ISSN 1392-5113 Nonlinear Analysis: Modelling and Control, 2017, Vol. 22, No. 3, 412–420
https://doi.org/10.15388/NA.2017.3.9

On an uniqueness theorem for characteristic functions

Saulius Norvidas

Institute of Mathematics and Informatics, Vilnius University,
Akademijos str. 4, LT-08663 Vilnius, Lithuania

Vilnius Gediminas Technical University,
Saulėtekio ave. 11, LT-10223 Vilnius, Lithuania
norvidas@gmail.com

Received: March 29, 2016 / **Revised:** October 21, 2016 / **Published online:** March 17, 2017

Abstract. Suppose that f is the characteristic function of a probability measure on the real line $\mathbb{R}$. We deal with the following open problem posed by N.G. Ushakov: Is it true that f is never determined by its imaginary part $\Im f$? In other words, is it true that for any characteristic function f, there exists a characteristic function g such that $\Im f = \Im g$, but $f \neq g$? The answer to this question is *no*. We give a characterization of those characteristic functions, which are uniquely determined by their imaginary parts. Also, several examples of characteristic functions, which are uniquely determined by their imaginary parts, are given.

Keywords: Bochner's theorem, characteristic function, Fourier algebra, positive definite function, imaginary part of the characteristic function.

1 Introduction

Let $M(\mathbb{R})$ be the Banach algebra of bounded regular complex-valued Borel measures μ on the real line $\mathbb{R}$ equipped with the total variation norm $\|\mu\|$. Throughout this paper, $\mathcal{B}(\mathbb{R})$ will denote the usual σ-algebra of all Borel subsets of $\mathbb{R}$. The Fourier–Stieltjes transform of $\mu \in M(\mathbb{R})$ is defined by

$$\hat{\mu}(\xi) = \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}\mu(x), \quad \xi \in \mathbb{R}.$$

We identify $L^1(\mathbb{R})$ with the closed ideal in $M(\mathbb{R})$ of all measures, which are absolutely continuous with respect to the Lebesgue measure $\mathrm{d}x$ on $\mathbb{R}$, i.e., if $\varphi \in L^1(\mathbb{R})$, then φ is associated with the following measure:

$$\mu_\varphi(E) = \int_E \varphi(x)\,\mathrm{d}x, \quad E \in \mathcal{B}(\mathbb{R}).$$

Therefore, the Fourier transform of $\varphi \in L^1(\mathbb{R})$ is defined by $\hat{\varphi}(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx}\varphi(x)\,\mathrm{d}x$, $t \in \mathbb{R}$. We normalize the inverse Fourier transform

$$\check{\varphi}(\xi) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi t}\varphi(t)\,\mathrm{d}t$$

so that the inversion formula $\check{\hat{\varphi}} = \varphi$ holds for suitable $\varphi \in L^1(\mathbb{R})$.

The family of Fourier–Stieltjes transforms $\hat{\mu}$ of $\mu \in M(\mathbb{R})$ forms the so-called Fourier–Stieltjes algebra $B(\mathbb{R})$. The norm in $B(\mathbb{R})$ is defined by

$$\|\hat{\mu}\|_{B(\mathbb{R})} := \|\mu\|_{M(\mathbb{R})}.$$

The closed ideal $L^1(\mathbb{R})$ of $M(\mathbb{R})$ generates the Fourier algebra $A(\mathbb{R}) = \{\hat{\varphi}\colon \varphi \in L^1(\mathbb{R})\}$. Of course, $A(\mathbb{R})$ is a closed ideal of $B(\mathbb{R})$, and the norm in $A(\mathbb{R})$ is defined by

$$\|\hat{\varphi}\|_{A(\mathbb{R})} := \|\varphi\|_{L^1(\mathbb{R})}.$$

A function $f : \mathbb{R} \to \mathbb{C}$ is said to be positive definite if an inequality

$$\sum_{j,k=1}^{n} f(x_j - x_k)c_j\overline{c}_k \geqslant 0$$

holds for all finite sets of complex numbers $c_1, \ldots, c_n$ and points $x_1, \ldots, x_n \in \mathbb{R}$. The Bochner theorem (see, e.g., [2, p. 121] or [7, p. 71]) states that a continuous function $f : \mathbb{R} \to \mathbb{C}$ is positive definite if and only if there exists a nonnegative $\mu \in M(\mathbb{R})$ such that $f = \hat{\mu}$. If, in addition, $\|\mu\| = 1$, then in the language of probability theory, these μ and f are called the probability measure and its characteristic function, respectively. Any characteristic function f is continuous on $\mathbb{R}$ and satisfies

$$f(-x) = \overline{f}(x) \quad \text{and} \quad \bigl|f(x)\bigr| \leqslant f(0) = 1 \tag{1}$$

for all $x \in \mathbb{R}$. In particular, such an f is real-valued if and only if it is the Fourier–Stieltjes transform of a symmetric distribution μ [7, p. 30], i.e., if μ satisfies $\mu(-A) = \mu(A)$ for any $A \in \mathcal{B}(\mathbb{R})$.

Some properties of characteristic functions are not only of theoretical interest in probability theory, but also helpful in solving applied problems as well. Here we deal with certain examples and assertions demonstrating the relationship between a characteristic function, its real part, and its imaginary part. It is known that f is not determined by $|f|$. More precisely (see [4, p. 506] and [8, p. 256]), there exist two different real-valued characteristic functions f and g such that $|f| = |g|$ everywhere. The same holds true for characteristic functions f and their real parts $\Re f$. Indeed, in [8, p. 259] it is shown that for any characteristic functions $f \neq 1$, there exists a characteristic function g such that $\Re f = \Re g$, but $f \neq g$.

In this context, it is natural to ask whether the same is true for f and its imaginary part $\Im f$? The following question as an open problem was given by Ushakov (see [8, p. 260] and the open problem No. 16 in [8, p. 334]):

(∗) *Is it true that the characteristic function is never determined by its imaginary part?*

For a characteristic function f, we say that $\Im f$ determines uniquely f if for each characteristic function g such that $\Im f = \Im g$, we have that $f = g$. It is not difficult to check that the answer to the question (∗) is *no*: For example, if

$$\Im f(x) = \Im g(x) = \sin x$$

for $x \in \mathbb{R}$, then necessarily $f(x) = g(x) = \mathrm{e}^{\mathrm{i}x}$, $x \in \mathbb{R}$. This can be verified directly by simple arguments. Even more, we give a characterization of those characteristic functions, which are uniquely determined by their imaginary parts. To simplify the proofs, we will study here the characteristic functions of absolutely continuous probability measures.

If $f = \hat{\varphi}$, where $\varphi \in L^1(\mathbb{R})$ is non-negative on $\mathbb{R}$ and $\|\varphi\|_{L^1(\mathbb{R})} = 1$, then

$$\begin{aligned} f(x) &= \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}xt}\varphi(t)\,\mathrm{d}t = \int_{\mathbb{R}} \cos(xt)\varphi(t)\,\mathrm{d}t + \mathrm{i}\int_{\mathbb{R}} \sin(-xt)\varphi(t)\,\mathrm{d}t \\ &= \Re f(x) + \mathrm{i}\Im f(x), \end{aligned}$$

$x \in \mathbb{R}$. Hence,

$$\Re f(x) = \Re\hat{\varphi}(x) = \frac{1}{2}\biggl[\int_{\mathbb{R}} \bigl(\mathrm{e}^{\mathrm{i}xt} + \mathrm{e}^{-\mathrm{i}xt}\bigr)\varphi(t)\,\mathrm{d}t\biggr]$$

and

$$\Im f(x) = \Im\hat{\varphi}(x) = \frac{1}{2i}\biggl[\int_{\mathbb{R}} \bigl(\mathrm{e}^{-\mathrm{i}xt} - \mathrm{e}^{\mathrm{i}xt}\bigr)\varphi(t)\,\mathrm{d}t\biggr].$$

Therefore, these $\Re f$ and $\Im f$ also are elements of $A(\mathbb{R})$. Moreover, $\|\Re f\|_{A(\mathbb{R})} = 1$ and $\Re f$ is an even characteristic function. On the other hand, the function $\Im f$ is odd and

$$\|\Im f\|_{A(\mathbb{R})} \leqslant 1. \tag{2}$$

By (1), this function is positive definite only in the trivial case if $\Im f \equiv 0$.

Now we state the main result of this paper.

Theorem 1. *Suppose that $\psi \in A(\mathbb{R})$ is real-valued and odd. Then the following three statements are equivalent*:

(A) *There is an unique characteristic function f such that $\Im f \equiv \psi$.*
(B) $\|\psi\|_{A(\mathbb{R})} = 1$.
(C) (i) *For any $c_1, \dots, c_m \in \mathbb{C}$ and $\gamma_1, \dots, \gamma_m \in \mathbb{R}$, the inequality*

$$\sup_{x\in\mathbb{R}}\Biggl|\sum_{k=1}^{m} c_k\psi(\gamma_k)\Biggr| \leqslant \sup_{x\in\mathbb{R}}\Biggl|\sum_{k=1}^{m} c_k\mathrm{e}^{\mathrm{i}\gamma_k x}\Biggr| \tag{3}$$

holds and (ii) *this inequality is sharp in the sense that there exist $c_1, \dots, c_m \in \mathbb{C}$ and $\gamma_1, \dots, \gamma_m \in \mathbb{R}$ such that inequality in* (3) *becomes an equality.*

Example 1. Several frequently used characteristic functions are uniquely determined by their imaginary parts. For example, that is true for the following characteristic functions:

(a) The characteristic functions of Beta and Gamma distributions.

(b) The characteristic function

$$f(x) = \mathrm{e}^{\mathrm{i}x/2} J_0\left(\frac{x}{2}\right) = \mathrm{e}^{\mathrm{i}x/2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}$$

of the arcsine distribution with density

$$p(t) = \begin{cases} \frac{1}{\pi\sqrt{t(1-t)}}, & 0 < t < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(c) The characteristic function

$$f(x) = \left(1 - \mathrm{i}x + \sqrt{(1-\mathrm{i}x)^2 - 1}\,\right)^{\varrho}$$

of the Bessel distribution with density

$$p(t) = \begin{cases} \frac{\varrho \mathrm{e}^{-t}}{t} I_{\varrho}(t) = \frac{\varrho \mathrm{e}^{-t}}{t} \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+\varrho+1)} \left(\frac{t}{2}\right)^{2^k+\varrho}, & t > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where ϱ is positive number.

(d) The characteristic function

$$f(x) = \frac{1}{(1-2\mathrm{i}x)^{n/2}}$$

of the χ^2-distribution with density (with n degrees of freedom)

$$p(t) = \begin{cases} \frac{1}{2^{n/2}\Gamma(n/2)} t^{(n/2)-1} \mathrm{e}^{-t/2}, & t > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where n is positive integer.

(e) The characteristic function

$$f(x) = \sum_{k=1}^{m} \frac{\alpha_k \lambda_k}{\lambda_k - xt}$$

of the hyperexponential distributions with density

$$p(t) = \begin{cases} \sum_{k=1}^{m} \alpha_k \lambda_k \mathrm{e}^{-\lambda_k t}, & t > 0, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_k, \lambda_k \geqslant 0$, and $\sum_{k=1}^{m} \alpha_k = 1$.

We end this section with a discussion of the Fourier algebra $A(\mathbb{R})$. If $\varphi \in L^1(\mathbb{R})$, then $\hat{\varphi}$ is uniformly continuous and vanishing at infinity, that is, it belongs to the space $C_0(\mathbb{R})$. Hence, $A(\mathbb{R})$ is a subset of $C_0(\mathbb{R})$. On the other hand, it is known that $A(\mathbb{R}) \not\equiv C_0(\mathbb{R})$. In the context of Theorem 1, it is important to know whether a given $C_0(\mathbb{R})$ function is an element of $A(\mathbb{R})$. Here we present two statements devoted to the sufficient conditions. The following basic result is due to Beurling [1, pp. 39-60]: If $f : \mathbb{R} \to \mathbb{C}$ is locally absolutely continuous and $f, f' \in L^2(\mathbb{R})$, then $f \in A(\mathbb{R})$.

Proposition 1. (*See* [6, p. 20].) *If $f \in C_0(\mathbb{R})$ satisfies the following Hölder type condition of order $\lambda \geqslant 1/2$, i.e.,*

$$\bigl|f(x) - f(y)\bigr| \leqslant C \frac{|x-y|^\lambda}{(|1+x|)^\lambda (|1+y|)^\lambda}, \quad x, y \in R,$$

then $f \in A(\mathbb{R})$.

These facts allow us to construct several classes of functions from $A(\mathbb{R})$. For example, if $S(\mathbb{R})$ denotes the Schwartz class of test functions, then $S(\mathbb{R}) \subset A(\mathbb{R})$. The class $S(\mathbb{R})$ is invariant with respect to the Fourier transform. Therefore, by using Theorem 1, it is very convenient to construct such characteristic functions $f \in S(\mathbb{R})$, which are uniquely determined by their imaginary parts.

2 Proofs

To begin, we recall some terminology and notation that will be used throughout this section. For $A \in \mathcal{B}(\mathbb{R})$, by definition, put

$$-A = \{x \in \mathbb{R}\colon -x \in A\}.$$

Given $f \in L^p(\mathbb{R})$, we associate with f two other functions, defined by

$$f_E(x) = \frac{1}{2}\bigl(f(x) + f(-x)\bigr) \quad \text{and} \quad f_O(x) = \frac{1}{2}\bigl(f(x) - f(-x)\bigr)$$

for $x \in \mathbb{R}$. Because we regard the space $L^p(\mathbb{R})$ as a normed space, i.e., the space of equivalence classes of functions, it is required to explain how to understand the value $f(x)$ of $f \in L^p(\mathbb{R})$ at the point $x \in \mathbb{R}$. We choose any function $\tilde{f} : \mathbb{R} \to \mathbb{C}$ in the class f and define $f(x) := \tilde{f}(x)$. Now it is easy to see that f_E and f_O defined in this manner give us well-defined elements of $L^p(\mathbb{R})$. For simplicity of language, we will call elements of $L^p(\mathbb{R})$ also functions. The functions f_E and f_O are called the even and the odd parts of f, respectively. Of course, we get $f = f_E + f_O$.

If $f \in L^p(\mathbb{R})$ is real-valued, then we set

$$f^+(x) = \max\bigl\{f(x); 0\bigr\} \quad \text{and} \quad f^-(x) = \max\bigl\{-f(x); 0\bigr\}.$$

The functions f^+ and f^- are called the positive and the negative parts of f, respectively. It is evident that both f^+ and f^- are non-negative on $\mathbb{R}$ and $f = f^+ - f^-$.

Proof of Theorem 1. By the definition of $A(\mathbb{R})$, there exists $v \in L^1(\mathbb{R})$ such that $\psi = \hat{v}$. Since ψ is odd, we have

$$\hat{v}(x) = \psi(x) = \psi_O(x) = \frac{1}{2}\biggl(\int\limits_{\mathbb{R}} v(t)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t - \int\limits_{\mathbb{R}} v(t)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t\biggr)$$
$$= \frac{1}{2}\int\limits_{\mathbb{R}} \bigl(v(t) - v(-t)\bigr)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t = \hat{v}_O(x)$$

for $x \in \mathbb{R}$. Therefore, v is also odd. In addition, ψ is real-valued. Therefore,

$$\int\limits_{\mathbb{R}} v(t)\mathrm{e}^{-\mathrm{i}xt}\,\mathrm{d}t = \psi(x) = \overline{\psi}(x) = \int\limits_{\mathbb{R}} \overline{v}(t)\mathrm{e}^{\mathrm{i}xt}\,\mathrm{d}t = \int\limits_{\mathbb{R}} \overline{v}(-s)\mathrm{e}^{-\mathrm{i}xs}\,\mathrm{d}s.$$

Here, using the fact that v is odd, we get that $\overline{v} = -v$. Thus, there exists a real-valued and odd function $v_1 \in L^1(\mathbb{R})$ such that

$$v = \mathrm{i}v_1. \tag{4}$$

Let $v_1 = v_1^+ - v_1^-$. We claim that there exists $E \in \mathcal{B}(\mathbb{R})$ such that $E \cap (-E) = \emptyset$ and

$$\|v_1^+\|_{L^1(\mathbb{R})} = \int\limits_E v_1^+(x)\,\mathrm{d}x = \int\limits_{-E} v_1^-(x)\,\mathrm{d}x = \|v_1^-\|_{L^1(\mathbb{R})} = \frac{1}{2}\|v_1\|_{L^1(\mathbb{R})}. \tag{5}$$

Indeed, we choose any odd function $\tilde{v}_1 : \mathbb{R} \to \mathbb{R}$ in the class $v_1 \in L^1(\mathbb{R})$ and take

$$E = \bigl\{x \in \mathbb{R}\colon\ \tilde{v}_1(x) > 0\bigr\}. \tag{6}$$

If $x \in E$, then

$$\tilde{v}_1(-x) = -\tilde{v}_1(x) < 0$$

since $\tilde{v}_1$ is odd. Therefore,

$$-E = \bigl\{x \in \mathbb{R}\colon\ \tilde{v}_1(x) < 0\bigr\} \quad \text{and} \quad E \cap (-E) = \emptyset.$$

Let $x \in E$. Then $\tilde{v}_1(x) = \tilde{v}_1^+(x)$ and $\tilde{v}_1(-x) = -\tilde{v}_1^-(-x)$. Hence, again using the fact that $\tilde{v}_1$ is odd, we get

$$\tilde{v}_1^+(x) = \tilde{v}_1^-(-x). \tag{7}$$

It is clear that is also true for each $x \in -E$ and $x \in \mathbb{R} \setminus (E \cup -E)$. Thus,

$$\|v_1\|_{L^1(\mathbb{R})} = \|\tilde{v}_1\|_{L^1(\mathbb{R})} = \int\limits_E \tilde{v}_1(x)\,\mathrm{d}x - \int\limits_{-E} \tilde{v}_1(x)\,\mathrm{d}x$$
$$= \int\limits_E \tilde{v}_1^+(x)\,\mathrm{d}x + \int\limits_{-E} \tilde{v}_1^-(x)\,\mathrm{d}x = \int\limits_{\mathbb{R}} \tilde{v}_1^+(x)\,\mathrm{d}x + \int\limits_{\mathbb{R}} \tilde{v}_1^-(x)\,\mathrm{d}x$$
$$= \bigl\|\tilde{v}_1^+\bigr\|_{L^1(\mathbb{R})} + \bigl\|\tilde{v}_1^-\bigr\|_{L^1(\mathbb{R})} = \bigl\|v_1^+\bigr\|_{L^1(\mathbb{R})} + \bigl\|v_1^-\bigr\|_{L^1(\mathbb{R})}.$$

By combining this with (7), we obtain (5). This proves our claim.

Let us show that (A) $\Rightarrow$ (B). To this end, we prove the contraposition: If $\psi \in A(\mathbb{R})$ is a real-valued odd function and $\|\psi\|_{A(\mathbb{R})} < 1$, then there are infinitely many characteristic functions f such that $\Im f \equiv \psi$. Let φ denote the Fourier transform of $|v_1| = v_1^+ + v_1^-$. By Bochner's theorem, φ is continuous positive definite function and $\varphi \in A(\mathbb{R})$. Moreover, φ is real-valued and even since $|v_1|$ is even. Now (4) implies

$$\varphi + \mathrm{i}\psi = \widehat{|v_1|} - \hat{v}_1 = 2\widehat{v_1^-}. \tag{8}$$

Therefore, $\varphi + \mathrm{i}\psi$ is continuous positive definite function with the imaginary part ψ. Combining (4), (5), and (8), we get

$$\begin{aligned}(\varphi + \mathrm{i}\psi)(0) &= 2\widehat{v_1^-}(0) = 2\int\limits_{\mathbb{R}} v_1^-(t)\,\mathrm{d}t = 2\|v_1^-\|_{\mathbb{R}} \\ &= \|v_1\|_{\mathbb{R}} = \|v\|_{\mathbb{R}} = \|\psi\|_{A(\mathbb{R})} < 1.\end{aligned}$$

Finally, let us denote by Ω the set of all continuous real-valued positive definite functions ζ such that

$$\|\zeta\|_{A(\mathbb{R})} = \zeta(0) = 1 - (\varphi + \mathrm{i}\psi)(0) > 0.$$

Then for each $\zeta \in \Omega$, the function $\zeta + \varphi + \mathrm{i}\psi$ is characteristic and has the prescribed imaginary part ψ.

(B) $\Rightarrow$ (A) By arguing as in the proof of (A) $\Rightarrow$ (B), we see that there exists at least one characteristic function f such that $f \in A(\mathbb{R})$ and $\Im f = \psi$. Let φ denote the real part of this function f. Then there exists a non-negative even function $u \in L^1(\mathbb{R})$, $\|u\|_{L^1(\mathbb{R})} = 1$, such that $\varphi = \hat{u}$ since $\varphi = \Re f$ is also characteristic function. Set

$$w = u - v_1 = u - v_1^+ + v_1^-, \tag{9}$$

where v_1 is defined in (4) with $\hat{v} = \psi$. If E is the same as in (6), then, by the definition of the function v_1^-, we get

$$\int\limits_E v_1^-(t)\,\mathrm{d}t = 0.$$

Recall that now $\|\psi\|_{A(\mathbb{R})} = \|v\|_{L^1(\mathbb{R})} = 1$. Therefore, by (5), we have

$$\begin{aligned}\int\limits_E w(t)\,\mathrm{d}t &= \int\limits_E u(t)\,\mathrm{d}t - \int\limits_E v_1^+(t)\,\mathrm{d}t + \int\limits_E v_1^-(t)\,\mathrm{d}t \\ &= \int\limits_E u(t)\,\mathrm{d}t - \int\limits_E v_1^+(t)\,\mathrm{d}t = \int\limits_E u(t)\,\mathrm{d}t - \frac{1}{2}.\end{aligned} \tag{10}$$

Since $f = \hat{w}$, we see that w is non-negative on $\mathbb{R}$. Therefore, from (10), and the fact that u is even, it is immediate that

$$\int\limits_{-E} u(t)\,\mathrm{d}t = \int\limits_E u(t)\,\mathrm{d}t \geqslant \frac{1}{2}.$$

On the other hand, we have $\|u\|_{L^1(\mathbb{R})} = 1$ and $E \cap (-E) = \emptyset$. Therefore, we see in fact that

$$\int\limits_{-E} u(t)\,\mathrm{d}t = \int\limits_{E} u(t)\,\mathrm{d}t = \frac{1}{2}. \tag{11}$$

This implies that $u(x) = 0$ for $x \in \mathbb{R} \setminus (E \cup (-E))$. On the other hand, w is non-negative on $\mathbb{R}$ and $w = u - v_1^+$ on E. Now using (5), we get

$$\int\limits_{E} v_1^+(t)\,\mathrm{d}t = \frac{1}{2}.$$

Combining this with (11), we have that $u \equiv v_1^+$ on E. Next, since w and v_1 are even and odd, respectively, it follows that $u = v_1^-$ on $-E$. Thus, $u = v_1^+ + v_1^-$, i.e., the real part of f is uniquely determined by $\Im f$.

(B) $\Leftrightarrow$ (C) Recall a characterization of elements $\varphi \in B(\mathbb{R})$ (see [5, p. 304]). A continuous function $\varphi : \mathbb{R} \to \mathbb{C}$ is the Fourier transform of some $\mu \in M(\mathbb{R})$ if and only if there exists a constant $K > 0$ such that

$$\sup_{x\in\mathbb{R}} \left| \sum_{k=1}^{m} c_k \varphi(\gamma_k) \right| \leqslant K \sup_{x\in\mathbb{R}} \left| \sum_{k=1}^{m} c_k \mathrm{e}^{\mathrm{i}\gamma_k x} \right| \tag{12}$$

for all finite sets of complex numbers $c_1, \dots, c_m$ and points $\gamma_1, \dots, \gamma_m \in$ in $\mathbb{R}$. Moreover, if $\varphi \in B(\mathbb{R})$, then the smallest value of K satisfying (12) is equal to $\|\varphi\|_{B(\mathbb{R})}$, (see e.g., [3, p. 465]). Now it is easy to check that (B) and (C) are equivalent. This finishes the proof of Theorem 1. □

Proof of Example 1. Let $p : \mathbb{R} \to \mathbb{R}$ be a probability density. Assume that p is continuous and $p(x) = 0$ for $x \in (-\infty, 0]$. Then the function $\tilde{p}(x) := p(-x)$ is also probability density supported on $[0, \infty)$. Therefore,

$$\|p_O\|_{L^1(\mathbb{R})} = \frac{1}{2}\|p - \tilde{p}\|_{L^1(\mathbb{R})} = \frac{1}{2}\big(\|p\|_{L^1(\mathbb{R})} + \|\tilde{p}\|_{L^1(\mathbb{R})}\big) = 1.$$

Combining this with the fact that

$$\Im\hat{p} = \frac{1}{2\mathrm{i}}\widehat{(p - \tilde{p})} = \frac{1}{\mathrm{i}}\hat{p}_O$$

and statement (B) of Theorem 1, we see that $\Im\hat{p}$ uniquely determines the characteristic function $f = \hat{p}$. Since all probability densities p given in our Examples 1(a)–1(e) are continuous and $p(x) = 0$ for $x \in (-\infty, 0]$, this complete the proof of Example 1. □

References

1. A. Beurling, *The Collected Works of Arne Beurling, Vol. 2: Harmonic analysis*, Birkhäuser, Boston, 1989.
2. V.I. Bogachev, *Measure Theory, Vol. II*, Springer, Berlin, New York, 2007.

3. W.F. Eberlein, Characterization of Fourier–Stieltjes transforms, *Duke Math. J.*, **22**:465–468, 1955.
4. W. Feller, *An Introduction to Probability and Its Applications, Vol. II,* 2nd ed., John Wiley & Sons, New York, 1971.
5. E. Hewitt, K.A. Ross, *Abstract Harmonic Analysis II*, Springer, Berlin, Heidelberg, 1997.
6. E. Liflyand, S. Samko, R. Trigub, The Wiener algebra of absolutely convergent Fourier integrals: An overview, *Anal. Math. Phys.*, **2**(1):1–68, 2012.
7. E. Lukacs, *Characteristic Functions,* 2nd ed., Hafner, New York, 1970.
8. N.G. Ushakov, *Selected Topics in Characteristic Functions*, VSP, Utrecht, 1999.