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Some existence, uniqueness results on positive solutions for a fractional differential equation with infinite-point boundary conditions*

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Abstract. We investigate a class of Riemann–Liouville's fractional differential equation with infinite-point boundary conditions. We give some new properties of the Green's function associated with the fractional differential equation boundary value problem. Based upon these new properties and by using Schauder's fixed point theorem, we establish some existence results on positive solutions for the boundary value problem. Further, by using a fixed point theorem of general concave operators, we also present an existence and uniqueness result on positive solutions for the boundary value problem.

Keywords: Riemann–Liouville's fractional derivative, infinite-point boundary value problem, positive solution, existence and uniqueness.

1 Introduction

Recently, there are several researchers who studied fractional differential equation with infinite-point boundary conditions, see [2, 4–7, 19, 21] and some references therein. In these papers, the existence of positive solutions was considered by using different methods, which include the fixed point theorem of cone expansion-compression, Avery–Peterson's fixed point theorem, Leggett–Williams's fixed point theorem, Leray–Schauder's nonlinear alternative, Leray–Schauder degree theory, and so on. However, we can see that the results on the infinite-point boundary value problems for fractional differential equations are still very few, and the uniqueness results on positive solutions for this type of boundary value problem are seldom obtained. So, this type of boundary value problem is worthwhile to discuss. In this article, we investigate the following boundary value problem

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of fractional differential equation with a boundary condition involving infinite points:

$$\begin{aligned} &D_{0+}^{\alpha}u(t) + f\bigl(t, u(t)\bigr) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ &u^{(i)}(1) = \sum_{j=1}^{\infty} \alpha_j u(\xi_j), \end{aligned} \tag{1}$$

where $\alpha > 2$, $n-1 < \alpha \leqslant n$, $i \in [1, n-2]$ is an integer, $\alpha_j \geqslant 0$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{j-1} < \xi_j < \cdots < 1$ ($j = 1, 2, \ldots$), and D_{0+}^{α} is Riemann–Liouville's fractional derivative of order α, i.e.,

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(k-\alpha)} \frac{\mathrm{d}^k}{\mathrm{d}t^k} \int_0^t \frac{y(s)}{(t-s)^{\alpha+1-k}}\,\mathrm{d}s, \quad k = [\alpha] + 1.$$

$[\alpha]$ is the integer part of the number α. A function $u \in C[0,1]$ is said to be a positive solution of problem (1) if $u(t) > 0$ on $(0,1)$ and u satisfies (1) on $[0,1]$. Set

$$\begin{aligned} \varDelta &= (\alpha-1)(\alpha-2)\cdots(\alpha-i), \\ p(s) &= \varDelta - \sum_{s \leqslant \xi_j} \alpha_j \left(\frac{\xi_j - s}{1-s}\right)^{\alpha-1} (1-s)^i. \end{aligned}$$

Throughout this paper, we always assume $\varDelta > \sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1}$ and $f : [0,1] \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, here $\mathbb{R}^+ = [0, +\infty)$. From [19] we know that the Green's function of problem (1) is

$$G(t,s) = \frac{1}{p(0)\Gamma(\alpha)} \begin{cases} t^{\alpha-1} p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1} p(s)(1-s)^{\alpha-1-i}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{2}$$

Moreover,

$$G(1,s) = \frac{1}{p(0)\Gamma(\alpha)} \bigl[p(s)(1-s)^{\alpha-1-i} - p(0)(1-s)^{\alpha-1}\bigr].$$

The paper [19] presents some important properties of $G(t,s)$ and $p(s)$.

Lemma 1. *Some properties of the function $G(t,s)$ are the following*:

(i) $G(t,s)$ *is a continuous function on* $[0,1] \times [0,1]$;
(ii) $G(t,s) > 0$, $0 < t, s < 1$;
(iii) $\max_{t \in [0,1]} G(t,s) = G(1,s)$, $0 \leqslant s \leqslant 1$;
(iv) $G(t,s) \geqslant t^{\alpha-1} G(1,s)$, $0 \leqslant t, s \leqslant 1$.

Lemma 2. $p(s) > 0, s \in [0,1]$, *and p is nondecreasing on* $[0,1]$.

In our paper, we will give some new properties of the Green's function $G(t,s)$ and use these new properties to study the existence, uniqueness of positive solutions for problem (1). By using Schauder's fixed point theorem, we first establish some existence results on positive solutions for problem (1). Second, we use a fixed point theorem of general concave operators to obtain an existence and uniqueness result on positive solutions for problem (1).

2 New properties of the Green's function

In this section, we give some new properties of the Green's function $G(t,s)$. First, we define a function g by

$$g(t) = \frac{t^{\alpha-2}[(\alpha-1)(1-t)+it]}{i}.$$

Lemma 3. *The following inequality holds*:

$$g(t) \geqslant t^{\alpha-1}, \quad 0 \leqslant t \leqslant 1.$$

Proof. From $\alpha > 2$ and Lemma 2 we have

$$t^{\alpha-2}\big[(\alpha-1)(1-t)+it\big] = t^{\alpha-2}(\alpha-1)(1-t)+it^{\alpha-1} \geqslant it^{\alpha-1}, \quad 0 \leqslant t \leqslant 1.$$

Note that $i \geqslant 1$, $g(t) \geqslant t^{\alpha-1}$ for $0 \leqslant t \leqslant 1$. □

Lemma 4. *For* $0 < t < s < 1$*, there are the following result*:

$$G(1,s)g(t) - G(t,s) \geqslant 0.$$

Proof. Set $m := 1-t$, $l := 1-s$, $\mu := \alpha - 1$, then $0 < l < m < 1$ and $\mu > i \geqslant 1$. Further,

$$\begin{aligned}
&G(1,s)g(t) - G(t,s)\\
&\quad = \frac{1}{p(0)\Gamma(\alpha)}\big[p(s)(1-s)^{\alpha-1-i} - p(0)(1-s)^{\alpha-1}\big]\frac{t^{\alpha-2}[(\alpha-1)(1-t)+it]}{i}\\
&\qquad - \frac{1}{p(0)\Gamma(\alpha)}t^{\alpha-1}p(s)(1-s)^{\alpha-1-i}\\
&\quad = \frac{p(s)t^{\alpha-2}(1-s)^{\alpha-1-i}}{ip(0)\Gamma(\alpha)}\left\{\left[1-\frac{p(0)}{p(s)}(1-s)^i\right]\left[(\alpha-1)(1-t)+it\right] - it\right\}\\
&\quad \geqslant \frac{p(s)t^{\alpha-2}(1-s)^{\alpha-1-i}}{ip(0)\Gamma(\alpha)}\big\{\big[1-(1-s)^i\big]\big[(\alpha-1)(1-t)+it\big] - it\big\}\\
&\quad = \frac{p(s)t^{\alpha-2}(1-s)^{\alpha-1-i}}{ip(0)\Gamma(\alpha)}\big\{\big(1-l^i\big)\big[m\mu+i(1-m)\big] - i(1-m)\big\}\\
&\quad = \frac{p(s)t^{\alpha-2}(1-s)^{\alpha-1-i}}{ip(0)\Gamma(\alpha)}\big[\big(1-l^i\big)m\mu - l^i i(1-m)\big]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{p(s)t^{\alpha-2}(1-s)^{\alpha-1-i}}{ip(0)\Gamma(\alpha)} il^i \mu m\left(\frac{1-l^i}{il^i} - \frac{1-m}{m\mu}\right)\\
&\geqslant \frac{p(s)t^{\alpha-2}(1-s)^{\alpha-1-i}}{p(0)\Gamma(\alpha)} l^i \mu m\left(\frac{1-l^i}{\mu l^i} - \frac{1-m}{m\mu}\right)\\
&= \frac{p(s)t^{\alpha-2}(1-s)^{\alpha-1-i}}{p(0)\Gamma(\alpha)} l^i m\left(\frac{1-l^i}{l^i} - \frac{1-m}{m}\right)\\
&\geqslant \frac{p(s)t^{\alpha-2}(1-s)^{\alpha-1-i}}{p(0)\Gamma(\alpha)} l^i m\left(\frac{1-m^i}{m^i} - \frac{1-m}{m}\right)\\
&= \frac{p(s)t^{\alpha-2}(1-s)^{\alpha-1-i}}{p(0)\Gamma(\alpha)} l^i m\left(\frac{m-m^i}{m^i m}\right) \geqslant 0.
\end{aligned}
$$

So, the conclusion holds. □

Lemma 5. *For $0 < s < t < 1$, the following conclusion holds*:

$$G(1,s)g(t) - G(t,s) \geqslant 0.$$

Proof. Set $l := 1-s$, $m := 1-t$, $\mu := \alpha - 1$, $w := 1/l$, $q := (1-wm)/(1-m)$. It is clear to see $0 < m < l < 1$, $\mu \geqslant n-2 \geqslant i \geqslant 1$, and $1 < w < 1/m$. Because $0 < s < t < 1$, we have

$$
\begin{aligned}
&G(1,s)g(t) - G(t,s)\\
&\quad= \frac{1}{p(0)\Gamma(\alpha)}\left[p(s)(1-s)^{\alpha-1-i} - p(0)(1-s)^{\alpha-1}\right]\frac{t^{\alpha-2}[(\alpha-1)(1-t)+it]}{i}\\
&\qquad - \frac{1}{p(0)\Gamma(\alpha)}\left[t^{\alpha-1}p(s)(1-s)^{\alpha-1-i} - p(0)(t-s)^{\alpha-1}\right]\\
&\quad= \frac{1}{i\Gamma(\alpha)}\left[\frac{p(s)}{p(0)}(1-s)^{\alpha-1-i} - (1-s)^{\alpha-1}\right]t^{\alpha-2}\left[(\alpha-1)(1-t)+it\right]\\
&\qquad - \frac{1}{i\Gamma(\alpha)}\left[\frac{p(s)}{p(0)}it^{\alpha-1}(1-s)^{\alpha-1-i} - i(t-s)^{\alpha-1}\right]\\
&\quad= \frac{1}{i\Gamma(\alpha)}\left[\frac{p(s)}{p(0)}(1-s)^{\alpha-1-i} - (1-s)^{\alpha-1}\right]t^{\alpha-2}(\alpha-1)(1-t)\\
&\qquad + \frac{1}{i\Gamma(\alpha)}it^{\alpha-1}\left[\frac{p(s)}{p(0)}(1-s)^{\alpha-1-i} - (1-s)^{\alpha-1}\right]\\
&\qquad - \frac{1}{i\Gamma(\alpha)}\frac{p(s)}{p(0)}it^{\alpha-1}(1-s)^{\alpha-1-i} + \frac{1}{i\Gamma(\alpha)}i(t-s)^{\alpha-1}\\
&\quad= \frac{1}{i\Gamma(\alpha)}\left\{\left[\frac{p(s)}{p(0)}(1-s)^{\alpha-1-i} - (1-s)^{\alpha-1}\right]t^{\alpha-2}(\alpha-1)(1-t)\right.\\
&\qquad \left. - it^{\alpha-1}(1-s)^{\alpha-1}\right\} + \frac{1}{i\Gamma(\alpha)}i(t-s)^{\alpha-1}
\end{aligned}
$$

$$\begin{aligned}
&\geqslant \frac{1}{i\Gamma(\alpha)}\bigl\{\bigl[(1-s)^{\alpha-1-i}-(1-s)^{\alpha-1}\bigr]t^{\alpha-2}(\alpha-1)(1-t)\\
&\quad -it^{\alpha-1}(1-s)^{\alpha-1}\bigr\}+\frac{1}{i\Gamma(\alpha)}i(t-s)^{\alpha-1}\\
&=\frac{1}{i\Gamma(\alpha)}\bigl[\bigl(l^{\alpha-1-i}-l^{\alpha-1}\bigr)(1-m)^{\alpha-2}(\alpha-1)m\\
&\quad -i(1-m)^{\alpha-1}l^{\alpha-1}+i(l-m)^{\alpha-1}\bigr]\\
&=\frac{1}{i\Gamma(\alpha)}\bigl[\bigl(l^{\mu-i}-l^{\mu}\bigr)(1-m)^{\mu-1}\mu m-i(1-m)^{\mu}l^{\mu}+i(l-m)^{\mu}\bigr]\\
&=\frac{l^{\mu}}{i\Gamma(\alpha)}\left[\bigl(l^{-i}-1\bigr)(1-m)^{\mu-1}\mu m-i(1-m)^{\mu}+i\left(1-\frac{m}{l}\right)^{\mu}\right]\\
&=\frac{l^{\mu}(1-m)^{\mu}}{i\Gamma(\alpha)}\left[\frac{l^{-i}-1}{1-m}\mu m-i+i\left(\frac{1-m/l}{1-m}\right)^{\mu}\right]\\
&=\frac{l^{\mu}(1-m)^{\mu}}{i\Gamma(\alpha)}\left[\frac{w^{i}-1}{1-m}\mu m-i+i\left(\frac{1-wm}{1-m}\right)^{\mu}\right]\\
&=\frac{l^{\mu}(1-m)^{\mu}\mu}{\Gamma(\alpha)}\left[\frac{w^{i}-1}{i}\frac{m}{1-m}-\frac{1-q^{\mu}}{\mu}\right]\\
&\geqslant\frac{l^{\mu}(1-m)^{\mu}\mu}{\Gamma(\alpha)}\left[(w-1)\frac{m}{1-m}-\frac{1-q^{\mu}}{\mu}\right]\\
&\geqslant\frac{l^{\mu}(1-m)^{\mu}\mu}{\Gamma(\alpha)}\left[(w-1)\frac{m}{1-m}-(1-q)\right]=0.
\end{aligned}$$

Hence, the proof is finished. □

From Lemmas 4 and 5 we can easily obtain the following result.

Theorem 1. *For* $t, s \in [0,1]$*, the following inequality holds*:

$$G(t,s)\leqslant G(1,s)g(t).$$

Further, from Lemma 1 and Theorem 1 we can also obtain the following conclusion.

Theorem 2. *For* $t, s \in [0,1]$*, the following inequality holds*:

$$G(1,s)t^{\alpha-1}\leqslant G(t,s)\leqslant G(1,s)g(t).$$

Theorem 3. *Assume* $u \in C[0,1]$ *and satisfies problem* (1)*, then*

$$t^{\alpha-1}u(1)\leqslant u(t)\leqslant u(1)g(t),\quad 0\leqslant t\leqslant 1.$$

Proof. From [19] the solution $u(t)$ of problem (1) can be expressed by

$$u(t)=\int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s.$$

By Theorem 2,

$$u(t) \geqslant t^{\alpha-1} \int_0^1 G(1,s) f\big(s, u(s)\big)\, \mathrm{d}s = t^{\alpha-1} u(1),$$

$$u(t) \leqslant g(t) \int_0^1 G(1,s) f\big(s, u(s)\big)\, \mathrm{d}s = g(t) u(1). \qquad \square$$

3 Existence of positive solutions for problem (1)

In the following, we use Theorem 2 to obtain several existence results for problem (1). Let $E = C[0,1]$ a Banach space, the norm is the standard maximum norm, i.e., $\|u\| = \max_{t\in[0,1]} |u(t)|$ for any $u \in E$.

First, we present a condition:

(H1) there are two constants $\tau_2 > \tau_1 > 0$ such that

$$\inf_{u\in\Omega} \int_0^1 G(1,s) f\big(s, u(s)\big)\, \mathrm{d}s \geqslant \tau_1, \qquad \sup_{u\in\Omega} \int_0^1 G(1,s) f\big(s, u(s)\big)\, \mathrm{d}s \leqslant \tau_2,$$

where $\Omega = \{u \in E\colon\ \tau_1 t^{\alpha-1} \leqslant u(t) \leqslant \tau_2 g(t),\ t \in [0,1]\}$.

Theorem 4. *Let condition* (H1) *be satisfied. Then problem* (1) *has at least one positive solution in* Ω.

Proof. Define an operator $A : E \to E$ by

$$Au(t) = \int_0^1 G(t,s) f\big(s, u(s)\big)\, \mathrm{d}s.$$

From [19] we know that u is a solution of problem (1) if and only if u is a fixed point of A. For every $u \in S$, by Theorem 2 and (H1),

$$Au(t) \geqslant t^{\alpha-1} \int_0^1 G(1,s) f\big(s, u(s)\big)\, \mathrm{d}s \geqslant \tau_1 t^{\alpha-1}$$

and

$$Au(t) \leqslant g(t) \int_0^1 G(1,s) f\big(s, u(s)\big)\, \mathrm{d}s \leqslant \tau_2 g(t).$$

Hence, $A(\Omega) \subseteq \Omega$. Further, it follows from the continuity of $G(t,s)$ and $f(t,u)$ that $A : \Omega \to \Omega$ is completely continuous. Therefore, A has a fixed point u^* in S by using Schauder's fixed point theorem. By Theorem 3, $u^*(t) \geqslant \tau_1 t^{\alpha-1} \geqslant 0$, $t \in [0,1]$, so, we claim that $u^*(t)$ is a positive solution. $\square$

From Theorem 4 we can easily obtain the following corollaries.

Corollary 1. *Suppose that there are $\tau_2 > \tau_1 > 0$ such that $f(t,\cdot)$ is increasing on $[0, \tau_2 g^*]$ for fixed $t \in [0,1]$, and*

$$\int_0^1 G(1,s) f\bigl(s, \tau_1 s^{\alpha-1}\bigr)\,\mathrm{d}s \geqslant \tau_1, \qquad \int_0^1 G(1,s) f\bigl(s, \tau_2 g(s)\bigr)\,\mathrm{d}s \leqslant \tau_2,$$

where $g^* = \max_{t\in[0,1]} g(t)$. Then problem (1) *has at least one positive solution in Ω.*

Corollary 2. *Suppose that there are $\tau_2 > \tau_1 > 0$ such that $f(t,\cdot)$ is decreasing on $[0, \tau_2 g^*]$ for fixed $t \in [0,1]$, and*

$$\int_0^1 G(1,s) f\bigl(s, \tau_2 g(s)\bigr)\,\mathrm{d}s \geqslant \tau_1, \qquad \int_0^1 G(1,s) f\bigl(s, \tau_1 s^{\alpha-1}\bigr)\,\mathrm{d}s \leqslant \tau_2,$$

where $g^* = \max_{t\in[0,1]} g(t)$. Then problem (1) *has at least one positive solution in Ω.*

Example 1. Consider the following infinite-point boundary value problem

$$\begin{aligned} &D_{0+}^{7/2} u(t) + t^2 \sqrt{u(t)} = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = u''(0) = 0, \qquad u'(1) = \sum_{j=1}^{\infty} \frac{2}{j^2} u\left(\frac{1}{j}\right), \end{aligned} \tag{3}$$

From this example we see that $\alpha = 7/2$, $n = 4$, $i = 1$, $\alpha_j = 2/j^2$, $\xi_j = 1/j$, $\Delta = 2.5$, $f(t,u) = t^2\sqrt{u}$. By a simple calculation, $\sum_{j=1}^{\infty} \alpha_j \xi_j^{\alpha-1} = \sum_{j=1}^{\infty} (2/j^2)(1/j)^{5/2} \approx 2.109 < \Delta$, $\Gamma(\alpha) = \Gamma(7/2) \approx 3.3234$, $p(0) \approx 0.391$. It is easily to see that $f(t,u)$ is continuous and is increasing in $u \in [0, +\infty)$. Take

$$0 < \tau_1 \leqslant \left(\frac{\int_0^1 [(1-s)^{3/2} - (1-s)^{5/2}] s^{13/4}\,\mathrm{d}s}{\Gamma(7/2)}\right)^2 \approx 2.116 \cdot 10^{-5},$$

$$\tau_2 \geqslant \left(\frac{2.5}{p(0)\Gamma(7/2)} \left(\frac{7}{2}\right)^{1/2} \int_0^1 (1-s)^{3/2} s^{11/4}\,\mathrm{d}s\right)^2 \approx 1.31 \cdot 10^{-2}.$$

Then $\tau_1 < \tau_2$. Moreover, $g(t) = t^{3/2}[(5/2)(1-t) + t]$. From (2) and Lemma 2,

$$\begin{aligned} &\int_0^1 G(1,s) f\bigl(s, \tau_1 s^{\alpha-1}\bigr)\,\mathrm{d}s \\ &\quad = \int_0^1 G(1,s) s^2 \bigl(\tau_1 s^{5/2}\bigr)^{1/2}\,\mathrm{d}s \\ &\quad = \frac{{\tau_1}^{1/2}}{p(0)\Gamma(7/2)} \int_0^1 \bigl[p(s)(1-s)^{\alpha-1-i} - p(0)(1-s)^{\alpha-1}\bigr] s^{13/4}\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&\geqslant \frac{{\tau_1}^{1/2}}{p(0)\Gamma(\frac{7}{2})} \int_0^1 \bigl[p(0)(1-s)^{3/2} - p(0)(1-s)^{5/2}\bigr] s^{13/4}\,\mathrm{d}s \\
&= \frac{{\tau_1}^{1/2}}{\Gamma(\frac{7}{2})} \int_0^1 \bigl[(1-s)^{3/2} - (1-s)^{5/2}\bigr] s^{13/4}\,\mathrm{d}s \geqslant \tau_1,
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^1 G(1,s) f\bigl(s, \tau_2 g(s)\bigr)\,\mathrm{d}s \\
&\quad= \int_0^1 G(1,s) s^2 \bigl(\tau_2 g(s)\bigr)^{1/2}\,\mathrm{d}s \\
&\quad\leqslant \frac{{\tau_2}^{1/2}}{p(0)\Gamma(\frac{7}{2})} \int_0^1 p(s)(1-s)^{\alpha-1-i} s^2 \bigl[s^{\alpha-2}\bigl((\alpha-1)(1-s)+s\bigr)\bigr]^{1/2}\,\mathrm{d}s \\
&\quad\leqslant \frac{{\tau_2}^{1/2}}{p(0)\Gamma(\frac{7}{2})} \int_0^1 \varDelta(1-s)^{\alpha-1-i} s^{\alpha/2+1} \bigl[(\alpha-1)+1\bigr]^{1/2}\,\mathrm{d}s \\
&\quad= \frac{2.5{\tau_2}^{1/2}}{p(0)\Gamma(\frac{7}{2})} \left(\frac{7}{2}\right)^{1/2} \int_0^1 (1-s)^{3/2} s^{11/4}\,\mathrm{d}s \leqslant \tau_2.
\end{aligned}
$$

Then, by using Corollary 1, problem (3) has at least one positive solution in $\varOmega$.

Example 2. In problem (3), we replace $f(t,u)$ by

$$
f(t,u) = t^2 \begin{cases} 16^{-1}, & u \in [0,4], \\ u^{-2}, & u \in (4,+\infty). \end{cases}
$$

Similar to Example 1, we can also easily prove that all the assumptions of Corollary 2 are satisfied.

4 Uniqueness of positive solutions for problem (1)

Let $(E, \|\cdot\|)$ be a real Banach space. θ is the zero element of E, and $P \subset E$ is a cone. Then E is partially ordered by cone P, i.e., $x \leqslant y$ if and only if $y - x \in P$. P is called normal if there exists a constant $K > 0$ such that, for $x, y \in E$ with $\theta \leqslant x \leqslant y$, $\|x\| \leqslant K\|y\|$. For $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. For $h > \theta$ (i.e., $h \geqslant \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E \mid x \sim h\}$.

Lemma 6. (*See Lemma* 2.1 *and Theorem* 2.1 *in* [18].) *Let $h > \theta$ and P be a normal cone. Assume*:

(D1) *$A : P \to P$ is increasing, and $Ah \in P_h$;*

(D2) *for $x \in P$ and $t \in (0,1)$, there exists $\varphi(t) \in (t,1)$ such that $A(tx) \geqslant \varphi(t)Ax$.*

Then:

(i) *there are $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $rv_0 \leqslant u_0 < v_0$, $u_0 \leqslant Au_0 \leqslant Av_0 \leqslant v_0$;*

(ii) *operator equation $x = Ax$ has a unique solution in P_h.*

Remark 1. An operator A is said to be *generalized concave* if it satisfies condition (D2).

Let $E = C[0,1]$, $P = \{x \in C[0,1] \mid x(t) \geqslant 0, t \in [0,1]\}$, then P is a normal cone in $C[0,1]$. From (2) we can obtain

$$G(t,s) \leqslant \frac{1}{p(0)\Gamma(\alpha)} t^{\alpha-1} p(s)(1-s)^{\alpha-1-i}, \quad 0 \leqslant t, s \leqslant 1. \tag{4}$$

Theorem 5. *Let $h(t) = t^{\alpha-1}, t \in [0,1]$ and suppose that*:

(H2) *$f(t,u)$ is increasing in u for each $t \in [0,1]$ with $f(t,0) \not\equiv 0$;*

(H3) *for $r \in (0,1)$, there is $\varphi(r) \in (r,1)$ such that*

$$f(t, ru) \geqslant \varphi(r) f(t,u), \quad t \in [0,1],\ u \in [0, +\infty).$$

Then, the following conclusions hold:

(i) *there exist $u_0, v_0 \in P_h$ such that*

$$u_0(t) \leqslant \int_0^1 G(t,s) f\bigl(s, u_0(s)\bigr)\,\mathrm{d}s, \quad v_0(t) \geqslant \int_0^1 G(t,s) f\bigl(s, v_0(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1];$$

(ii) *problem* (1) *has a unique positive solution u^* in P_h.*

Remark 2. In Theorem 5, $P_h = \{x \in C[0,1] \mid \exists \lambda(x,h), \mu(x,h) > 0\colon\ \lambda t^{\alpha-1} = \lambda h(t) \leqslant x(t) \leqslant \mu h(t) = \mu t^{\alpha-1},\ t \in [0,1]\}$, we can see that λ, μ are not constants and depend on x, h. In Ω, τ_1, τ_2 are two constants. So, $\Omega \neq P_h$.

Proof of Theorem 5. We also consider the following operator:

$$Au(t) = \int_0^1 G(t,s) f\bigl(s, u(s)\bigr)\,\mathrm{d}s.$$

Clearly, $A : P \to P$. Further, from (H2) we can easily obtain that $A : P \to P$ is increasing. Now we prove that A is generalized concave. For $r \in (0,1)$ and $u \in P$, from (H3),

$$\begin{aligned} A(ru)(t) &= \int_0^1 G(t,s) f\bigl(s, ru(s)\bigr)\,\mathrm{d}s \geqslant \varphi(r) \int_0^1 G(t,s) f\bigl(s, u(s)\bigr)\,\mathrm{d}s \\ &= \varphi(r) Au(t), \quad t \in [0,1]. \end{aligned}$$

Hence, $A(ru) \geqslant \varphi(r)Au$ for all $u \in P$, $r \in (0,1)$.

Next, we show $Ah \in P_h$. From Theorem 2, (H2), and Lemma 2,

$$\begin{aligned} Ah(t) &= \int_0^1 G(t,s) f\bigl(s, h(s)\bigr)\,\mathrm{d}s = \int_0^1 G(t,s) f\bigl(s, s^{\alpha-1}\bigr)\,\mathrm{d}s \\ &\geqslant \int_0^1 G(1,s) t^{\alpha-1} f(s,0)\,\mathrm{d}s \\ &= \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 \bigl[p(s)(1-s)^{\alpha-1-i} - p(0)(1-s)^{\alpha-1}\bigr] f(s,0)\,\mathrm{d}s \cdot t^{\alpha-1} \\ &\geqslant \frac{1}{\Gamma(\alpha)} \int_0^1 \bigl[(1-s)^{\alpha-1-i} - (1-s)^{\alpha-1}\bigr] f(s,0)\,\mathrm{d}s \cdot t^{\alpha-1}, \quad t \in [0,1]. \end{aligned}$$

Also from (H2) and (4) we obtain

$$\begin{aligned} Ah(t) &\leqslant \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 t^{\alpha-1} p(s)(1-s)^{\alpha-1-i} f(s,1)\,\mathrm{d}s \\ &= \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 p(s)(1-s)^{\alpha-1-i} f(s,1)\,\mathrm{d}s \cdot t^{\alpha-1}, \quad t \in [0,1]. \end{aligned}$$

Put

$$\begin{aligned} r_1 &= \frac{1}{\Gamma(\alpha)} \int_0^1 \bigl[(1-s)^{\alpha-1-i} - (1-s)^{\alpha-1}\bigr] f(s,0)\,\mathrm{d}s, \\ r_2 &= \frac{1}{p(0)\Gamma(\alpha)} \int_0^1 p(s)(1-s)^{\alpha-1-i} f(s,1)\,\mathrm{d}s. \end{aligned}$$

Note that f is continuous and $f(t,0) \not\equiv 0$, we can get $0 < r_1 \leqslant r_2$ by using Lemma 2 and (H2). So, we get

$$Ah(t) \geqslant r_1 h(t), \quad Ah(t) \leqslant r_2 h(t), \quad t \in [0,1].$$

Consequently, $r_1 h \leqslant Ah \leqslant r_2 h$ and thus $Ah \in P_h$. Finally, by Lemma 6, there are $u_0, v_0 \in P_h$ and $r \in (0,1)$ such that $r v_0 \leqslant u_0 < v_0$, $u_0 \leqslant A u_0 \leqslant A v_0 \leqslant v_0$; and we can claim that A has a unique fixed point in P_h. That is,

$$u_0(t) \leqslant \int_0^1 G(t,s) f\bigl(s, u_0(s)\bigr)\,\mathrm{d}s, \quad v_0(t) \geqslant \int_0^1 G(t,s) f\bigl(s, v_0(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1];$$

and problem (1) has a unique positive solution u^* in P_h. □

Example 3. Consider the following infinite-point boundary value problem:

$$\begin{aligned} &D_{0+}^{7/2}u(t) + \bigl(\sqrt{u(t)} + 1\bigr)\sin\pi t = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = u''(0) = 0, \qquad u'(1) = \sum_{j=1}^{\infty}\frac{2}{j^2}u\left(\frac{1}{j}\right), \end{aligned} \tag{5}$$

From this example we see that $\alpha = 7/2$, $n = 4$, $i = 1$, $\alpha_j = 2/j^2$, $\xi_j = 1/j$, $\Delta = 2.5$, $f(t,u) = (\sqrt{u} + 1)\sin\pi t$. It is clear to see that $f(t,u)$ is continuous and is increasing in $u \in [0,+\infty)$. In addition, $f(t,0) = \sin\pi t \not\equiv 0$. Take $\varphi(t) = \sqrt{t}$, $t \in (0,1)$. Then $\varphi(t) \in (t,1)$ and, for $r \in (0,1)$,

$$\begin{aligned} f(t,ru) &= (\sqrt{ru} + 1)\sin\pi t \geqslant (\sqrt{ru} + \sqrt{r})\sin\pi t \\ &= \sqrt{r}(\sqrt{u} + 1)\sin\pi t = \varphi(r)f(t,u). \end{aligned}$$

By using Theorem 5, problem (5) has a unique positive solution in $P_{t^{5/2}}$.

Remark 3. The uniqueness of positive solutions for fractional boundary value problems appeared in many papers, see [1, 3, 8–17, 20] for example. However, to our knowledge, there are very few unique results on fractional infinite-point boundary value problems. So, our results are new and then complement previously known results.

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