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Dynamics of a host-pathogen model with constant mortality rate

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Abstract. In this paper, we propose a discrete-time host-pathogen model and study its qualitative behavior. The model is for the spread of an infectious disease with constant mortality rate of hosts. Moreover, the time-step is equal to the duration of the infectious phase, and the host mortality is taken at some constant rate $d > 0$. This two-dimensional discrete-time epidemic model has complex dynamical behavior. More precisely, we investigate the existence and uniqueness of positive equilibrium point, boundedness character, local and global asymptotic stability of unique positive equilibrium point, and the rate of convergence of positive solutions that converge to unique positive equilibrium point. Numerical simulations are provided to illustrate our theoretical results.

Keywords: host-pathogen model, boundedness, local stability, global stability.

1 Introduction

Differential and difference equations are used to study a wide range of population models. For more detail of some interesting population models both in differential equations as well as in difference equations, we refer the interested reader to [1, 2, 15]. When the population remains small over a number of generations or remains essentially constant over a generation, it would seem that the dynamics of the population is best described by a discrete-time model [17].

It is well known fact that in population growth disease is an important agent to control the population dynamics. Many experiments show that parasites can reasonably reduce host population and even take host population to complete annihilation. This natural phenomenon is successfully modeled by many simple SI-type host-parasite models. The most interesting properties of such models are their ability of generating host annihilation dynamics with ideal parametric values and initial conditions. This is possible because such models naturally contain the proportion transmission term, which is often referred to as ratio-dependent functional response in the case of predator–prey models. In the SI-type model, the population is subdivided into two classes, susceptibles S and infectives I.

The notation SI means that there is a transfer from the susceptible to infective class, susceptibles become infective and do not recover from the infection. Thus, the transfer continues until all individuals become infected. This type of model is very simple, but may represent some complicated dynamical properties. Most of the SI-type models consist of the mass action principle, i.e., the assumption that new cases arise in a simple proportion to the product of the number of individuals which are susceptible and the number of individuals which are infectious. However, this principle has limited validity, and in discrete models, this principle leads to biologically irrelevant results unless some restrictions are suggested for the parameters. It is more appropriate for discrete epidemic models to include an exponential factor in the rate of transmission. In [16, 19–21], authors studied qualitative behavior of population models in exponential form of difference equations.

Din et al. [12] investigated the qualitative behavior of the following discrete-time host-pathogen model for spread of an infectious disease with permanent immunity:

$$I_{n+1} = S_n\bigl(1 - \mathrm{e}^{-\alpha I_n}\bigr), \qquad S_{n+1} = S_n + \beta - I_{n+1} = S_n \mathrm{e}^{-\alpha I_n} + \beta,$$

where the time-step is equal to the duration of the infectious phase, the state variables are S_n, the number of susceptible individuals at time n, and I_n representing the number of individuals getting the disease (new cases) between times $n - 1$ and n. Moreover, β is the number of births between n and $n + 1$, all added to the susceptible class and assumed to be constant over time. So the difference equation $S_{n+1} = S_n \mathrm{e}^{-\alpha I_n} + \beta$ is just "conservation of mass" for the susceptible class. The first part $I_{n+1} = S_n(1 - \mathrm{e}^{-\alpha I_n})$ of the model is just like Nicholson–Bailey, it comes from assuming that each susceptible escapes infection with probability $\mathrm{e}^{-\alpha I_n}$, the more infectives there are, the lower the chance of escape. The model ignores mortality in the susceptible class on the assumption that everyone gets the disease while young and mortality occurs later in life.

It is a natural fact to assume the mortality of host at some constant rate $d > 0$, say. In this paper, we want to investigate stability analysis of the case where there is host mortality at some constant rate $d > 0$ and the susceptible dynamics become $S_{n+1} = (1 - d)S_n + \beta - I_n$, where $0 < d < 1$. In this case, the host-pathogen model with constant mortality rate of host is given by

$$I_{n+1} = S_n\bigl(1 - \mathrm{e}^{-\alpha I_n}\bigr), \qquad S_{n+1} = (1 - d)S_n + \beta - I_n. \tag{1}$$

More precisely, our aim is to investigate boundedness character, local asymptotic stability of unique positive equilibrium point, the global asymptotic character of equilibrium point, and the rate of convergence of positive solutions of system (1). Although, system (1) seems to be very simple 2-dimensional discrete dynamical system, but it has extremely complex behavior. For some interesting results related to the qualitative behavior of difference equations, we refer the reader to [3–14].

2 Boundedness

The following theorem shows that every positive solution $\{(I_n, S_n)\}_{n=0}^{\infty}$ of system (1) is bounded.

Theorem 1. *Assume that* $0 < d < 1$*, then every positive solution* $\{(I_n, S_n)\}$ *of* (1) *is bounded.*

Proof. Let $\{(I_n, S_n)\}_{n=0}^{\infty}$ be an arbitrary positive solution of system (1). From $S_{n+1} = (1-d)S_n + \beta - I_n$ one has

$$S_{n+1} \leqslant (1-d)S_n + \beta$$

for all $n = 0, 1, 2, \dots$. Consider the difference equation $x_{n+1} = (1-d)x_n + \beta$ with an initial condition $x_0 = S_0$, then its solution is given by

$$x_n = (1-d)^n x_0 + \beta \frac{(1-(1-d)^n)}{d}.$$

Assume that $0 < d < 1$, then

$$x_n \leqslant x_0 + \frac{\beta}{d}$$

for all $n = 1, 2, \dots$. Thus, by comparison one has $S_n \leqslant S_0 + \beta/d$ for all $n = 1, 2, \dots$. Similarly, from $I_{n+1} = S_n(1 - \mathrm{e}^{-\alpha I_n})$ we have

$$I_{n+1} \leqslant S_n \leqslant S_0 + \frac{\beta}{d}$$

for all $n = 1, 2, \dots$. It follows that

$$0 < I_n \leqslant \frac{\beta}{d} + S_0, \qquad 0 < S_n \leqslant \frac{\beta}{d} + S_0$$

for all $n = 1, 2, \dots$. □

Theorem 2. *Let* $\{(I_n, S_n)\}$ *be a positive solution of system* (1)*. Then* $[0, \beta/d] \times [0, \beta/d]$ *is an invariant set for system* (1).

Proof. Let $\{(I_n, S_n)\}$ be a positive solution of system (1) with initial conditions $I_0, S_0 \in I = [0, \beta/d]$. Then, from system (1),

$$I_1 = S_0\bigl(1 - \mathrm{e}^{-\alpha I_0}\bigr) \leqslant S_0 \leqslant \frac{\beta}{d}$$

and

$$S_1 = (1-d)S_0 + \beta - I_0 \leqslant (1-d)S_0 + \beta \leqslant (1-d)\frac{\beta}{d} + \beta = \frac{\beta}{d}.$$

Hence, $I_1, S_1 \in I$. Then it follows by induction that

$$0 < S_n \leqslant \frac{\beta}{d}, \qquad 0 < I_n \leqslant \frac{\beta}{d}$$

for all $n = 1, 2, \dots$. Hence, the proof is completed. □

3 Existence and local stability of positive equilibrium

The following result shows the existence and uniqueness of positive equilibrium point of system (1).

Theorem 3. *Assume that $\alpha\beta > d$, then system* (1) *has a unique positive equilibrium point* $(\bar{x}, \bar{y})$ *in* $[0, \beta/d] \times [0, \beta/d]$.

Proof. Consider the following system:

$$x = y\bigl(1 - \mathrm{e}^{-\alpha x}\bigr), \qquad y = (1-d)y + \beta - x. \tag{2}$$

It follows from (2) that

$$y = \frac{x}{1 - \mathrm{e}^{-\alpha x}}, \qquad x = \beta - dy. \tag{3}$$

Moreover, taking

$$F(x) = \beta - df(x) - x,$$

where

$$f(x) = \frac{x}{1 - \mathrm{e}^{-\alpha x}}.$$

Then it follows that

$$\lim_{x \to 0} F(x) = \frac{\alpha\beta - d}{\alpha} > 0 \quad \text{if } \alpha\beta > d$$

and

$$F\left(\frac{\beta}{d}\right) = \beta\left(1 - \frac{1}{1 - \mathrm{e}^{-\alpha\beta/d}} - \frac{1}{d}\right) < 0.$$

Hence, $F(x)$ has at least one root in $[0, \beta/d]$. Furthermore, we have

$$F'(x) = -df'(x) - 1,$$

where

$$f'(x) = \frac{\mathrm{e}^{\alpha x}(\mathrm{e}^{\alpha x} - \alpha x - 1)}{(\mathrm{e}^{\alpha x} - 1)^2} > 0.$$

It follows that $F'(x) < 0$ for every $x \in [0, \beta/d]$. Hence, $F(x)$ has a unique positive root in $[0, \beta/d]$. This completes the proof. □

Consider the two-dimensional discrete dynamical system of the form

$$x_{n+1} = f(x_n, y_n), \quad y_{n+1} = g(x_n, y_n), \quad n = 0, 1, \dots, \tag{4}$$

where $f : I \times J \to I$ and $g : I \times J \to J$ are continuously differentiable functions and I, J are some intervals of real numbers. Furthermore, a solution $\{(x_n, y_n)\}_{n=0}^{\infty}$ of system (4) is uniquely determined by initial conditions $(x_0, y_0) \in I \times J$. The linearized system of (4) about the equilibrium point $(\bar{x}, \bar{y})$ is

$$X_{n+1} = \mathbf{M}X_n,$$

where $X_n = \binom{x_n}{y_n}$ and $\mathbf{M}$ is Jacobian matrix of system (4) about the equilibrium point $(\bar{x}, \bar{y})$.

Let $(\bar{I}, \bar{S})$ is the equilibrium point of system (1), then one has

$$\bar{I} = \bar{S}\big(1 - \mathrm{e}^{-\alpha \bar{I}}\big), \qquad \bar{S} = (1-d)\bar{S} - \bar{I} + \beta.$$

Moreover, the Jacobian matrix $\mathbf{M}(\bar{I}, \bar{S})$ of system (1) about the equilibrium point $(\bar{I}, \bar{S})$ is given by

$$\mathbf{M}(\bar{I}, \bar{S}) = \begin{bmatrix} \alpha \bar{S} \mathrm{e}^{-\alpha \bar{I}} & 1 - \mathrm{e}^{-\alpha \bar{I}} \\ -1 & 1-d \end{bmatrix}.$$

Lemma 1 [Jury condition]. *Consider the second-degree polynomial equation*

$$\lambda^2 + p\lambda + q = 0, \tag{5}$$

where p and q are real numbers. Then the necessary and sufficient condition for both roots of equation (5) to lie inside the open disk $|\lambda| < 1$ is

$$|p| < 1 + q < 2.$$

Arguing as in [15], we take the following theorems for local asymptotic stability of positive equilibrium point of system (1).

Theorem 4. *Assume that $\alpha\beta > d$. Then the unique positive equilibrium point $(\bar{x}, \bar{y})$ in $[0, \beta/d] \times [0, \beta/d]$ is locally asymptotically stable if*

$$(1-d)\left(1 + \frac{\alpha\beta}{d} - \alpha r(1+d)\right) + \frac{\alpha\beta}{d} + \frac{rd^2}{\beta - rd} - \alpha r(1+d) < 1,$$

where r is the ratio of the steady-state host density with its constant mortality rate d, i.e., $r = \bar{I}/d$.

Proof. As pointed out in [15], it is convenient to discuss stability behavior in terms of the quantity $r = \bar{I}/d$. The equilibrium value $r = \bar{I}/d$ is of interest in modeling as being the ratio of the steady-state host densities with its constant mortality rate. For the positive equilibrium point $(\bar{I}, \bar{S})$ of system (1), we have from system (3)

$$\bar{S} = \frac{\beta}{d} - \frac{\bar{I}}{d} = \frac{\beta}{d} - r$$

and

$$\bar{S}\mathrm{e}^{-\alpha \bar{I}} = \frac{\beta}{d} - r(1+d), \qquad 1 - \mathrm{e}^{-\alpha \bar{I}} = \frac{rd^2}{\beta - rd}.$$

In terms of ratio r, the unique positive equilibrium point of (1) is given by

$$(\bar{I}, \bar{S}) = \left(rd, \frac{\beta - rd}{d}\right).$$

For the consistency of r, it is enough to show that

$$\left(rd, \frac{\beta - rd}{d}\right) \in \left[0, \frac{\beta}{d}\right] \times \left[0, \frac{\beta}{d}\right].$$

The inequality $0 < \exp(-\alpha \bar{I}) < 1$ implies that

$$0 < 1 - \frac{rd^2}{\beta - rd} < 1. \tag{6}$$

From (6) it follows that

$$0 < rd^2 < \beta - rd < \beta.$$

Hence, $0 < rd < \beta/d$. Similarly, one can obtain that $0 < \beta - rd/d < \beta/d$. The characteristic polynomial of $\mathbf{M}(\bar{I}, \bar{S})$ is given by

$$P(\lambda) = \lambda^2 - \operatorname{Tr}\big[\mathbf{M}(\bar{I}, \bar{S})\big]\lambda + \det\big[\mathbf{M}(\bar{I}, \bar{S})\big], \tag{7}$$

where $\operatorname{Tr}[\mathbf{M}(\bar{I}, \bar{S})] = 1 - d + \alpha \bar{S} \mathrm{e}^{-\alpha \bar{I}} > 0$ and $\det[\mathbf{M}(\bar{I}, \bar{S})] = (1-d)\alpha \bar{S} \mathrm{e}^{-\alpha \bar{I}} + 1 - \mathrm{e}^{-\alpha \bar{I}} > 0$. Moreover, we assume that:

$$\Phi(\lambda) = \lambda^2, \qquad \Psi(\lambda) = \big(1 - d + \alpha \bar{S} \mathrm{e}^{-\alpha \bar{I}}\big)\lambda - (1-d)\alpha \bar{S} \mathrm{e}^{-\alpha \bar{I}} - 1 + \mathrm{e}^{-\alpha \bar{I}}.$$

Assume that $(1-d)(1 + \alpha\beta/d - \alpha r(1+d)) + \alpha\beta/d + rd^2/(\beta - rd) - \alpha r(1+d) < 1$, and $|\lambda| = 1$. Then we obtain

$$\begin{aligned} \big|\Psi(\lambda)\big| &\leqslant 1 - d + \alpha \bar{S} \mathrm{e}^{-\alpha \bar{I}} + (1-d)\alpha \bar{S} \mathrm{e}^{-\alpha \bar{I}} + 1 - \mathrm{e}^{-\alpha \bar{I}} \\ &= (1-d)\left(1 + \frac{\alpha\beta}{d} - \alpha r(1+d)\right) + \frac{\alpha\beta}{d} + \frac{rd^2}{\beta - rd} - \alpha r(1+d) < 1. \end{aligned}$$

Hence, by Rouche's theorem $\Phi(\lambda)$ and $\Phi(\lambda) - \Psi(\lambda)$ have same number of zeroes in an open unit disk $|\lambda| < 1$. Hence, both roots of (7) lie in an open disk $|\lambda| < 1$, and it follows that the equilibrium point $(\bar{I}, \bar{S})$ in $[0, \beta/d] \times [0, \beta/d]$ is locally asymptotically stable. □

The following result shows necessary and sufficient condition for local asymptotic stability of unique positive equilibrium point of system (1).

Theorem 5. *The unique positive equilibrium point of system* (1) *is locally asymptotically stable if and only if*

$$1 - d + \frac{\alpha\beta}{d} - \alpha r(1+d) < 1 + \alpha(1-d)\left(\frac{\beta}{d} - r(1+d)\right) + \frac{rd^2}{\beta - rd} < 2. \tag{8}$$

4 Global stability analysis

In this section, we will determine the global character of the unique positive equilibrium point of system (1). Similar methods can be found in [18].

Lemma 2. *Let $I = [a, b]$ and $J = [c, d]$ be real intervals, and let $f : I \times J \to I$ and $g : I \times J \to J$ be continuous functions. Consider system* (4) *with initial conditions $(x_0, y_0) \in I \times J$. Suppose that following statements are true*:

(i) *$f(x, y)$ is non-decreasing in both arguments*;
(ii) *$g(x, y)$ is non-increasing in x and non-decreasing in y*;
(iii) *$(m_1, M_1, m_2, M_2) \in I^2 \times J^2$ is a solution of the system*

$$m_1 = f(m_1, m_2), \qquad M_1 = f(M_1, M_2),$$
$$m_2 = g(M_1, m_2), \qquad M_2 = g(m_1, M_2)$$

such that $m_1 = M_1$ and $m_2 = M_2$.

Then there exists exactly one equilibrium point $(\bar{x}, \bar{y})$ of system (4) *such that $\lim_{n\to\infty}(x_n, y_n) = (\bar{x}, \bar{y})$.*

Proof. According to Brouwer fixed point theorem, the function $F : I \times J \to I \times J$ defined by $F(x, y) = F(f(x, y), g(x, y))$ has a fixed point $(\bar{x}, \bar{y})$, which is a fixed point of system (4).

Assume that $m_1^0 = a$, $M_1^0 = b$, $m_2^0 = c$, $M_2^0 = d$ such that

$$m_1^{i+1} = f\bigl(m_1^i, m_2^i\bigr), \qquad M_1^{i+1} = f\bigl(M_1^i, M_2^i\bigr)$$

and

$$m_2^{i+1} = g\bigl(M_1^i, m_2^i\bigr), \qquad M_2^{i+1} = g\bigl(m_1^i, M_2^i\bigr).$$

Then

$$m_1^0 = a \leqslant f\bigl(m_1^0, m_2^0\bigr) \leqslant f\bigl(M_1^0, M_2^0\bigr) \leqslant b = M_1^0$$

and

$$m_2^0 = c \leqslant g\bigl(M_1^0, m_2^0\bigr) \leqslant g\bigl(m_1^0, M_2^0\bigr) \leqslant d = M_2^0.$$

Moreover, one has

$$m_1^0 \leqslant m_1^1 \leqslant M_1^1 \leqslant M_1^0$$

and

$$m_2^0 \leqslant m_2^1 \leqslant M_2^1 \leqslant M_2^0.$$

We similarly have

$$m_1^1 = f\bigl(m_1^0, m_2^0\bigr) \leqslant f\bigl(m_1^1, m_2^1\bigr) \leqslant f\bigl(M_1^1, M_2^1\bigr) \leqslant f\bigl(M_1^0, M_2^0\bigr) = M_1^1$$

and

$$m_2^1 = g\bigl(M_1^0, m_2^0\bigr) \leqslant g\bigl(M_1^1, m_2^1\bigr) \leqslant g\bigl(m_1^1, M_2^1\bigr) \leqslant g\bigl(m_1^0, M_2^0\bigr) = M_2^1.$$

Now observe that for each $i \geqslant 0$,

$$a = m_1^0 \leqslant m_1^1 \leqslant \cdots \leqslant m_1^i \leqslant M_1^i \leqslant M_1^{i-1} \leqslant \cdots \leqslant M_1^0 = b$$

and

$$c = m_2^0 \leqslant m_2^1 \leqslant \cdots \leqslant m_2^i \leqslant M_2^i \leqslant M_2^{i-1} \leqslant \cdots \leqslant M_2^0 = d.$$

Hence, $m_1^i \leqslant x_n \leqslant M_1^i$ and $m_2^i \leqslant y_n \leqslant M_2^i$ for $n \geqslant 2i+1$. Let $m_1 = \lim_{n\to\infty} m_1^i$, $M_1 = \lim_{n\to\infty} M_1^i$, $m_2 = \lim_{n\to\infty} m_2^i$, and $M_2 = \lim_{n\to\infty} M_2^i$. Then $a \leqslant m_1 \leqslant M_1 \leqslant b$ and $c \leqslant m_2 \leqslant M_2 \leqslant d$. By continuity of f and g one has

$$m_1 = f(m_1, m_2), \qquad M_1 = f(M_1, M_2),$$
$$m_2 = g(M_1, m_2), \qquad M_2 = g(m_1, M_2).$$

Hence, $m_1 = M_1$, $m_2 = M_2$. □

Lemma 3. *The unique positive equilibrium point of system* (1) *is a global attractor if the following condition is satisfied*:

$$\mathrm{e}^{\alpha\beta-d} + \alpha\beta\mathrm{e}^{1-d} < 1 + d\mathrm{e}^{\alpha\beta-d} + (1-d)(\alpha\beta-d). \tag{9}$$

Proof. Let $f(x,y) = y(1-\mathrm{e}^{-\alpha x})$ and $g(x,y) = (1-d)y + \beta - x$. Then it is easy to see that $f(x,y)$ is non-decreasing in both x and y. Moreover, $g(x,y)$ is non-increasing in x and non-decreasing in y if $0<d<1$. Let (m_1, M_1, m_2, M_2) be a solution of the system

$$m_1 = f(m_1, m_2), \qquad M_1 = f(M_1, M_2),$$
$$m_2 = g(M_1, m_2), \qquad M_2 = g(m_1, M_2).$$

Then one has

$$m_1 = m_2\bigl(1-\mathrm{e}^{-\alpha m_1}\bigr), \qquad M_1 = M_2\bigl(1-\mathrm{e}^{-\alpha M_1}\bigr) \tag{10}$$

and

$$m_2 = (1-d)m_2 + \beta - M_1, \qquad M_2 = (1-d)M_2 + \beta - m_1. \tag{11}$$

Furthermore, it suffices to suppose that

$$0 < m_1 \leqslant M_1, \qquad 0 < m_2 \leqslant M_2.$$

(11) implies that

$$M_1 = \beta - dm_2, \qquad m_1 = \beta - dM_2. \tag{12}$$

Moreover, using the inequality $x/(1+x) \leqslant (1-\mathrm{e}^{-x}) \leqslant x$ for all $x \geqslant 0$, we obtain from (10) and (12)

$$\frac{1}{\alpha}(1-d)(\alpha\beta-d) \leqslant m_1 \leqslant M_1 \leqslant \beta - \frac{d}{\alpha} \tag{13}$$

and

$$\frac{1}{\alpha} \leqslant m_2 \leqslant M_2 \leqslant \frac{1}{\alpha}(1+\alpha\beta-d). \tag{14}$$

On subtracting (12), we have

$$M_1 - m_1 = d(M_2 - m_2). \tag{15}$$

Subtracting (10) and using (12), one has

$$\begin{aligned}
M_1 - m_1 &= M_2 - m_2 + m_2 \mathrm{e}^{-\alpha m_1} - M_2 \mathrm{e}^{-\alpha M_1} \\
&= M_2 - m_2 + \frac{\beta - M_1}{d} \mathrm{e}^{-\alpha m_1} - \frac{\beta - m_1}{d} \mathrm{e}^{-\alpha M_1} \\
&= M_2 - m_2 + \frac{\beta}{d}\big(\mathrm{e}^{-\alpha m_1} - \mathrm{e}^{-\alpha M_1}\big) + \frac{1}{d}\big(m_1 \mathrm{e}^{-\alpha M_1} - M_1 \mathrm{e}^{-\alpha m_1}\big) \\
&= M_2 - m_2 + \frac{\alpha\beta}{d} \mathrm{e}^{\alpha(\theta_1 - m_1 - M_1)}(M_1 - m_1) \\
&\quad + \frac{1}{d}(\alpha\theta_2 + 1)\mathrm{e}^{\alpha(\theta_2 - m_1 - M_1)}(m_1 - M_1),
\end{aligned} \tag{16}$$

where $\theta_1, \theta_2 \in [m_1, M_1]$. Then from (13), (15), and (16) it follows that

$$\big(1 + d\mathrm{e}^{\alpha\beta - d} + (1-d)(\alpha\beta - d) - \mathrm{e}^{\alpha\beta - d} - \alpha\beta \mathrm{e}^{1-d}\big)(M_1 - m_1) \leqslant 0. \tag{17}$$

Finally, from (9) and (17) we have $m_1 = M_1$. Then (15) implies that $m_2 = M_2$. Hence, from Lemma 2 the equilibrium point of system (1) is a global attractor. □

Theorem 6. *Under conditions* (8) *and* (9)*, the unique positive equilibrium point of system* (1) *is globally asymptotically stable.*

5 Rate of convergence

In this section, we will determine the rate of convergence of a solution that converges to the unique positive equilibrium point of system (1).

The following result gives the rate of convergence of solutions of a system of difference equations:

$$X_{n+1} = \big(A + B(n)\big)X_n, \tag{18}$$

where X_n is an m-dimensional vector, $A \in C^{m\times m}$ is a constant matrix, and $B : \mathbb{Z}^+ \to C^{m\times m}$ is a matrix function satisfying

$$\big\|B(n)\big\| \to 0 \tag{19}$$

as $n \to \infty$, where $\|\cdot\|$ denotes any matrix norm, which is associated with the vector norm

$$\big\|(x, y)\big\| = \sqrt{x^2 + y^2}.$$

Proposition 1 [Perron's theorem]. (*See* [22].) *Suppose that condition* (19) *holds. If X_n is a solution of* (18)*, then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \|X_n\|^{1/n} \tag{20}$$

exists and is equal to the modulus of one the eigenvalues of matrix A.

Proposition 2. (*See* [22].) *Suppose that condition* (19) *holds. If X_n is a solution of* (18), *then either $X_n = 0$ for all large n or*

$$\rho = \lim_{n\to\infty} \frac{\|X_{n+1}\|}{\|X_n\|} \tag{21}$$

exists and is equal to the modulus of one the eigenvalues of matrix A.

Let $\{(I_n, S_n)\}$ be any solution of system (1) such that $\lim_{n\to\infty} I_n = \bar{I}$ and $\lim_{n\to\infty} S_n = \bar{S}$. To find the error terms, it follows from system (1)

$$\begin{aligned} I_{n+1} - \bar{I} &= S_n\bigl(1 - \mathrm{e}^{-\alpha I_n}\bigr) - \bar{S}\bigl(1 - \mathrm{e}^{-\alpha \bar{I}}\bigr) \\ &= \frac{\bar{S}(\mathrm{e}^{-\alpha \bar{I}} - \mathrm{e}^{-\alpha I_n})}{I_n - \bar{I}}(I_n - \bar{I}) + \bigl(1 - \mathrm{e}^{-\alpha I_n}\bigr)(S_n - \bar{S}), \end{aligned}$$

and

$$\begin{aligned} S_{n+1} - \bar{S} &= S_n(1-d) - I_n - \bar{S} + \bar{I} \\ &= (1-d)(S_n - \bar{S}) - (I_n - \bar{I}). \end{aligned}$$

Let $e_n^1 = I_n - \bar{I}$, and $e_n^2 = S_n - \bar{S}$, then one has

$$e_{n+1}^1 = a_n e_n^1 + b_n e_n^2 \quad \text{and} \quad e_{n+1}^2 = c_n e_n^1 + d_n e_n^2,$$

where

$$\begin{aligned} a_n &= \frac{\bar{S}(\mathrm{e}^{-\alpha \bar{I}} - \mathrm{e}^{-\alpha I_n})}{I_n - \bar{I}}, & b_n &= 1 - \mathrm{e}^{-\alpha I_n}, \\ c_n &= -1, & d_n &= 1 - d. \end{aligned}$$

Moreover,

$$\begin{aligned} \lim_{n\to\infty} a_n &= \alpha \bar{S} \mathrm{e}^{-\alpha \bar{I}}, & \lim_{n\to\infty} b_n &= 1 - \mathrm{e}^{-\alpha \bar{I}}, \\ \lim_{n\to\infty} c_n &= -1, & \lim_{n\to\infty} d_n &= 1 - d. \end{aligned}$$

Now the limiting system of error terms can be written as

$$\begin{bmatrix} e_{n+1}^1 \\ e_{n+1}^2 \end{bmatrix} = \begin{bmatrix} \alpha \bar{S} \mathrm{e}^{-\alpha \bar{I}} & 1 - \mathrm{e}^{-\alpha \bar{I}} \\ -1 & 1-d \end{bmatrix} \begin{bmatrix} e_n^1 \\ e_n^2 \end{bmatrix},$$

which is similar to linearized system of (1) about the equilibrium point $(\bar{I}, \bar{S})$.

Using Proposition 1, one has the following result.

Theorem 7. *Assume that $\{(I_n, S_n)\}$ be a positive solution of system* (1) *such that $\lim_{n\to\infty} I_n = \bar{I}$ and $\lim_{n\to\infty} S_n = \bar{S}$, where $(\bar{I}, \bar{S})$ be unique positive equilibrium point of the system* (1). *Then the error vector $e_n = \binom{e_n^1}{e_n^2}$ of every solution of* (1) *satisfies both of the following asymptotic relations*:

$$\lim_{n\to\infty} \|e_n\|^{1/n} = |\lambda_{1,2}|, \qquad \lim_{n\to\infty} \frac{\|e_{n+1}\|}{\|e_n\|} = |\lambda_{1,2}|,$$

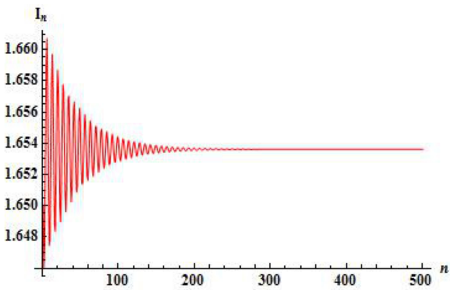
where $\lambda_{1,2}$ are the characteristic roots of Jacobian matrix $F_{\mathrm{J}}(\bar{I}, \bar{S})$.

6 Numerical simulations and discussion

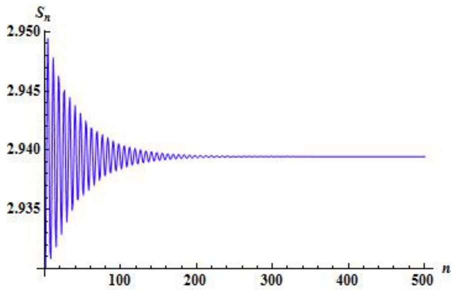
In order to verify our theoretical results and to support our theoretical discussions, we consider some interesting numerical examples in this section. These examples represent different types of qualitative behavior of solutions of system (1). *Mathematica* is used for numerical simulation.

Example 1. Let $\alpha = 0.5$, $d = 0.39$, and $\beta = 2.8$ with initial conditions $I_0 = 1.65$, $S_0 = 2.93$, then system (1) has a unique positive equilibrium point, which is locally asymptotically stable. In this case, $(\bar{I}, \bar{S}) = (1.65361, 2.93946)$ be unique positive equilibrium point of system (1). Moreover, the plot of I_n is shown in Fig. 1(a), the plot of S_n is shown in Fig. 1(b), and a phase portrait of system (1) is shown in Fig. 1(c). This example shows that both host and pathogen survive and positive equilibrium point is locally asymptotically stable.

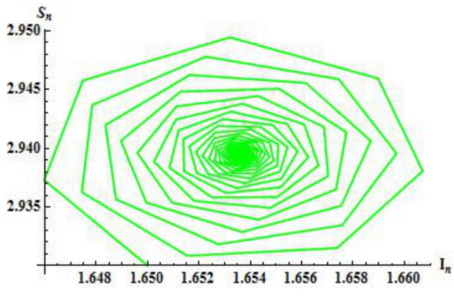
Example 2. Let $\alpha = 0.9$, $d = 0.5$, and $\beta = 2.85$ with initial conditions $I_0 = 1.74$, $S_0 = 2.2$, then system (1) has a unique positive equilibrium point, which is locally asymptotically stable. In this case, $(\bar{I}, \bar{S}) = (1.74751, 2.20498)$ be unique positive equilibrium point of system (1). Moreover, the plot of I_n is shown in Fig. 2(a), the plot of S_n is shown in Fig. 2(b), and a phase portrait of system (1) is shown in Fig. 2(c). This



(a) I_n for system (1)

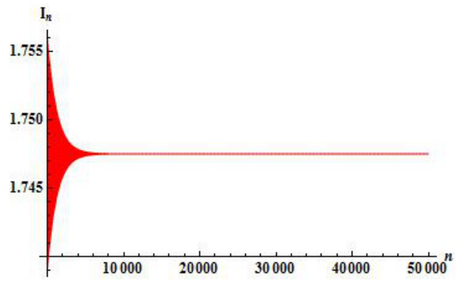


(b) S_n for system (1)

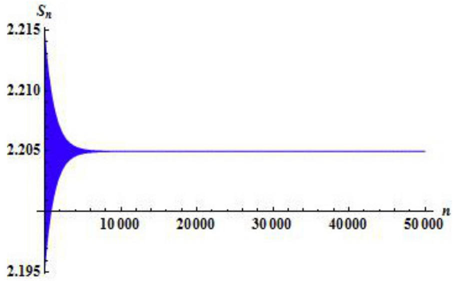


(c) Phase portrait for system (1)

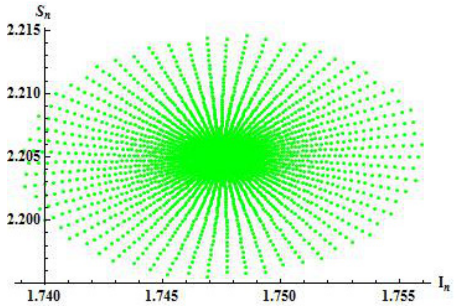
Figure 1. Plots for system (1) with $\alpha = 0.5$, $d = 0.39$, $\beta = 2.8$ and initial conditions $I_0 = 1.65$, $S_0 = 2.93$.



(a) I_n for system (1)



(b) S_n for system (1)



(c) Phase portrait for system (1)

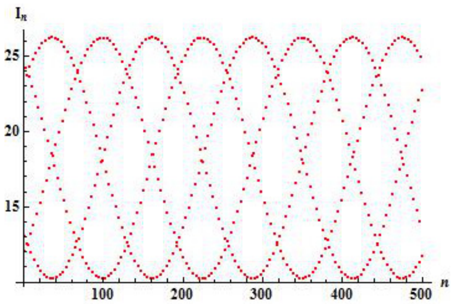
Figure 2. Plots for system (1) with $\alpha = 0.9$, $d = 0.5$, $\beta = 2.85$ and initial conditions $I_0 = 1.74$, $S_0 = 2.2$.

example shows that both host and pathogen survive and positive equilibrium point is locally asymptotically stable.

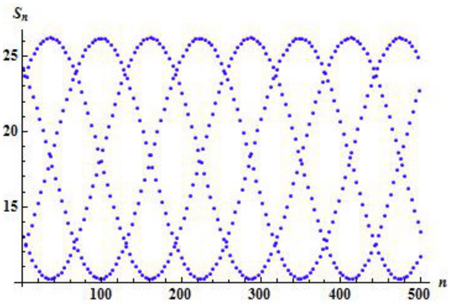
Example 3. Let $\alpha = 0.98$, $d = 0.95$, and $\beta = 35.5$ with initial conditions $I_0 = 12$, $S_0 = 13$, then unique positive equilibrium point of system (1) is unstable. Moreover, the plot of I_n is shown in Fig. 3(a), the plot of S_n is shown in Fig. 3(b), and a phase portrait of system (1) is shown in Fig. 3(c). This example shows that both host and pathogen survive, but positive equilibrium point is unstable.

Example 4. In this example, we take β as bifurcation parameter. Let $\alpha = 0.9$, $d = 0.87$, and $6 \leqslant \beta \leqslant 18$ with initial conditions $I_0 = S_0 = 5$, then system (1) undergoes bifurcation. Moreover, bifurcation diagram of I_n is shown in Fig. 4(a), and bifurcation diagram of S_n is shown in Fig. 4(b).

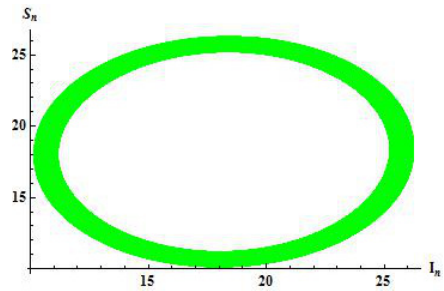
Example 5. Finally, we investigate the sensitivity of parameters β, d and α, respectively. First, we take $\alpha = 0.5$, $d = 0.6$, and $\beta = 3.3, 3.01, 3.02, \dots, 3.09$ with initial conditions $I_0 = 1.56$, $S_0 = 2.88$. Figures 5(a) and 5(b) show that both infected and susceptible individuals are directly proportional to birth rate β. Next, we take $\alpha = 0.5$, $\beta = 3.3$ and vary mortality rate $d = 0.6, 0.6001, 0.6002, \dots, 0.6009$. It is easy to see from Figs. 6(a) and 6(b) that both populations are inversely proportional to mortality rate d. Finally, $d = 0.6$, $\beta = 3.3$, and initial conditions $I_0 = 1.56$, $S_0 = 2.88$ are kept fixed, while



(a) I_n for system (1)

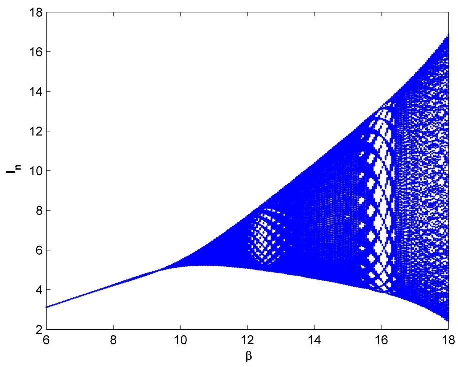


(b) S_n for system (1)

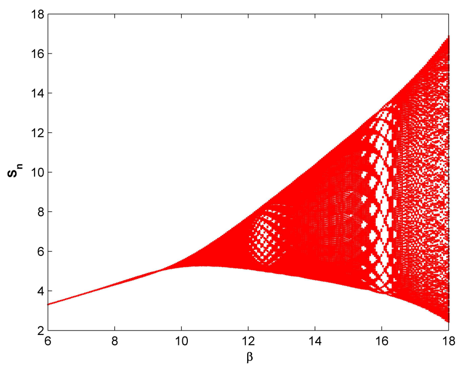


(c) Phase portrait for system (1)

Figure 3. Plots for system (1) with $\alpha = 0.98$, $d = 0.95$, $\beta = 35.5$ and initial conditions $I_0 = 12$, $S_0 = 13$.



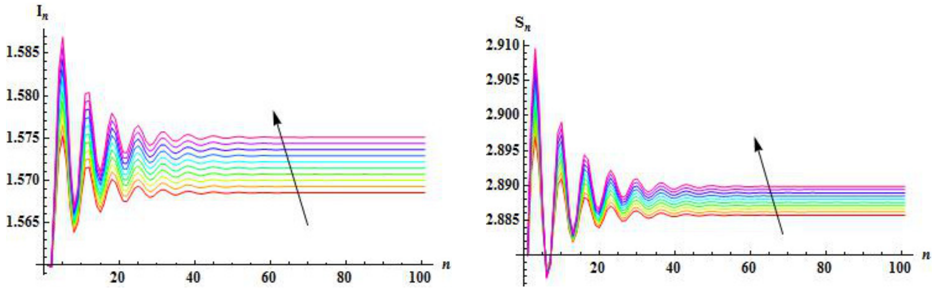
(a) Bifurcation diagram for I_n



(b) Bifurcation diagram for S_n

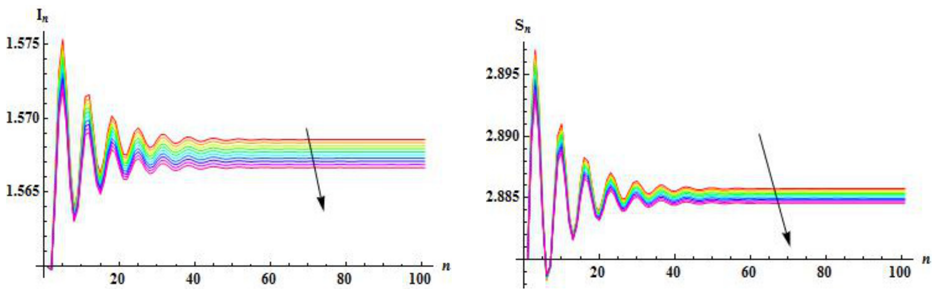
Figure 4. Bifurcation diagrams for system (1) with $\alpha = 0.9$, $d = 0.87$, $6 \leqslant \beta \leqslant 18$ and initial conditions $I_0 = S_0 = 5$.

parameter α is taken as $\alpha = 0.5, 0.501, \ldots, 0.509$. In this case, infected class is directly proportional to α, whereas susceptible population is inversely proportional to α (see Figs. 7(a) and 7(b)).



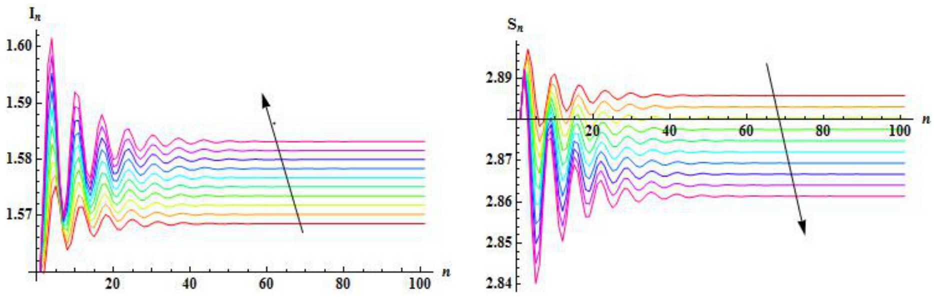
(a) I_n with variation in β (b) S_n with variation in β

Figure 5. Plots for system (1) with $\alpha = 0.5$, $d = 0.6$, $\beta = 3.3, 3.01, 3.02, \ldots, 3.09$ and initial conditions $I_0 = 1.56$, $S_0 = 2.88$.



(a) I_n with variation in d (b) S_n with variation in d

Figure 6. Plots for system (1) with $\alpha = 0.5$, $\beta = 3.3$, $d = 0.6, 0.6001, 0.6002, \ldots, 0.6009$ and initial conditions $I_0 = 1.56$, $S_0 = 2.88$.



(a) I_n with variation in α (b) S_n with variation in α

Figure 7. Plots for system (1) with $d = 0.6$, $\beta = 3.3$, $\alpha = 0.5, 0.501, 0.502, \ldots, 0.509$ and initial conditions $I_0 = 1.56$, $S_0 = 2.88$.

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