ISSN 1392-5113 Nonlinear Analysis: Modelling and Control, 2017, Vol. 22, No. 2, 160–172
https://doi.org/10.15388/NA.2017.2.2

Existence and uniqueness of positive solutions for a class of fractional differential equation with integral boundary conditions*

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Received: July 25, 2015 / **Revised:** January 22, 2016 / **Published online:** January 19, 2017

Abstract. The purpose of this paper is to investigate the existence and uniqueness of positive solutions for a class of fractional differential equation with integral boundary conditions. Our analysis relies on two fixed point theorems of a sum operator in partial ordering Banach space. The main results obtained can not only guarantee the existence of a unique positive solution, but also be applied to construct an iterative scheme for approximating it.

Keywords: existence and uniqueness, positive solution, fractional differential equation, integral boundary value condition, fixed point theorems for a sum operator.

1 Introduction

It is well known that fractional differential equations arise in many fields, such as physics, mechanics, chemistry, economics and biological sciences, etc.; see [1–4, 6–17, 19, 20, 22–24] and the references therein. In recent years, the study of positive solutions for fractional differential equation boundary value problems has attracted considerable attention; see [1, 3, 8, 14, 20, 23] and the references therein. On the other hand, the uniqueness of positive solution for nonlinear fractional differential equation boundary value problems has been studied by some authors; see [2, 9, 17, 19, 22] and the references therein. In [24], by using Guo–Krasnosel'skii's fixed point theorem for completely continuous operators, Zhao et al. obtained the existence and nonexistence results of positive solutions for a class of

*This paper was supported financially by the Youth Science Foundation of China (11201272), Shanxi Province Science Foundation (2015011005), and the Youth Science Foundation of Shanxi University of Finance and Economic (2014026).

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fractional differential equation with integral boundary conditions, where the nonlinear term satisfies super-linearity or sub-linearity conditions. But it is not able to construct iterative schemes for approximating the positive solutions. In a recent paper [15], Sun and Zhao constructed a completely continuous operator and utilized monotone iteration method to study the following fractional differential equation with integral boundary conditions

$$D_{0+}^{\alpha}u(t) + g(t)f\big(t, u(t)\big) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0, \qquad u(1) = \int_0^1 q(s)u(s)\,\mathrm{d}s,$$

where $2 < \alpha \leqslant 3$, D_{0+}^{α} is the standard Riemann–Liouville fractional derivative of order α. The authors established the existence of one positive solution for this problem, and can construct an iterative sequence for approximating the positive solution for a given initial value. But the uniqueness of positive solutions is not treated in [15, 24].

Motivated by [15], in present paper we consider the following form of fractional differential equation with integral boundary conditions

$$\begin{aligned} &D_{0+}^{\alpha}u(t) + f\big(t, u(t)\big) + g\big(t, u(t)\big) = 0, \quad 0 < t < 1, \\ &u(0) = u'(0) = 0, \qquad u(1) = \int_0^1 q(s)u(s)\,\mathrm{d}s, \end{aligned} \tag{1}$$

where $2 < \alpha \leqslant 3$, D_{0+}^{α} is also the Riemann–Lioville fractional derivative of order α. The function $q(t)$ satisfies the following conditions:

(Q) $q : [0,1] \to [0,\infty)$ with $q \in L^1[0,1]$ and $\sigma_1 = \int_0^1 s^{\alpha-1}(1-s)q(s)\,\mathrm{d}s > 0$, $\sigma_2 = \int_0^1 s^{\alpha-1}q(s)\,\mathrm{d}s < 1$.

Our main interest in this paper is to give some alternative answers to the main results of these papers [12, 15, 16, 24]. We will use two fixed point theorems for a sum operator to show the existence and uniqueness of positive solutions for problem (1). Moreover, we can construct some sequences for approximating the unique solution. Comparing our main results in this paper with ones in [15,24], we can get the existence and uniqueness of positive solutions for problem (1). For any initial value in a special set, we can construct an iterative scheme for approximating the unique solution. In addition, we do not assume different requirements of super-linearity, sub-linearity or boundness of nonlinear terms.

2 Preliminaries and previous results

For the convenience, here we list some definitions, lemmas and fixed point theorems that will be used in the proofs of our main results.

Definition 1. (See [13, Def. 2.1].) The integral

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} \, \mathrm{d}t, \quad x > 0,$$

is called the Riemann–Liouville fractional integral of order α, where $\alpha > 0$ and $\Gamma(\alpha)$ denotes the gamma function.

Definition 2. (See [13, pp. 36–37].) For a function $f(x)$ given in the interval $[0, \infty)$, the expression

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^n \int_0^x \frac{f(t)}{(x-t)^{\alpha-n+1}} \, \mathrm{d}t,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, is called the Riemann–Liouville fractional derivative of order α.

Lemma 1. (*See* [24].) *Assume* (Q) *holds. Let* $y \in C[0, 1]$, $2 < \alpha \leqslant 3$, *then the following integral boundary value problem*

$$D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1,$$
$$u(0) = u'(0) = 0, \qquad u(1) = \int_0^1 q(t) u(t) \, \mathrm{d}t,$$

has the solution

$$u(t) = \int_0^1 G(t, s) y(s) \, \mathrm{d}s,$$

where

$$G(t, s) = G_1(t, s) + G_2(t, s), \quad (t, s) \in [0, 1] \times [0, 1], \tag{2}$$

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leqslant s \leqslant t \leqslant 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \leqslant t \leqslant s \leqslant 1, \end{cases} \tag{3}$$

$$G_2(t, s) = \frac{t^{\alpha-1}}{1-\sigma_2} \int_0^1 G_1(\tau, s) q(\tau) \, \mathrm{d}\tau. \tag{4}$$

Lemma 2. (*See* [20].) *The function* $G_1(t, s)$ *defined by* (3) *has the following properties*:

$$\frac{t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}}{\Gamma(\alpha)} \leqslant G_1(t, s) \leqslant \frac{s(1-s)^{\alpha-1}}{\Gamma(\alpha-1)}, \quad t, s \in [0, 1].$$

From [15] we have

$$G(t,s) \leqslant \frac{t^{\alpha-1}(1-s)^{\alpha-1}}{(1-\sigma_2)\Gamma(\alpha)}, \quad t,s \in [0,1]. \tag{5}$$

On the other hand, from (2)–(4) and Lemma 2,

$$\begin{aligned}
G(t,s) = G_1(t,s) + G_2(t,s) \geqslant G_2(t,s) = \frac{t^{\alpha-1}}{1-\sigma_2}\int_0^1 G_1(\tau,s)q(\tau)\,\mathrm{d}\tau \\
\geqslant \frac{t^{\alpha-1}}{1-\sigma_2}\int_0^1 \frac{\tau^{\alpha-1}(1-\tau)s(1-s)^{\alpha-1}}{\Gamma(\alpha)}q(\tau)\,\mathrm{d}\tau \\
= \frac{t^{\alpha-1}s(1-s)^{\alpha-1}}{(1-\sigma_2)\Gamma(\alpha)}\int_0^1 \tau^{\alpha-1}(1-\tau)q(\tau)\,\mathrm{d}\tau.
\end{aligned}$$

Therefore, we have

$$G(t,s) \geqslant \frac{\sigma_1 s(1-s)^{\alpha-1}t^{\alpha-1}}{(1-\sigma_2)\Gamma(\alpha)}. \tag{6}$$

In the rest of this section, we introduce some notations and known results. For convenience of readers, we suggest that one refer to [5, 18, 21] for details.

Let $(E, \|\cdot\|)$ be a real Banach space and θ be the zero element of E. A non-empty closed convex set $P \subset E$ is a cone if it satisfies: (a) $x \in P$, $\lambda \geqslant 0 \Rightarrow \lambda x \in P$; (b) $x \in P$, $-x \in P \Rightarrow x = \theta$. E is partially ordered by cone P, i.e., $x \leqslant y$ if and only if $y - x \in P$. A cone P is called normal if there exists a constant $N > 0$ such that, for all $x, y \in E$, $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$; in this case, N is called the normality constant of P. We say that an operator $A : E \to E$ is increasing (decreasing) if $x \leqslant y$ implies $Ax \leqslant Ay$ ($Ax \geqslant Ay$).

For all $x, y \in E$, the notation $x \sim y$ means that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e., $h \geqslant \theta$ and $h \neq \theta$), we denote by P_h the set $P_h = \{x \in E\colon\ x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 3. Let γ be a real number with $0 < \gamma < 1$. An operator $A : P \to P$ is said to be γ-concave if it satisfies $A(tx) \geqslant t^\gamma Ax$ for all $t \in (0,1)$, $x \in P$. An operator $A : E \to E$ is said to be homogeneous if it satisfies $A(tx) = tAx$ for all $t > 0$, $x \in E$. An operator $A : P \to P$ is said to be sub-homogeneous if it satisfies $A(tx) \geqslant tAx$ for all $t > 0$, $x \in P$.

In recent papers [18, 21], the authors considered the following sum operator equation:

$$Ax + Bx = x, \tag{7}$$

where A, B are monotone operators. They established the existence and uniqueness of positive solutions for (7) and present the following interesting results.

Theorem 1. (*See* [21].) *Let P be a normal cone in a real Banach space E, $A : P \to P$ be an increasing γ-concave operator, and $B : P \to P$ be an increasing sub-homogeneous operator. Assume that*

(i) *there is $h > \theta$ such that $Ah \in P_h$ and $Bh \in P_h$;*
(ii) *there exists a constant $\delta_0 > 0$ such that $Ax \geqslant \delta_0 Bx$ for all $x \in P$.*

Then the operator equation (7) *has a unique solution x^* in P_h. Moreover, constructing successively the sequence $y_n = Ay_{n-1} + By_{n-1}$, $n = 1, 2 \dots$, for any initial value $y_0 \in P_h$, we have $y_n \to x^*$ as $n \to \infty$.*

Theorem 2. (*See* [18].) *Let P be a normal cone a real Banach space E, $A : P \to P$ be an increasing operator, and $B : P \to P$ be a decreasing operator. Assume that:*

(i) *for any $x \in P$ and $t \in (0, 1)$, there exist $\varphi_i(t) \in (t, 1)$ $(i = 1, 2)$ such that*

$$A(tx) \geqslant \varphi_1(t)Ax, \qquad B(tx) \leqslant \frac{1}{\varphi_2(t)}Bx; \tag{8}$$

(ii) *there exists $h_0 \in P_h$ such that $Ah_0 + Bh_0 \in P_h$.*

Then the operator equation (7) *has a unique solution x^* in P_h. Moreover, for any initial values $x_0, y_0 \in P_h$, constructing successively the sequences*

$$x_n = Ax_{n-1} + By_{n-1}, \quad y_n = Ay_{n-1} + Bx_{n-1}, \quad n = 1, 2, \dots,$$

we have $x_n \to x^*$, $y_n \to x^*$ as $n \to \infty$.

Remark 1. When B is a null operator, Theorems 1, 2 also hold.

3 Existence and uniqueness of positive solutions for problem (1)

In this section, we use Theorems 1, 2 to study problem (1), and we obtain some new results on the existence and uniqueness of positive solutions.

Throughout this section, we work in the Banach space $C[0, 1] = \{x : [0, 1] \to R$ is continuous$\}$ with the standard norm $\|x\| = \sup\{|x(t)|\colon t \in [0, 1]\}$. Let $P = \{x \in C[0, 1]\colon x(t) \geqslant 0,\ t \in [0, 1]\}$, then it is a normal cone in $C[0, 1]$ and the normality constant is 1. We know that this space can be equipped with a partial order given by

$$x \leqslant y, \quad y \in C[0, 1] \quad \Longleftrightarrow \quad x(t) \leqslant y(t), \quad t \in [0, 1].$$

Theorem 3. *Assume* (Q) *and*

(H1) *$f, g : [0, 1] \times [0, +\infty) \to [0, +\infty)$ are continuous and increasing with respect to the second argument, $g(t, 0) \not\equiv 0$;*
(H2) *$g(t, \lambda x) \geqslant \lambda g(t, x)$ for $\lambda \in (0, 1)$, $t \in [0, 1]$, $x \in [0, +\infty)$, and there exists a constant $\gamma \in (0, 1)$ such that $f(t, \lambda x) \geqslant \lambda^\gamma f(t, x)$ for all $t \in [0, 1]$, $\lambda \in (0, 1)$, $x \in [0, +\infty)$;*
(H3) *there exists a constant $\delta_0 > 0$ such that $f(t, x) \geqslant \delta_0 g(t, x)$, $t \in [0, 1]$, $x \geqslant 0$.*

Then problem (1) *has a unique positive solution u^* in P_h, where $h(t) = t^{\alpha-1}$, $t \in [0,1]$. And, for any initial value $u_0 \in P_h$, constructing successively the sequence*

$$u_{n+1}(t) = \int_0^1 G(t,s)\bigl[f\bigl(s,u_n(s)\bigr) + g\bigl(s,u_n(s)\bigr)\bigr]\,\mathrm{d}s, \quad n = 0,1,2\dots,$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$, where $G(t,s)$ is given as (2)*.*

Proof. From Lemma 1 we know that problem (1) has an integral formulation given by

$$u(t) = \int_0^1 G(t,s)\bigl[f\bigl(s,u(s)\bigr) + g\bigl(s,u(s)\bigr)\bigr]\,\mathrm{d}s,$$

where $G(t,s)$ is given as in (2).

Define two operators $A : P \to E$ and $B : P \to E$ by

$$Au(t) = \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s, \qquad Bu(t) = \int_0^1 G(t,s)g\bigl(s,u(s)\bigr)\,\mathrm{d}s.$$

Then we can see that u is the solution of problem (1) if and only if $u = Au + Bu$. From (H1), (2)–(4) we know that $A : P \to P$ and $B : P \to P$. In the following, we check that A, B satisfy all assumptions of Theorem 1.

Firstly, we show that A, B are two increasing operators. For $u, v \in P$ with $u \geqslant v$, we have $u(t) \geqslant v(t)$, $t \in [0,1]$, and, by (H1), (2)–(4),

$$Au(t) = \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s \geqslant \int_0^1 G(t,s)f\bigl(s,v(s)\bigr)\,\mathrm{d}s = Av(t).$$

That is, $Au \geqslant Av$. Similarly, $Bu \geqslant Bv$.

Secondly, we prove that A is a γ-concave operator and B is a sub-homogeneous operator. For any $\lambda \in (0,1)$ and $u \in P$, from (H2) we obtain

$$A(\lambda u)(t) = \int_0^1 G(t,s)f\bigl(s,\lambda u(s)\bigr)\,\mathrm{d}s \geqslant \lambda^\gamma \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s = \lambda^\gamma Au(t).$$

That is, $A(\lambda(u)) \geqslant \lambda^\gamma Au$ for $\lambda \in (0,1)$, $u \in P$. So the operator A is a γ-concave operator. Also, for any $\lambda \in (0,1)$ and $u \in P$, from (H2) we obtain

$$B(\lambda u)(t) = \int_0^1 G(t,s)g\bigl(s,\lambda u(s)\bigr)\,\mathrm{d}s \geqslant \lambda \int_0^1 G(t,s)g\bigl(s,u(s)\bigr)\,\mathrm{d}s = \lambda Bu(t).$$

That is, $B(\lambda u)(t) \geqslant \lambda Bu$ for $\lambda \in (0,1)$, $u \in P$. So the operator B is sub-homogeneous.

Next, we prove that $Ah \in P_h$ and $Bh \in P_h$. From (H1), (5), and (6),

$$Ah(t) = \int_0^1 G(t,s) f\bigl(s, s^{\alpha-1}\bigr)\,\mathrm{d}s \leqslant \frac{t^{\alpha-1}}{(1-\sigma_2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,1)\,\mathrm{d}s,$$

$$Ah(t) = \int_0^1 G(t,s) f\bigl(s, s^{\alpha-1}\bigr)\,\mathrm{d}s \geqslant \frac{\sigma_1 t^{\alpha-1}}{(1-\sigma_2)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} f(s,0)\,\mathrm{d}s.$$

From (H3) and (H1) we have

$$f(s,1) \geqslant f(s,0) \geqslant \delta_0 g(s,0) \geqslant 0.$$

Note that $\alpha - 1 > 0$ and $g(t,0) \not\equiv 0$, we can get

$$\int_0^1 (1-s)^{\alpha-1} f(s,1)\,\mathrm{d}s \geqslant \int_0^1 s(1-s)^{\alpha-1} f(s,0)\,\mathrm{d}s \geqslant \delta_0 \int_0^1 s(1-s)^{\alpha-1} g(s,0)\,\mathrm{d}s > 0.$$

Let

$$l_1 := \frac{\sigma_1}{(1-\sigma_2)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} f(s,0)\,\mathrm{d}s > 0,$$

$$l_2 := \frac{1}{(1-\sigma_2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s,1)\,\mathrm{d}s > 0.$$

Then $l_2 \geqslant l_1 > 0$ and thus $l_1 h(t) \leqslant Ah(t) \leqslant l_2 h(t)$, $t \in [0,1]$. So we have $Ah \in P_h$. Similarly,

$$Bh(t) = \int_0^1 G(t,s) g\bigl(s, s^{\alpha-1}\bigr)\,\mathrm{d}s \leqslant \frac{t^{\alpha-1}}{(1-\sigma_2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} g(s,1)\,\mathrm{d}s$$

$$Bh(t) = \int_0^1 G(t,s) g\bigl(s, s^{\alpha-1}\bigr)\,\mathrm{d}s \geqslant \frac{\sigma_1 t^{\alpha-1}}{(1-\sigma_2)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1} g(s,0)\,\mathrm{d}s,$$

also from $g(t,0) \not\equiv 0$ we can easily prove $Bh \in P_h$. That is, condition (i) of Theorem 1 holds.

Further, we prove that condition (ii) of Theorem 1 is also satisfied. For $u\in P$, by (H3),

$$Au(t) = \int_0^1 G(t,s)f\bigl(s,u(s)\bigr)\,\mathrm{d}s \geqslant \delta_0 \int_0^1 G(t,s)g\bigl(s,u(s)\bigr)\,\mathrm{d}s = \delta_0 Bu(t).$$

So we obtain $Au \geqslant \delta_0 Bu$, $u \in P$.

Finally, from Theorem 1 we know that operator equation $Au + Bu = u$ has a unique solution u^* in P_h; for any initial value $u_0 \in P_h$, constructing successively the sequence $u_n = Au_{n-1} + Bu_{n-1}$, $n = 1, 2, \dots$, we have $u_n \to u^*$ as $n \to \infty$. That is, problem (1) has a unique positive solution u^* in P_h. And, for any initial value $u_0 \in P_h$, constructing successively the sequence

$$u_{n+1}(t) = \int_0^1 G(t,s)\bigl[f\bigl(s,u_n(s)\bigr) + g\bigl(s,u_n(s)\bigr)\bigr]\,\mathrm{d}s, \quad n = 0, 1, 2 \dots,$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$. □

Corollary 1. *Assume* (Q) *and*

- (H1′) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and increasing with respect to the second argument,* $f(t,0) \not\equiv 0$;
- (H2′) *there exists a constant* $\gamma \in (0,1)$ *such that* $f(t,\lambda x) \geqslant \lambda^\gamma f(t,x)$ *for all* $t \in [0,1]$, $\lambda \in (0,1)$, $x \in [0,+\infty)$.

Then the following problem

$$D_{0+}^{\alpha} u(t) + f\bigl(t,u(t)\bigr) = 0, \quad 0 < t < 1,\ 2 < \alpha \leqslant 3,$$
$$u(0) = u'(0) = 0, \qquad u(1) = \int_0^1 q(s)u(s)\,\mathrm{d}s$$

has a unique positive solution u^* *in* P_h*, where* $h(t) = t^{\alpha-1}$, $t \in [0,1]$. *And, for any initial value* $u_0 \in P_h$*, constructing successively the sequence*

$$u_{n+1}(t) = \int_0^1 G(t,s)f\bigl(s,u_n(s)\bigr)\,\mathrm{d}s, \quad n = 0, 1, 2 \dots,$$

we have $u_n(t) \to u^*(t)$ *as* $n \to \infty$*, where G(t,s) is given as* (2).

Proof. From Remark 1 and Theorem 3 the conclusions hold. □

Theorem 4. *Assume* (Q) *and*

- (H4′) $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and increasing with respect to the second argument,* $f(t,0) \not\equiv 0$;

(H5′) $g : [0,1] \times [0,+\infty) \to [0,+\infty)$ *is continuous and decreasing with respect to the second argument,* $g(t,1) \not\equiv 0$;

(H6′) *for* $\lambda \in (0,1)$*, there exist* $\varphi_i(\lambda) \in (\lambda,1)$ $(i=1,2)$ *such that*

$$f(t,\lambda x) \geqslant \varphi_1(\lambda) f(t,x), \quad g(t,\lambda x) \leqslant \frac{1}{\varphi_2(\lambda)} g(t,x)$$

for $t \in [0,1]$, $x \in [0,+\infty)$.

Then problem (1) *has a unique positive solution* u^* *in* P_h*, where* $h(t) = t^{\alpha-1}$, $t \in [0,1]$. *And, for any initial values* $x_0, y_0 \in P_h$*, constructing successively the sequences*

$$x_{n+1}(t) = \int_0^1 G(t,s)\bigl[f\bigl(s,x_n(s)\bigr) + g\bigl(s,y_n(s)\bigr)\bigr]\,\mathrm{d}s,$$

$$y_{n+1}(t) = \int_0^1 G(t,s)\bigl[f\bigl(s,y_n(s)\bigr) + g\bigl(s,x_n(s)\bigr)\bigr]\,\mathrm{d}s,$$

$n = 0,1,2\dots$*, we have* $x_n(t) \to u^*(t)$*,* $y_n(t) \to u^*(t)$ *as* $n \to \infty$*, where* $G(t,s)$ *is given as* (2)*.*

Proof. Similar to the proof of Theorem 3, we consider two operators $A : P \to E$ and $B : P \to E$ by

$$Au(t) = \int_0^1 G(t,s) f\bigl(s,u(s)\bigr)\,\mathrm{d}s, \qquad Bu(t) = \int_0^1 G(t,s) g\bigl(s,u(s)\bigr)\,\mathrm{d}s.$$

From (H4), (H5) we know that $A : P \to P$ is increasing and $B : P \to P$ is decreasing. Further, from (H6) we can prove that A, B satisfy (8). So we only need to prove that $Ah + Bh \in P_h$. From (H4), (H5), (5), and (6),

$$\begin{aligned} Ah(t) + Bh(t) &= \int_0^1 G(t,s)\bigl[f\bigl(s,s^{\alpha-1}\bigr) + g\bigl(s,s^{\alpha-1}\bigr)\bigr]\,\mathrm{d}s \\ &\leqslant \frac{t^{\alpha-1}}{(1-\sigma_2)\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1}\bigl[f(s,1) + g(s,0)\bigr]\,\mathrm{d}s, \\ Ah(t) + Bh(t) &= \int_0^1 G(t,s)\bigl[f\bigl(s,s^{\alpha-1}\bigr) + g\bigl(s,s^{\alpha-1}\bigr)\bigr]\,\mathrm{d}s \\ &\geqslant \frac{\sigma_1 t^{\alpha-1}}{(1-\sigma_2)\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-1}\bigl[f(s,0) + g(s,1)\bigr]\,\mathrm{d}s. \end{aligned}$$

From (H4) and (H5) we have

$$f(s,1)+g(s,0)\geqslant f(s,0)+g(s,1)\geqslant 0.$$

Note that $\alpha-1>0$ and $f(t,0)+g(t,1)\not\equiv 0$, we can get

$$\int_0^1 (1-s)^{\alpha-1}\big[f(s,1)+g(s,0)\big]\,\mathrm{d}s \geqslant \int_0^1 s(1-s)^{\alpha-1}\big[f(s,0)+g(s,1)\big]\,\mathrm{d}s>0.$$

Let

$$l_3:=\frac{\sigma_1}{(1-\sigma_2)\Gamma(\alpha)}\int_0^1 s(1-s)^{\alpha-1}\big[f(s,0)+g(s,1)\big]\,\mathrm{d}s>0,$$
$$l_4:=\frac{1}{(1-\sigma_2)\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\big[f(s,1)+g(s,0)\big]\,\mathrm{d}s>0.$$

Then $l_4\geqslant l_3>0$ and thus $l_3h(t)\leqslant Ah(t)+Bh(t)\leqslant l_4h(t)$, $t\in[0,1]$. So we have $Ah+Bh\in P_h$.

Finally, from Theorem 2 we know that operator equation $Au+Bu=u$ has a unique solution u^* in P_h; for any initial values $x_0,y_0\in P_h$, constructing successively the sequences

$$x_n=Ax_{n-1}+By_{n-1},\quad y_n=Ay_{n-1}+Bx_{n-1},\quad n=1,2,\dots,$$

we have $x_n\to x^*$, $y_n\to x^*$ as $n\to\infty$. That is, problem (1) has a unique positive solution u^* in P_h, where $h(t)=t^{\alpha-1}$, $t\in[0,1]$. And, for any initial values $x_0,y_0\in P_h$, constructing successively the sequences

$$x_{n+1}(t)=\int_0^1 G(t,s)\big[f\big(s,x_n(s)\big)+g\big(s,y_n(s)\big)\big]\,\mathrm{d}s,$$
$$y_{n+1}(t)=\int_0^1 G(t,s)\big[f\big(s,y_n(s)\big)+g\big(s,x_n(s)\big)\big]\,\mathrm{d}s,$$

$n=0,1,2\dots$, we have $x_n(t)\to u^*(t)$, $y_n(t)\to u^*(t)$ as $n\to\infty$. □

Corollary 2. *Assume* (Q), (H1′) *and*

(H7′) *for* $\lambda\in(0,1)$*, there exist* $\varphi(\lambda)\in(\lambda,1)$ *such that* $f(t,\lambda x)\geqslant\varphi(\lambda)f(t,x)$ *for* $t\in[0,1]$, $x\in[0,+\infty)$.

Then the following problem

$$D_{0+}^{\alpha}u(t) + f\big(t, u(t)\big) = 0, \quad 0 < t < 1,\ 2 < \alpha \leqslant 3,$$
$$u(0) = u'(0) = 0, \qquad u(1) = \int_0^1 q(s)u(s)\,\mathrm{d}s$$

has a unique positive solution u^* *in* P_h*, where* $h(t) = t^{\alpha-1}$, $t \in [0,1]$. *And, for any initial value* $u_0 \in P_h$*, constructing successively the sequence*

$$u_{n+1}(t) = \int_0^1 G(t,s)f\big(s, u_n(s)\big)\,\mathrm{d}s, n = 0,1,2\ldots,$$

we have $u_n(t) \to u^*(t)$ *as* $n \to \infty$*, where* $G(t,s)$ *is given as* (2)*.*

Proof. From Remark 1 and Theorem 4 the conclusions hold. □

It is easy to see that there are many functions which satisfy the conditions of Theorems 3, 4. Here we present two simple examples.

Example 1. Consider the following problem:

$$D_{0+}^{2.2}u(t) + u^{1/4}(t) + \frac{u(t)}{1+u(t)}\mathrm{e}^t + a = 0, \quad t \in (0,1),$$
$$u(0) = u'(0) = 0, \qquad u(1) = \int_0^1 s^2 u(s)\,\mathrm{d}s, \tag{9}$$

where $a > 0$ is a constant. In this example, $\alpha = 2.2$, $q(t) = t^2$. Then $q : [0,1] \to [0,\infty)$ with $q \in L^1[0,1]$ and $\sigma_1 = \int_0^1 s^{1.2}(1-s)s^2\,\mathrm{d}s = 25/546 > 0$, $\sigma_2 = \int_0^1 s^{1.2}s^2\,\mathrm{d}s = 5/21 < 1$. Take $0 < b < a$, and let

$$f(t,x) = x^{1/4} + b, \qquad g(t,x) = \frac{x}{1+x}\mathrm{e}^t + a - b, \qquad \gamma = \frac{1}{4}.$$

Clearly, $f, g : [0,1] \times [0,\infty) \to [0,\infty)$ are continuous and increasing with respect to the second argument, $g(t,0) = a - b > 0$. In addition, for $\lambda \in (0,1)$, $t \in [0,1]$, $x \in [0,\infty)$, we have

$$g(t,\lambda x) = \frac{\lambda x}{1+\lambda x}\mathrm{e}^t + a - b \geqslant \frac{\lambda x}{1+x}\mathrm{e}^t + \lambda(a-b) = \lambda g(t,x),$$
$$f(t,\lambda x) = \lambda^{1/4}x^{1/4} + b \geqslant \lambda^{1/4}(x^{1/4} + b) = \lambda^{\gamma} f(t,x).$$

Moreover, if we take $\delta_0 \in (0, b/(e+a-b)]$, then we obtain

$$f(t,x) = x^{1/4} + b \geqslant b = \frac{b}{e+a-b} \cdot (e+a-b) \geqslant \delta_0\left[\frac{x}{1+x}\mathrm{e}^t + a - b\right]$$
$$= \delta_0 g(t,x).$$

So all the conditions of Theorem 3 are satisfied. Therefore, problem (9) has a unique positive solution in P_h, where $h(t) = t^{1.2}$, $t \in [0,1]$.

Example 2. Consider the following problem:

$$\begin{aligned} &D_{0^+}^{2.2}u(t) + u^{1/2}(t) + \frac{t}{1+u^{1/3}(t)} + a = 0, \quad t \in (0,1), \\ &u(0) = u'(0) = 0, \qquad u(1) = \int_0^1 s^2 u(s)\,\mathrm{d}s, \end{aligned} \tag{10}$$

where $a > 0$ is a constant. In this example, α, $q(t)$ are the same with Example 1. Let

$$f(t,x) = x^{1/2} + a, \qquad g(t,x) = \frac{t}{1+x^{1/3}}.$$

Clearly, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and increasing with respect to the second argument, $f(t,0) = a > 0$. $g : [0,1] \times [0,\infty) \to [0,\infty)$ is continuous and decreasing with respect to the second argument, $g(t,1) = t/2 \not\equiv 0$. In addition, let $\varphi_1(\lambda) = \lambda^{1/2}$, $\varphi_2(\lambda) = \lambda^{1/3}$. Then $\varphi_1(\lambda), \varphi_2(\lambda) \in (\lambda, 1)$ for $\lambda \in (0,1)$. Further, we have

$$\begin{aligned} f(t,\lambda x) &= \lambda^{1/2}x^{1/2} + a \geqslant \lambda^{1/2}\bigl(x^{1/2} + a\bigr) = \varphi_1(\lambda) f(t,x), \\ g(t,\lambda x) &= \frac{t}{1+(\lambda x)^{1/3}} \leqslant \frac{t}{\lambda^{1/3}(1+x^{1/3})} = \frac{1}{\varphi_2(\lambda)} g(t,x). \end{aligned}$$

So all the conditions of Theorem 4 are satisfied. Therefore, problem (10) has a unique positive solution in P_h, where $h(t) = t^{1.2}$, $t \in [0,1]$.

Remark 2. In [15,24], the nonlinear terms were required super-linearity, sub-linearity or boundness. Here our nonlinear terms f, g in Examples 1, 2 do not satisfy these conditions. So the conclusions of Examples 1, 2 cannot been obtained by the main results in [15,24].

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