^2,$$

where

$$A = \frac{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)\tau_1}{\lambda_3 - \lambda_2}, \qquad C = \frac{2\lambda_1 - \lambda_2 - \lambda_3}{\lambda_3 - \lambda_2}, \qquad \tilde{l} = \tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}.$$

Here, A and $\tilde{l}$ are respectively the amplitude and velocity of the soliton, while B is the inverse width of the soliton. Thus, we can say that the solitons exist for $\tau_1 < 0$. Furthermore, if we take $\tau_0 = -\tau_1\lambda_2$ and $\xi_0 = 0$, the Jacobi elliptic function solutions (33) and (34) can be written as

$$u(x,t) = \frac{A_1}{C_1 + \mathrm{sn}^2[B_i(x - \tilde{l}t), \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}]}, \tag{35}$$
$$v(x,t) = \left(\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}\right)\left[\frac{A_1}{C_1 + \mathrm{sn}^2[B_i(x - \tilde{l}t), \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}]}\right] - \frac{1}{2}\left[\frac{A_1}{C_1 + \mathrm{sn}^2[B_i(x - \tilde{l}t), \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}]}\right]^2, \tag{36}$$

where

$$A_1 = \frac{\tau_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_1 - \lambda_4}, \qquad C_1 = \frac{\lambda_4 - \lambda_2}{\lambda_1 - \lambda_4}.$$

Remark 1. When the modulus $k \to 1$, then solutions (35) and (36) can be reduced to singular solitary wave solutions

$$u(x,t) = \frac{A_1}{C_1 + \tanh^2[B_i(x - \tilde{l}t)]},$$
$$v(x,t) = \left(\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}\right)\left[\frac{A_1}{C_1 + \tanh^2[B_i(x - \tilde{l}t)]}\right] - \frac{1}{2}\left[\frac{A_1}{C_1 + \tanh^2[B_i(x - \tilde{l}t)]}\right]^2,$$

where $\lambda_3 = \lambda_4$.

Remark 2. When the modulus $k \to 0$, then solutions (35) and (36) can be turned to periodic-singular wave solutions

$$\begin{aligned} u(x,t) &= \frac{A_1}{C_1 + \sin^2[B_i(x - \tilde{l}t)]}, \\ v(x,t) &= \left(\tau_0 - \frac{\mu_3}{3\zeta_0\tau_1}\right)\left[\frac{A_1}{C_1 + \sin^2[B_i(x - \tilde{l}t)]}\right] \\ &\quad - \frac{1}{2}\left[\frac{A_1}{C_1 + \sin^2[B_i(x - \tilde{l}t)]}\right]^2, \end{aligned}$$

where $\lambda_2 = \lambda_3$.

3 Lie symmetry analysis

In this section, we will perform Lie classical method [5, 13, 24] on system of Eqs. (9).

Let us consider one parameter Lie group of transformation

$$\begin{aligned} u^* &\longrightarrow u + \epsilon\eta_1(x,t,u,v), \qquad v^* \longrightarrow v + \epsilon\eta_2(x,t,u,v), \\ x^* &\longrightarrow x + \epsilon\xi(x,t,u,v), \qquad t^* \longrightarrow t + \epsilon\tau(x,t,u,v) \end{aligned}$$

with small parameter $\epsilon \ll 1$.

The associated vector field can be written as

$$V = \xi(x,t,u,v)\frac{\partial}{\partial x} + \tau(x,t,u,v)\frac{\partial}{\partial t} + \eta_1(x,t,u,v)\frac{\partial}{\partial u} + \eta_2(x,t,u,v)\frac{\partial}{\partial v}.$$

Now, applying the second prolongation $\mathrm{pr}^2 V$ of V to system of Eq. (9), we find that the coefficient functions ξ, τ, η_1, and η_2 must satisfy the invariance condition

$$\begin{aligned} &\eta_1^t + \eta_1 u_x + u\eta_1^x + \eta_2^x = 0, \\ &\eta_2^t + \eta_2 u_x + v\eta_1^x + \eta_1 v_x + u\eta_2^x + \frac{1}{3}\eta_1^{xxx} = 0, \end{aligned} \tag{37}$$

where η_1^t, η_1^x, η_2^x, η_2^t, and η_1^{xxx} are extended infinitesimals.

Substituting the infinitesimals η_1^t, η_1^x, η_2^x, η_2^t, and η_1^{xxx} into Eqs. (37), then using the system of equations (9), and equating the coefficients of the various derivative terms, we obtain a system of NLEEs. Solving this system, we obtain following form of infinitesimals:

$$\xi = \frac{x}{2}c_1 + tc_3 + c_4, \qquad \tau = tc_1 + c_2, \qquad \eta_1 = c_3 - \frac{u}{2}c_1, \qquad \eta_2 = -vc_1,$$

where c_1, c_2, c_3, and c_4 are arbitrary real constants.

Corresponding vector fields are

$$\begin{aligned} &V_1 = \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} - \frac{u}{2}\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \qquad V_2 = \frac{\partial}{\partial t}, \\ &V_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \qquad V_4 = \frac{\partial}{\partial x}. \end{aligned} \tag{38}$$

Commutator table for vector fields (38) is as follows:

	V_1	V_2	V_3	V_4
V_1	0	$-V_2$	$\frac{V_3}{2}$	$-\frac{V_4}{2}$
V_2	V_2	0	V_4	0
V_3	$-\frac{V_3}{2}$	$-V_4$	0	0
V_4	$\frac{V_4}{2}$	0	0	0

For the reduction of system of equations (9), let us consider following vector fields:

$$\text{(i)}\quad V_1, \qquad \text{(iii)}\quad V_2 + \lambda V_4,$$
$$\text{(ii)}\quad V_3 + \mu V_2, \qquad \text{(iv)}\quad V_3.$$

where μ and λ are arbitrary constants.

For each case, one can get the similarity variables using characteristic equations:

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}t}{\tau} = \frac{\mathrm{d}u}{\eta_1} = \frac{\mathrm{d}v}{\eta_2}. \tag{39}$$

Vector field V_1

Using (39), we obtain similarity variables are as follows:

$$\sigma = \frac{x^2}{t}, \qquad u(x,t) = \frac{F(\sigma)}{x}, \qquad v(x,t) = \frac{G(\sigma)}{t}, \tag{40}$$

where σ is new independent variable, and F, G are new dependent variable.

Substituting the similarity variables (40) into system of equations (9), we obtain following system of ODEs

$$\begin{aligned} &-\sigma^2 F' - F^2 + 2\sigma F F' + 2\sigma^2 G' = 0 \\ &-\sigma^2 G - \sigma^3 G' - \sigma G F + 2\sigma^2 G F' + 2\sigma^2 F G' - 2F + 2\sigma F' + \frac{8}{3}\sigma^3 F''' = 0, \end{aligned} \tag{41}$$

where $'$ denotes derivatives with respect to σ.

Integrating the first equation of (41), we obtain

$$-F + \frac{F^2}{\sigma} + 2G - C_1 = 0, \tag{42}$$

where C_1 is arbitrary constant.

Using (42) into second equation of (41), we have

$$\begin{aligned} &-3\left(\sigma^2 F + \sigma F^2 + C_1\sigma^2\right) - 3\sigma^3 F' + 18\sigma^2 F F' + 9F^3 - 3C_1\sigma F - 18\sigma F^2 F' \\ &\quad + 6C_1\sigma^2 F' - 12F + 12\sigma F' + 16\sigma^4 F''' = 0. \end{aligned} \tag{43}$$

We obtain following solutions of Eq. (43):

$$\begin{aligned}
&\text{(i)} \quad F = \pm\frac{2\sqrt{3}}{3} + \frac{2}{3}\sigma \quad \text{with } C_1 = \pm\frac{2\sqrt{3}}{3},\\
&\text{(ii)} \quad F = \pm\frac{2\sqrt{3}}{3} + \sigma \quad \text{with } C_1 = \pm\frac{4\sqrt{3}}{3},\\
&\text{(iii)} \quad F = \sigma \quad \text{with arbitrary } C_1,\\
&\text{(iv)} \quad F = \pm\frac{2\sqrt{3}}{3} \quad \text{with } C_1 = \mp\frac{2\sqrt{3}}{3}.
\end{aligned} \tag{44}$$

Using (44) and (42) in (40), we obtain following solutions of main system of equations (9):

$$\begin{aligned}
&\text{(i)} \quad u(x,t) = \frac{2}{3}\frac{\sqrt{3}t + x^2}{xt}, \qquad v(x,t) = \frac{2\sqrt{3}tx^2 + x^4 - 6t^2}{9t^2x^2},\\
&\text{(ii)} \quad u(x,t) = \frac{3x^2 \pm 2\sqrt{3}t}{3xt}, \qquad v(x,t) = -\frac{\sqrt{3}(2\sqrt{3}t \pm 3x^2)}{9tx^2},\\
&\text{(iii)} \quad u(x,t) = \frac{x}{t}, \qquad v(x,t) = \frac{C_1}{2t},\\
&\text{(iv)} \quad u(x,t) = \pm\frac{2\sqrt{3}}{3x}, \qquad v(x,t) = -\frac{2}{3x^2}.
\end{aligned}$$

Vector field $V_3 + \mu V_2$

Corresponding similarity variable are

$$\zeta = \mu x - \frac{t^2}{2}, \qquad u = \frac{t}{\mu} + H(\zeta), \qquad v = J(\zeta), \tag{45}$$

where ζ is new independent variable, and H, J are new dependent variables.

Using (45) in (9), we obtain the following system of ODEs:

$$\begin{aligned}
&1 + \mu^2 HH' + \mu^2 J' = 0,\\
&3(HJ)' + \mu^2 H''' = 0,
\end{aligned} \tag{46}$$

where $'$ denotes derivatives with respect to ζ.

Integrating first equation of system (46), we have

$$J = \frac{1}{\mu^2}\left(C_1 - \zeta - \frac{\mu^2}{2}H^2\right), \tag{47}$$

where C_1 is constant of integration.

Integrating second equation of system (46) and using (47) in that, we have

$$(\zeta - C_1)H + 3\mu^2 H^3 - 2\mu^4 H'' - 2C_2\mu^2 = 0, \tag{48}$$

where C_2 is constant of integration.

Solving Eq. (48), we obtain following solutions:

$$\begin{aligned}
&\text{(i)} \quad H = +\frac{2\sqrt{3}\mu}{3\zeta} \quad \text{with } C_1 = 0,\ C_2 = +\frac{2\sqrt{3}}{\mu},\\
&\text{(ii)} \quad H = -\frac{2\sqrt{3}\mu}{3\zeta} \quad \text{with } C_1 = 0,\ C_2 = -\frac{2\sqrt{3}}{\mu},\\
&\text{(iii)} \quad H = 0 \quad \text{with } C_2 = 0.
\end{aligned}$$

Corresponding solutions of main system of equations (9) are as follows:

$$\begin{aligned}
\text{(i)} \quad & u(x,t) = \frac{t}{\mu} + \frac{2\sqrt{3}\mu}{3\mu x - \frac{t^2}{2}},\\
& v(x,t) = -\frac{24\mu^3x^3 - 36\mu^2x^2t^2 + 18\mu x t^4 - 3t^6 + 16\mu^4}{6\mu^2(2\mu x - t^2)^2},\\
\text{(ii)} \quad & u(x,t) = \frac{t}{\mu} - \frac{2\sqrt{3}\mu}{3\mu x - \frac{t^2}{2}},\\
& v(x,t) = -\frac{1}{6\mu^2(2\mu x - t^2)^2}\big(-24C_2\mu^2x^2 + 24C_2\mu x t^2 - 6C_2t^4,\\
&\qquad + 24\mu^3x^3 - 36\mu^2x^2t^2 + 18\mu x t^4 - 3t^6 + 16\mu^4\big),\\
\text{(iii)} \quad & u(x,t) = \frac{t}{\mu}, \qquad v(x,t) = -\frac{-2C_2 + 2\mu x - t^2}{2\mu^2}.
\end{aligned}$$

Vector field $V_2 + \lambda V_4$

Corresponding similarity variables are

$$\rho = x - \lambda t, \qquad u = P(\rho), \qquad v = Q(\rho), \tag{49}$$

where ρ and P, Q are new independent and dependent variables, respectively.

Substituting (49) in (9), we have

$$\begin{aligned}
&-\lambda P' + PP' + Q' = 0,\\
&-\lambda Q' + (PQ)' + \frac{1}{3}P''' = 0,
\end{aligned} \tag{50}$$

where $'$ denotes derivatives with respect to ρ.

Integrating first equation of (50), we have

$$-\lambda P + \frac{P^2}{2} + Q + C_1 = 0, \tag{51}$$

where C_1 is constant of integration.

Now integrating the second of system (50) and using (51), we have

$$(P-\lambda)\left(\lambda P - \frac{P^2}{2} - C_1\right) + \frac{P''}{3} + C_2 = 0, \tag{52}$$

where C_2 is constant of integration.

Solving the ODE (52), we obtain

$$\begin{aligned}
&\text{(i)}\quad P = \lambda + \frac{2\sqrt{3}}{3\rho} \quad \text{with } C_1 = \frac{\lambda^2}{2},\ C_2 = 0,\\
&\text{(ii)}\quad P = \lambda + \frac{\sqrt{-12a_3^2 + 18\lambda^2}}{3}\operatorname{sn}\left(a_2 + a_3\rho, \frac{\sqrt{-4a_3^2 + 6\lambda^2}}{2a_3}\right)\\
&\qquad \text{with } C_1 = C_2 = 0,\\
&\text{(iii)}\quad P = \lambda + \frac{\sqrt{12a_3^2 + 18\lambda^2}}{3}\operatorname{dn}\left(a_2 + a_3\rho, \frac{\sqrt{8a_3^2 + 6\lambda^2}}{2a_3}\right)\\
&\qquad \text{with } C_1 = C_2 = 0,\\
&\text{(iv)}\quad P = \lambda + \lambda\tanh\left(a_1 + \frac{\sqrt{3}}{2}\lambda\rho\right) \quad \text{with } C_1 = C_2 = 0,\\
&\text{(v)}\quad P = \lambda + \sqrt{2}\lambda\csc\left(a_1 + \frac{\sqrt{6}}{2}\lambda\rho\right) \quad \text{with } C_1 = C_2 = 0,
\end{aligned} \tag{53}$$

where a_1, a_2, and a_3 are arbitrary constants.

Now using (53) and (51), we have following solutions of system (9):

$$\begin{aligned}
&\text{(i)}\quad u(x,t) = \lambda + \frac{2\sqrt{3}}{3(x-\lambda t)}, \qquad v(x,t) = -\frac{2}{3(-x+\lambda t)^2},\\
&\text{(ii)}\quad u(x,t) = \lambda + \frac{\sqrt{-12a_3^2 + 18\lambda^2}}{3}\operatorname{sn}\left(a_2 + a_3(-x+\lambda t)\frac{\sqrt{-4a_3^2 + 6\lambda^2}}{2a_3}\right),\\
&\qquad v(x,t) = -\frac{1}{2}\left(\lambda + \frac{\sqrt{-12a_3^2 + 18\lambda^2}}{3}\operatorname{sn}\left(a_2 + a_3(-x+\lambda t)\frac{\sqrt{-4a_3^2 + 6\lambda^2}}{2a_3}\right)\right)^2\\
&\qquad\qquad + \lambda\left(\lambda + \frac{\sqrt{-12a_3^2 + 18\lambda^2}}{3}\operatorname{sn}\left(a_2 + a_3(-x+\lambda t)\frac{\sqrt{-4a_3^2 + 6\lambda^2}}{2a_3}\right)\right),
\end{aligned}$$

$$\text{(iii)}\quad u(x,t) = \lambda + \frac{\sqrt{12a_3^2 + 18\lambda^2}}{3}\operatorname{dn}\left(a_2 + a_3(-x+\lambda t), \frac{\sqrt{8a_3^2 + 6\lambda^2}}{2a_3}\right),$$

$$v(x,t) = -\frac{1}{2}\left(\lambda + \frac{\sqrt{12a_3^2 + 18\lambda^2}}{3}\operatorname{dn}\left(a_2 + a_3(-x+\lambda t), \frac{\sqrt{8a_3^2 + 6\lambda^2}}{2a_3}\right)\right)^2 + \lambda\left(\lambda + \frac{\sqrt{12a_3^2 + 18\lambda^2}}{3}\operatorname{dn}\left(a_2 + a_3(-x+\lambda t), \frac{\sqrt{8a_3^2 + 6\lambda^2}}{2a_3}\right)\right),$$

$$\text{(iv)}\quad u(x,t) = \lambda + \lambda\tanh\left(a_1 + \frac{\sqrt{3}}{2}\lambda(-x+\lambda t)\right),$$

$$v(x,t) = -\frac{1}{2}\left(\lambda + \lambda\tanh\left(a_1 + \frac{\sqrt{3}}{2}\lambda(-x+\lambda t)\right)\right)^2 + \lambda\left(\lambda + \lambda\tanh\left(a_1 + \frac{\sqrt{3}}{2}\lambda(-x+\lambda t)\right)\right),$$

$$\text{(v)}\quad u(x,t) = \lambda + \sqrt{2}\lambda\csc\left(a_1 + \frac{\sqrt{6}}{2}\lambda(-x+\lambda t)\right)$$

$$v(x,t) = -\frac{1}{2}\left(\lambda + \sqrt{2}\lambda\csc\left(a_1 + \frac{\sqrt{6}}{2}\lambda(-x+\lambda t)\right)\right)^2 + \lambda\left(\lambda + \sqrt{2}\lambda\csc\left(a_1 + \frac{\sqrt{6}}{2}\lambda(-x+\lambda t)\right)\right),$$

where a_1, a_2, and a_3 are arbitrary constants.

Vector field V_3

Corresponding similarity variables are

$$\theta = t, \qquad u(x,t) = \frac{x}{t} + R(\theta), \qquad v(x,t) = S(\theta), \tag{54}$$

where θ is new independent variable, and R, S are new dependent variables.

Using (54) in (9), we obtain

$$\theta R' + R = 0, \qquad \theta S' + S = 0, \tag{55}$$

where $'$ denotes derivative with respect to θ.

Solution of Eqs. (55) is

$$R = \frac{a_1}{t}, \qquad S = \frac{a_2}{t},$$

where a_1, a_2 are arbitrary constants.

Corresponding solution of system of equations (9) is

$$u(x,t) = \frac{x}{t} + \frac{a_1}{t}, \qquad v(x,t) = \frac{a_2}{t}.$$

4 Mapping method

In this section, we give an analysis of the mapping method, which will be employed in this paper.

Consider a nonlinear coupled PDE with two dependent variables u and v and two independent variables x and t given by

$$F(u, v, u_t, v_t, u_x, v_x, u_{xxx}, v_{xxx}, \dots) = 0, \tag{56}$$

where subscripts denote partial derivatives with respect to the corresponding independent variables, and F is a polynomial function of the indicated variables.

Step 1. Assume that Eq. (56) has a travelling wave solution (TWS) in the form

$$u(x,t) = u(\xi) = \sum_{i=0}^{l_1} A_i f^i(\xi), v(x,t) = v(\xi) = \sum_{i=0}^{l_2} B_i f^i(\xi), \tag{57}$$

where $\xi = x - \eta t$, A_i, B_i, and η are arbitrary constants, l_1 and l_2 are integers, and f^i represents integer powers of f.

The first derivative of f with respect to ξ denoted by f' can be expressed in powers of f in the form

$$f'^2 = pf^2 + \frac{1}{2}qf^4 + r, \tag{58}$$

where p, q, and r are arbitrary constants.

The motivation for Eq. (58) was that the squares of the first derivatives of JEFs can be expressed in even powers of themselves.

Step 2. Substituting Eq. (57) into Eq. (56), the PDE reduces to an ODE. Balancing the highest order derivative term and the highest order nonlinear term of the ODE, the values of l_1 and l_2 can be found.

Step 3. Substituting for u and v and using Eq. (58), the ODE gives rise to a set of algebraic equations by setting the coefficients of various powers of f to zero.

Step 4. From the values of the parameters A_i, B_i, p, q, and r, the solution of Eq. (56) can be derived.

Thus, a mapping relation is established through Eq. (57) between the solution to Eq. (58) and that of Eq. (56).

It is to be noted that if the values of l_1 and l_2 are integers, we can use the method directly to get a variety of solutions in terms of hyperbolic functions or JEFs. If they are non integers, the equation may still have solutions as rational expressions involving hyperbolic functions or JEFs.

Applying this method to Eq. (13), we can assume the solution in the form

$$U = A_0 + A_1 f(\xi). \tag{59}$$

Substituting Eq. (59) into Eq. (13) and using Eq. (58), we arrive at a set of algebraic equations by equating various powers of f to zero. From these equations, we obtain

$$A_0 = l = \pm\sqrt{-\frac{2p}{3}}, \qquad A_1 = \pm\sqrt{\frac{2q}{3}}.$$

Case 1. If $f = \tanh(\xi)$, Eq. (58) gives $p = -2$, $q = 2$, $r = 1$. So, we get $A_0 = A_1 = \pm 2/\sqrt{3}$.

Therefore, the solutions of Eqs. (9) can be written as

$$u(x,t) = \pm\frac{2}{\sqrt{3}}\left(1 + \tanh\left(x \mp \frac{2}{\sqrt{3}}t\right)\right), \tag{60}$$

$$v(x,t) = \frac{2}{3}\left(\operatorname{sech}^2\left(x \mp \frac{2}{\sqrt{3}}t\right)\right). \tag{61}$$

These are shock waves and solitary waves, respectively.

Case 2. If $f = \operatorname{sn}\xi$, Eq. (58) gives $p = -(1+m^2)$, $q = 2m^2$, $r = 1$. So, we get $A_0 = \pm\sqrt{2(1+m^2)/3}$, $A_1 = \pm 2m/\sqrt{3}$.

Therefore, the solutions of Eqs. (9) can be written, as

$$u(x,t) = \pm\sqrt{\frac{2(1+m^2)}{3}} \pm \frac{2m}{\sqrt{3}}\operatorname{sn}\left(x \mp \sqrt{\frac{2(1+m^2)}{3}}\,t\right), \tag{62}$$

$$v(x,t) = \frac{1}{3}(1-m^2) + \frac{2m^2}{3}\operatorname{cn}^2\left(x \mp \sqrt{\frac{2(1+m^2)}{3}}\,t\right). \tag{63}$$

which represents cnoidal and snoidal waves. As $m \to 1$, Eqs. (62) and (63) collapse to shock waves and solitary wave solutions as indicated in Eqs. (60) and (61).

Case 3. If $f = \operatorname{ns}\xi$, Eq. (58) gives $p = -(1+m^2)$, $q = 2$, $r = m^2$. So, we get $A_0 = \pm\sqrt{2(1+m^2)/3}$, $A_1 = \pm 2/\sqrt{3}$.

Therefore, the solutions of Eqs. (9) can be written as

$$u(x,t) = \pm\sqrt{\frac{2(1+m^2)}{3}} \pm \frac{2}{\sqrt{3}}\operatorname{ns}\left(x \mp \sqrt{\frac{2(1+m^2)}{3}}t\right), \tag{64}$$

$$v(x,t) = \frac{1}{3}\bigl(1+m^2\bigr) + \frac{2m^2}{3}\operatorname{ns}^2\left(x \mp \sqrt{\frac{2(1+m^2)}{3}}t\right). \tag{65}$$

As $m \to 1$, Eqs. (64) and (65) will lead us to singular soliton solutions

$$u(x,t) = \pm\frac{2}{\sqrt{3}}\left(1 + \coth\left(x \mp \frac{2}{\sqrt{3}}t\right)\right),$$
$$v(x,t) = -\frac{2}{3}\operatorname{csch}^2\left(x \mp \frac{2}{\sqrt{3}}t\right).$$

5 Conclusions

This paper is a sequel to previously reported results to Wu–Zhang system during 2012 [29]. The extended trial solution method, Lie symmetry analysis as well as mapping methods obtained several forms of solutions to the model. These are shock waves, solitary waves as well as singular solitary waves. Additional forms of solutions that these algorithms recover are periodic-singular solutions, cnoidal waves which, as a special case, leads to solitary waves and shock wave solutions. The spectrum of solutions that are reported in this paper will be of immense value in the context of dispersive long waves. In future, there are additional avenues that will be explored. This model will be studied with fractional temporal evolution, time-dependent coefficients as well as stochastic coefficients. These modifications will lead to a closer to reality situations. The results of those research will be disseminated elsewhere.

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