ISSN 1392-5113 Nonlinear Analysis: Modelling and Control, 2017, Vol. 22, No. 2, 188–209
https://doi.org/10.15388/NA.2017.2.4

The existence and numerical solution for a k-dimensional system of multi-term fractional integro-differential equations

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Received: August 23, 2015 / **Revised:** June 2, 2016 / **Published online:** January 19, 2017

Abstract. First, we investigate the existence and uniqueness of solution for a k-dimensional system of multi-term fractional integro-differential equations. Also, we apply shifted Chebyshev and shifted Legendre polynomials to obtain an approximation solution for the k-dimensional system. Finally, we provide some examples illustrating the presented methods.

Keywords: Chebyshev polynomials, fixed point, Legendre polynomials, numerical solution, k-dimensional system of multi-term fractional differential equations.

1 Introduction

Many researchers have investigated on fractional differential equations and inclusions by using different views and techniques (see, for example, [1, 2, 5–7, 12, 22, 23, 30, 34, 35, 37] and the references there in). In the last decade, several methods have been used to solve fractional differential equations such differential transform method [15], Adomians decomposition method [9] and variational iteration method [29]. On the other hand, it has published many works about numerical solution for fractional differential equations (see, for example, [3, 8, 11, 13, 16–20, 24, 25, 27, 28, 31, 36] and the references there in). A few works have been published on systems of fractional integro-differential equations (see, for example, [4, 10, 21] and [26]). In 2014, a k-dimensional system of fractional integro-differential equations has been investigated [6]. By using main idea of [6], we first study the existence and uniqueness of solution for the k-dimensional system of multi-term fractional integro-differential equations

$$\begin{aligned} {}^cD^{\alpha_1}x_1(t) &= f_1\bigl(t, x_1(t), x_2(t), \dots, x_k(t), I^{\beta_{11}}x_1(t), I^{\beta_{12}}x_2(t), \dots, I^{\beta_{1k}}x_k(t)\bigr),\\ {}^cD^{\alpha_2}x_2(t) &= f_2\bigl(t, x_1(t), x_2(t), \dots, x_k(t), I^{\beta_{21}}x_1(t), I^{\beta_{22}}x_2(t), \dots, I^{\beta_{2k}}x_k(t)\bigr),\\ &\vdots\\ {}^cD^{\alpha_k}x_k(t) &= f_k\bigl(t, x_1(t), x_2(t), \dots, x_k(t), I^{\beta_{k1}}x_1(t), I^{\beta_{k2}}x_2(t), \dots, I^{\beta_{kk}}x_k(t)\bigr), \end{aligned} \tag{1}$$

with boundary conditions

$$x_i(0) + x_i(1) = a_i, \qquad \sum_{j=1}^{k} I^{\beta_{ij}}x_i(\xi_j) + \sum_{j=1}^{k} I^{\beta_{ij}}x_i(\eta_j) = b_i\int_0^1 x_i(s)\,\mathrm{d}s$$

for $i = 1, 2, \dots, k$, where k is an natural number, $I = [0, 1]$, ${}^cD$ denotes the Caputo fractional derivative, $1 \leqslant \alpha_i < 2$, $\beta_{ij} > 0\,(i, j = 1, \dots, k)$, $0 < \xi_1 < \dots < \xi_k$, $0 < \eta_1 < \eta_2 < \dots < \eta_k$, $a_i, b_i \in \mathbb{R}$, $t \in I$, $f_i \in C(I \times \mathbb{R}^{2k}, \mathbb{R})$ is continuous functions for all $i = 1, 2, \dots, k$. In fact, the main difference between our results and similar results of [6] are that here the right-hand side of the k-dimensional system have mutual couplings between Riemann–Liouville fractional integrals, while in [6], the couplings are between Caputo fractional derivatives. Another main difference is that the results of [6] are analytical ones, while we provide numerical study via some new numerical examples.

By combining the main idea of the papers [6, 14, 19, 20], mixing it with the method of [21] and using the shifted Chebyshev and Legendre polynomials, our approach in this paper is to obtain numerical solutions for the k-dimensional system (1). Consider the Banach space $X = C(I)$ endowed with the sup norm $\|x\| = \sup_{t\in I}|x(t)|$. Note that the product space X^k endowed with the norm $\|(x_1, x_2, \dots, x_k)\|_* = \|x_1\| + \|x_2\| + \dots + \|x_k\|$ is a Banach space. Recall that the Riemann–Liouville fractional integral of order q is defined by $I^qf(t) = \Gamma^{-1}(q)\int_0^t f(s)/(t-s)^{1-q}\,\mathrm{d}s$ $(q > 0)$ provided the integral exists. The Caputo derivative of order q for a function $f \in C^n([0,\infty), \mathbb{R})$ is defined by ${}^cD^qf(t) = \Gamma^{-1}(n-q)\int_0^t f^{(n)}(s)/(t-s)^{q+1-n}\,\mathrm{d}s$ for $n - 1 \leqslant q < n$ [6].

In what follows, we denote

$$F_i^{\beta}\bigl(t, \boldsymbol{x}(t)\bigr) = f_i\bigl(t, x_1(t), x_2(t), \dots, x_k(t), I^{\beta_{i1}}x_1(t), I^{\beta_{i2}}x_2(t), \dots, I^{\beta_{ik}}x_k(t)\bigr).$$

2 Main results

First, we investigate the existence and uniqueness of solution for problem (1). First, we give the following well-known result [6].

Lemma 1. *Let $k \geqslant 1$, $1 \leqslant \alpha < 2$, $\beta_1, \dots, \beta_k > 0$, $0 < \xi_1 < \xi_2 < \dots < \xi_k$, $0 < \eta_1 < \eta_2 < \dots < \eta_k$, $a, b \in \mathbb{R}$, and $y \in C([0,1], \mathbb{R})$. Then the unique solution of the boundary value problem ${}^cD^{\alpha}x(t) = y(t)$ with the boundary conditions $x(0) + x(1) = a$*

and $\sum_{j=1}^{k} I^{\beta_j} x(\xi_j) + \sum_{j=1}^{k} I^{\beta_j} x(\eta_j) = b\int_0^1 x(s)\,\mathrm{d}s$ *is given by*

$$x(t) = I^{\alpha} y(t) + \frac{1}{\Lambda_1 - \Lambda_2}\Biggl[\Lambda_1(1-t) I^{\alpha} y(1) + b(1-t) I^{\alpha+1} y(1) + (t-1)\sum_{j=1}^{k}\bigl(I^{\alpha+\beta_j} y(\eta_j) + I^{\alpha+\beta_j} y(\xi_j)\bigr) + a\Lambda_1(t-1)\Biggr],$$

where $\Lambda_1 = -b + \sum_{j=1}^{k}(\xi_j^{\beta_j} + \eta_j^{\beta_j})/\Gamma(\beta_j+1)$ *and* $\Lambda_2 = -b/2 + \sum_{j=1}^{k}(\xi_j^{\beta_j+1} + \eta_j^{\beta_j+1})/\Gamma(\beta_j+2)$ *with* $\Lambda_1 - \Lambda_2 \neq 0$.

Now, put

$$\Lambda_{i1} = -b_i + \sum_{j=1}^{k}\frac{\xi_j^{\beta_{ij}} + \eta_j^{\beta_{ij}}}{\Gamma(\beta_{ij}+1)}, \qquad \Lambda_{i2} = \frac{-b_i}{2} + \sum_{j=1}^{k}\frac{\xi_j^{\beta_{ij}+1} + \eta_j^{\beta_{ij}+1}}{\Gamma(\beta_{ij}+2)}$$

for all $i = 1, \dots, k$. Define the operator $T : X^k \to X^k$ by

$$T(x_1, x_2, \dots, x_k)(t) = \begin{pmatrix} T_1(x_1, x_2, \dots, x_k)(t) \\ T_2(x_1, x_2, \dots, x_k)(t) \\ \vdots \\ T_k(x_1, x_2, \dots, x_k)(t) \end{pmatrix},$$

where

$$\begin{aligned} &T_i(x_1, x_2, \dots, x_k)(t) \\ &\quad = \frac{\int_0^t (t-s)^{\alpha_i-1} F_i^{\beta}(s, \boldsymbol{x}(s))\,\mathrm{d}s}{\Gamma(\alpha_i)} + \Biggl[\Lambda_{i1}(1-t)\frac{\int_0^t (t-s)^{\alpha_i-1} F_i^{\beta}(s, \boldsymbol{x}(s))\,\mathrm{d}s}{\Gamma(\alpha_i)} \\ &\qquad + b_i(1-t)\frac{\int_0^t (t-s)^{\alpha_i} F_i^{\beta}(s, \boldsymbol{x}(s))\,\mathrm{d}s}{\Gamma(\alpha_i+1)} \\ &\qquad + (t-1)\sum_{j=1}^{k}\biggl(\frac{\int_0^{\eta_j} (\eta_j-s)^{\alpha_i+\beta_{ij}-1} F_i^{\beta}(s, \boldsymbol{x}(s))\,\mathrm{d}s}{\Gamma(\alpha_i+\beta_{ij})} \\ &\qquad + \frac{\int_0^{\xi_j} (\xi_j-s)^{\alpha_i+\beta_{ij}-1} F_i^{\beta}(s, \boldsymbol{x}(s))\,\mathrm{d}s}{\Gamma(\alpha_i+\beta_{ij})}\biggr) + a_i\Lambda_{i1}(t-1)\Biggr]\frac{1}{\Lambda_{i1} - \Lambda_{i2}} \end{aligned}$$

for $i = 1, 2, \dots, k$. The uniqueness of the solution for problem (1) is proved in the subsequent result which uses Lemma 1.

Theorem 1. *Suppose that there exists* $L > 0$ *such that*

$$\bigl|f_i(t, x_1, x_2, \dots, x_{2k}) - f_i(t, x_1', x_2', \dots, x_{2k}')\bigr| \leqslant L\sum_{j=1}^{2k}|x_j - x_j'|$$

and

$$\sum_{i=1}^{k}\left[\left(1+\sum_{s=1}^{k}\frac{1}{\Gamma(\beta_{is}+1)}\right)\left[\frac{1}{\Gamma(\alpha_i+1)}+\frac{|\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)}\right.\right.$$
$$\left.\left.+\frac{|b_i|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)}+\frac{1}{|\Lambda_{i1}-\Lambda_{i2}|}\sum_{j=1}^{k}\frac{\eta_j^{\alpha_i}+\xi_j^{\alpha_i}}{\Gamma(\alpha_i+\beta_{ij}+1)}\right]\right]<\frac{1}{L}$$

for all $i=1,2,\dots,k$, $t\in[0,1]$, $x_j,x_j'\in\mathbb{R}$ *and* $j=1,2,\dots,2k$. *Then problem* (1) *has a unique solution.*

Proof. Let $(x_1,x_2,\dots,x_k),(y_1,y_2,\dots,y_k)\in X^k$ and $t\in I$ be given. Then we have

$$\begin{aligned}
&\bigl|T_i(x_1,x_2,\dots,x_k)(t)-T_i(y_1,y_2,\dots,y_k)(t)\bigr|\\
&\quad\leqslant\frac{\int_0^t(t-s)^{\alpha_i-1}|F_i^{\beta}(s,\boldsymbol{x}(s))-F_i^{\beta}(s,\boldsymbol{y}(s))|\,\mathrm{d}s}{\Gamma(\alpha_i)}\\
&\qquad+\frac{1}{|\Lambda_{i1}-\Lambda_{i2}|}\left[|\Lambda_{i1}|(1-t)\frac{\int_0^1(1-s)^{\alpha_i-1}|F_i^{\beta}(s,\boldsymbol{x}(s))-F_i^{\beta}(s,\boldsymbol{y}(s))|\,\mathrm{d}s}{\Gamma(\alpha_i)}\right.\\
&\qquad+|b_i|(1-t)\frac{\int_0^1(1-s)^{\alpha_i}|F_i^{\beta}(s,\boldsymbol{x}(s))-F_i^{\beta}(s,\boldsymbol{y}(s))|\,\mathrm{d}s}{\Gamma(\alpha_i+1)}\\
&\qquad+|t-1|\sum_{j=1}^{k}\left(\frac{\int_0^{\eta_j}(\eta_j-s)^{\alpha_i+\beta_{ij}-1}|F_i^{\beta}(s,\boldsymbol{x}(s))-F_i^{\beta}(s,\boldsymbol{y}(s))|\,\mathrm{d}s}{\Gamma(\alpha_i+\beta_{ij})}\right.\\
&\qquad\left.\left.+\frac{\int_0^{\xi_j}(\xi_j-s)^{\alpha_i+\beta_{ij}-1}|F_i^{\beta}(s,\boldsymbol{x}(s))-F_i^{\beta}(s,\boldsymbol{y}(s))|\,\mathrm{d}s}{\Gamma(\alpha_i+\beta_{ij})}\right)\right]\\
&\quad\leqslant L\left(1+\sum_{s=1}^{k}\frac{1}{\Gamma(\beta_{is}+1)}\right)\left[\frac{1}{\Gamma(\alpha_i+1)}+\frac{|\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)}\right.\\
&\qquad\left.+\frac{|c_i|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)}+\frac{1}{|\Lambda_{i1}-\Lambda_{i2}|}\sum_{j=1}^{k}\frac{\eta_j^{\alpha_i}+\xi_j^{\alpha_i}}{\Gamma(\alpha_i+\beta_{ij}+1)}\right]\sum_{j=1}^{k}\|x_j-y_j\|
\end{aligned}$$

for $i=1,\dots,k$. This implies that

$$\begin{aligned}
&\bigl\|T_i(x_1,\dots,x_k)-T_i(y_1,\dots,y_k)\bigr\|\\
&\quad\leqslant L\left(1+\sum_{s=1}^{k}\frac{1}{\Gamma(\beta_{is}+1)}\right)\left[\frac{1}{\Gamma(\alpha_i+1)}+\frac{|\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)}\right.\\
&\qquad\left.+\frac{|b_i|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)}+\frac{1}{|\Lambda_{i1}-\Lambda_{i2}|}\sum_{j=1}^{k}\frac{\eta_j^{\alpha_i}+\xi_j^{\alpha_i}}{\Gamma(\alpha_i+\beta_{ij}+1)}\right]\sum_{j=1}^{k}\|x_j-y_j\|
\end{aligned}$$

for $i = 1, \dots, k$ and so

$$\begin{aligned}
&\big\|T(x_1,\dots,x_k) - T(y_1,\dots,y_k)\big\|_* \\
&\quad = \sum_{i=1}^{k} \big\|T_i(x_1,\dots,x_k) - T_i(y_1,\dots,y_k)\big\| \\
&\quad \leqslant L \sum_{i=1}^{k} \Bigg[\Bigg(1 + \sum_{s=1}^{k} \frac{1}{\Gamma(\beta_{is}+1)}\Bigg) \Bigg(\frac{1}{\Gamma(\alpha_i+1)} + \frac{|\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)} \\
&\qquad + \frac{|b_i|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)} + \frac{1}{|\Lambda_{i1}-\Lambda_{i2}|} \sum_{j=1}^{k} \frac{\eta_j^{\alpha_i} + \xi_j^{\alpha_i}}{\Gamma(\alpha_i+\beta_{ij}+1)} \Bigg) \Bigg] \\
&\qquad \times \big\|(x_1,\dots,x_k) - (y_1,\dots,y_k)\big\|_*.
\end{aligned}$$

Hence, T is a contraction, and so by using the Banach contraction principle, T has a unique fixed point $x^* \in X^k$. By using Lemma 1, one can easily get that x^* is the unique solution for problem (1). □

One can find next result in [32].

Lemma 2. *Let E be a Banach space, C a closed and convex subset of E and V an open subset of C such that $0 \in V$. Suppose that $T : \overline{V} \to E$ is continuous and completely continuous operator. Then either T has a fixed point in $\overline{V}$ or there exist $v \in \partial V$ and $\lambda \in (0,1)$ such that $v = \lambda T v$.*

Now, we present a different conditions for the existence of solution for problem (1).

Theorem 2. *Suppose that there exist continuous nondecreasing functions $\psi_1,\dots,\psi_k : [0,\infty) \to (0,\infty)$ and continuous functions $h_1,\dots,h_k : [0,1] \to (0,\infty)$ such that*

$$\big|f_i(t, x_1, x_2, \dots, x_{2k})\big| \leqslant h_i(t) \sum_{j=1}^{2k} \psi_i\big(|x_j|\big)$$

for $i = 1,\dots,k$, and there exist a number $M > 0$ such that

$$M \Bigg(\sum_{i=1}^{k} \Bigg[\Phi_i \|h_i\| \sum_{j=1}^{k} \Bigg(\psi_i(M) + \psi_i\Bigg(\frac{M}{\Gamma(\beta_{ij}+1)} \Bigg) \Bigg) + \frac{|a_i||\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|} \Bigg] \Bigg)^{-1} > 1,$$

where

$$\begin{aligned}
\Phi_i = {}& \frac{1}{\Gamma(\alpha_i+1)} + \frac{|\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)} + \frac{|b_i|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)} \\
& + \frac{1}{|\Lambda_{i1}-\Lambda_{i2}|} \sum_{s=1}^{k} \frac{\eta_s^{\alpha_i} + \xi_s^{\alpha_i}}{\Gamma(\alpha_i+\beta_{is}+1)}
\end{aligned}$$

for all $x_1,\dots,x_{2k} \in \mathbb{R}$, $t \in I$ and $i = 1,2,\dots,k$. Then problem (1) has at least one solution.

Proof. First, we show that the operator $T : X^k \to X^k$ is completely continuous. Let $\{(x_1^n, x_2^n, \dots, x_k^n)\}$ be a sequence in X^k with $(x_1^n, x_2^n, \dots, x_k^n) \to (x_1^0, x_2^0, \dots, x_k^0)$. Then we have

$$\begin{aligned}
&\sup_{t\in I}\bigl|I^{\beta_{ij}}x_j^n(t) - I^{\beta_{ij}}x_j^0(t)\bigr| \\
&\quad = \sup_{t\in I}\left|\frac{1}{\Gamma(\beta_{ij})}\int_0^t (t-s)^{\beta_{ij}-1}x_j^n(s)\,\mathrm{d}s - \frac{1}{\Gamma(\beta_{ij})}\int_0^t (t-s)^{\beta_{ij}-1}x_j^0(s)\,\mathrm{d}s\right| \\
&\quad \leqslant \sup_{t\in I}\frac{1}{\Gamma(\beta_{ij})}\int_0^t (t-s)^{\beta_{ij}-1}\bigl|x_j^n(s) - x_j^0(s)\bigr|\,\mathrm{d}s \\
&\quad \leqslant \frac{1}{\Gamma(\beta_{ij}+1)}\sup_{t\in I}\bigl|x_j^n(t) - x_j^0(t)\bigr| = \frac{1}{\Gamma(\beta_{ij}+1)}\bigl\|x_j^n - x_j^0\bigr\|
\end{aligned}$$

for $i, j = 1, 2, \dots, k$. Since $\|x_j^n - x_j^0\| \to 0$ for all $j = 1, 2, \dots, k$, $(I^{\beta_{ij}})x_j^n(t)$ converges uniformly to $(I^{\beta_{ij}})x_j^0(t)$ on $[0, 1]$ for $i, j = 1, 2, \dots, k$. Since

$$\begin{aligned}
&\bigl\|T\bigl(x_1^n, x_2^n, \dots, x_k^n\bigr) - T\bigl(x_1^0, x_2^0, \dots, x_k^0\bigr)\bigr\|_* \\
&\quad = \sup_{t\in[0,1]}\bigl|T_1\bigl(x_1^n, x_2^n, \dots, x_k^n\bigr)(t) - T_1\bigl(x_1^0, x_2^0, \dots, x_k^0\bigr)(t)\bigr| \\
&\qquad + \sup_{t\in[0,1]}\bigl|T_2\bigl(x_1^n, x_2^n, \dots, x_k^n\bigr)(t) - T_2\bigl(x_1^0, x_2^0, \dots, x_k^0\bigr)(t)\bigr| + \cdots \\
&\qquad + \sup_{t\in[0,1]}\bigl|T_k\bigl(x_1^n, x_2^n, \dots, x_k^n\bigr)(t) - T_k\bigl(x_1^0, x_2^0, \dots, x_k^0\bigr)(t)\bigr|,
\end{aligned}$$

by using above inequalities and the continuity of the functions $f_1, \dots, f_k$, we get

$$\bigl\|T\bigl(x_1^n, x_2^n, \dots, x_k^n\bigr) - T\bigl(x_1^0, x_2^0, \dots, x_k^0\bigr)\bigr\|_* \to 0.$$

Thus, T is continuous on X^k. Let $r > 0$, $B(r) = \{(x_1, \dots, x_k) \in X^k\colon \|(x_1, \dots, x_k)\|_* < r\}$ be a bounded ball in X^k, $(x_1, x_2, \dots, x_k) \in B(r)$ and $t \in [0, 1]$. Then we have $|T(x_1, x_2, \dots, x_k)(t)| = \sum_{i=1}^k |T_i(x_1, x_2, \dots, x_k)(t)|$ and

$$\begin{aligned}
&\bigl|T(x_1, x_2, \dots, x_k)(t)\bigr| \\
&\quad \leqslant \sum_{i=1}^k \Biggl[\frac{\int_0^t (t-s)^{\alpha_i-1}|F_i^\beta(s, \boldsymbol{x}(s))|\,\mathrm{d}s}{\Gamma(\alpha_i)} + \Biggl(|\varLambda_{i1}|(1-t)\frac{\int_0^1 (1-s)^{\alpha_i-1}|F_i^\beta(s, \boldsymbol{x}(s))|\,\mathrm{d}s}{\Gamma(\alpha_i)} \\
&\qquad + |b_i|(1-t)\frac{\int_0^1 (1-s)^{\alpha_i}|F_i^\beta(s, \boldsymbol{x}(s))|\,\mathrm{d}s}{\Gamma(\alpha_i+1)} \\
&\qquad + |t-1|\sum_{j=1}^k\biggl(\frac{\int_0^{\eta_j}(\eta_j - s)^{\alpha_i+\beta_{ij}-1}|F_i^\beta(s, \boldsymbol{x}(s))|\,\mathrm{d}s}{\Gamma(\alpha_i+\beta_{ij})} \\
&\qquad + \frac{\int_0^{\xi_j}(\xi_j - s)^{\alpha_i+\beta_{ij}-1}|F_i^\beta(s, \boldsymbol{x}(s))|\,\mathrm{d}s}{\Gamma(\alpha_i+\beta_{ij})}\biggr) + \bigl|a_i\varLambda_{i1}(t-1)\bigr|\Biggr)\frac{1}{|\varLambda_{i1} - \varLambda_{i2}|}\Biggr]
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \sum_{i=1}^{k}\Biggl[\Biggl(\|h_i\|\sum_{j=1}^{k}\psi_i\bigl(\bigl\|(x_1,x_2,\dots,x_k)\bigr\|_*\bigr)+\psi_i\biggl(\frac{\|(x_1,x_2,\dots,x_k)\|_*}{\Gamma(\beta_{ij}+1)}\biggr)\Biggr) \\
&\quad\times\biggl(\frac{1}{\Gamma(\alpha_i+1)}+\frac{|\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)}+\frac{|b_i|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)} \\
&\quad+\frac{1}{|\Lambda_{i1}-\Lambda_{i2}|}\sum_{s=1}^{k}\frac{\eta_s^{\alpha_i}+\xi_s^{\alpha_i}}{\Gamma(\alpha_i+\beta_{is}+1)}\biggr)+\frac{|a_i||\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|}\Biggr] \\
&\leqslant \sum_{i=1}^{k}\Biggl[\Biggl(\|h_i\|\sum_{j=1}^{k}\psi_i(r)+\psi_i\biggl(\frac{r}{\Gamma(\beta_{ij}+1)}\biggr)\Biggr)\biggl(\frac{1}{\Gamma(\alpha_i+1)}+\frac{|\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)} \\
&\quad+\frac{|b_i|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)}+\frac{1}{|\Lambda_{i1}-\Lambda_{i2}|}\sum_{s=1}^{k}\frac{\eta_s^{\alpha_i}+\xi_s^{\alpha_i}}{\Gamma(\alpha_i+\beta_{is}+1)}\biggr)+\frac{|a_i||\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|}\Biggr].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\bigl\|T(x_1,x_2,\dots,x_k)\bigr\|_* \\
&\quad\leqslant \sum_{i=1}^{k}\Biggl[\Biggl(\|h_i\|\sum_{j=1}^{k}\psi_i(r)+\psi_i\biggl(\frac{r}{\Gamma(\beta_{ij}+1)}\biggr)\Biggr)\biggl(\frac{1}{\Gamma(\alpha_i+1)}+\frac{|\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)} \\
&\quad\quad+\frac{|b_i|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)}+\frac{1}{|\Lambda_{i1}-\Lambda_{i2}|}\sum_{s=1}^{k}\frac{\eta_s^{\alpha_i}+\xi_s^{\alpha_i}}{\Gamma(\alpha_i+\beta_{is}+1)}\biggr)+\frac{|a_i||\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|}\Biggr].
\end{aligned}
$$

This implies that the operator T is uniformly bounded. Now, we show that T maps bounded sets on equicontinuous sets of X^k. Let $0 \leqslant t_1 < t_2 \leqslant 1$ and $(x_1, x_2, \dots, x_k) \in B(r)$. Then we have

$$
\begin{aligned}
&\bigl|T_i(x_1,x_2,\dots,x_k)(t_2)-T_i(x_1,x_2,\dots,x_k)(t_1)\bigr| \\
&\quad=\Biggl|\frac{\int_0^{t_2}(t_2-s)^{\alpha_i-1}F_i^{\beta}(s,\boldsymbol{x}(s)))\,\mathrm{d}s}{\Gamma(\alpha_i)}-\frac{\int_0^{t_1}(t_1-s)^{\alpha_i-1}F_i^{\beta}(s,\boldsymbol{x}(s))\,\mathrm{d}s}{\Gamma(\alpha_i)} \\
&\quad\quad+\frac{1}{\Lambda_{i1}-\Lambda_{i2}}\biggl[\Lambda_{i1}(t_1-t_2)\frac{\int_0^1(1-s)^{\alpha_i-1}F_i^{\beta}(s,\boldsymbol{x}(s))\,\mathrm{d}s}{\Gamma(\alpha_i)} \\
&\quad\quad+b_i(t_1-t_2)\frac{\int_0^1(1-s)^{\alpha_i}F_i^{\beta}(s,\boldsymbol{x}(s))\,\mathrm{d}s}{\Gamma(\alpha_i+1)} \\
&\quad\quad+(t_1-t_2)\sum_{j=1}^{k}\biggl(\frac{\int_0^{\eta_j}(\eta_j-s)^{\alpha_i+\beta_{ij}-1}F_i^{\beta}(s,\boldsymbol{x}(s))\,\mathrm{d}s}{\Gamma(\alpha_i+\beta_{ij})} \\
&\quad\quad+\frac{\int_0^{\xi_j}(\xi_j-s)^{\alpha_i+\beta_{ij}-1}F_i^{\beta}(s,\boldsymbol{x}(s))\,\mathrm{d}s}{\Gamma(\alpha_i+\beta_{ij})}\biggr)+a_i\Lambda_{i1}(t_1-t_2)\biggr]\Biggr|
\end{aligned}
$$

$$\leqslant \left(\|h_i\|\sum_{j=1}^{k}\psi_i(r)+\psi_i\left(\frac{r}{\Gamma(\beta_{ij}+1)}\right)\right)\left(\frac{t_2^{\alpha_i}-t_1^{\alpha_i}}{\Gamma(\alpha_i+1)}+\frac{|\Lambda_{i1}|(t_2-t_1)}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)}\right.$$
$$\left.+\frac{|b_i|(t_2-t_1)}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)}+\frac{t_2-t_1}{|\Lambda_{i1}-\Lambda_{i2}|}\sum_{s=1}^{k}\frac{\eta_s^{\alpha_i}+\xi_s^{\alpha_i}}{\Gamma(\alpha_i+\beta_{is}+1)}\right)+\frac{|a_i\Lambda_{i1}|(t_1-t_2)}{|\Lambda_{i1}-\Lambda_{i2}|}$$

for all $i = 1,\dots,k$. Obviously, the right-hand side of above inequality tends to zero as $t_2 \to t_1$. Now by using the Arzela–Ascoli theorem, one can conclude that the operator $T : X^k \to X^k$ is completely continuous. Let $V = \{(x_1,x_2,\dots,x_k) \in X^k\colon \|(x_1,x_2,\dots,x_k)\|_* < M\}$ and $(x_1,x_2,\dots,x_k) \in V$. Then we have

$$\begin{aligned}
&\big\|T(x_1,x_2,\dots,x_k)\big\|_*\\
&\quad\leqslant \sum_{i=1}^{k}\left[\left(\|h_i\|\sum_{j=1}^{k}\psi_i\big(\big\|(x_1,x_2,\dots,x_k)\big\|_*\big)+\psi_i\left(\frac{\|(x_1,x_2,\dots,x_k)\|_*}{\Gamma(\beta_{ij}+1)}\right)\right)\right.\\
&\qquad\times\left(\frac{1}{\Gamma(\alpha_i+1)}+\frac{|\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)}+\frac{|b_i|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)}\right.\\
&\qquad\left.\left.+\frac{1}{|\Lambda_{i1}-\Lambda_{i2}|}\sum_{s=1}^{k}\frac{\eta_s^{\alpha_i}+\xi_s^{\alpha_i}}{\Gamma(\alpha_i+\beta_{is}+1)}\right)+\frac{|a_i||\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|}\right]\\
&\quad\leqslant \sum_{i=1}^{k}\left[\left(\|h_i\|\sum_{j=1}^{k}\psi_i(M)+\psi_i\left(\frac{M}{\Gamma(\beta_{ij}+1)}\right)\right)\right.\\
&\qquad\times\left(\frac{1}{\Gamma(\alpha_i+1)}+\frac{|\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)}+\frac{|b_i|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)}\right.\\
&\qquad\left.\left.+\frac{1}{|\Lambda_{i1}-\Lambda_{i2}|}\sum_{s=1}^{k}\frac{\eta_s^{\alpha_i}+\xi_s^{\alpha_i}}{\Gamma(\alpha_i+\beta_{is}+1)}\right)+\frac{|a_i||\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|}\right] < M.
\end{aligned}$$

If there exist $(x_1,\dots,x_k)\in\partial V$ and $\lambda\in(0,1)$ such that $(x_1,\dots,x_k)=\lambda T(x_1,\dots,x_k)$, then

$$\begin{aligned}
M &= \big\|(x_1,\dots,x_k)\big\|_* = \lambda\big\|T(x_1,\dots,x_k)\big\|_*\\
&\leqslant \lambda\sum_{i=1}^{k}\left[\left(\|h_i\|\sum_{j=1}^{k}\psi_i\big(\big\|(x_1,x_2,\dots,x_k)\big\|_*\big)+\psi_i\left(\frac{\|(x_1,x_2,\dots,x_k)\|_*}{\Gamma(\beta_{ij}+1)}\right)\right)\right.\\
&\quad\times\left(\frac{1}{\Gamma(\alpha_i+1)}+\frac{|\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+1)}+\frac{|b_i|}{|\Lambda_{i1}-\Lambda_{i2}|\Gamma(\alpha_i+2)}\right.\\
&\quad\left.\left.+\frac{1}{|\Lambda_{i1}-\Lambda_{i2}|}\sum_{s=1}^{k}\frac{\eta_j^{\alpha_i}+\xi_s^{\alpha_i}}{\Gamma(\alpha_i+\beta_{is}+1)}\right)+\frac{|a_i||\Lambda_{i1}|}{|\Lambda_{i1}-\Lambda_{i2}|}\right]
\end{aligned}$$

$$= \lambda \sum_{i=1}^{k} \Phi \|h_i\| \sum_{j=1}^{k} \left(\psi_i(M) + \psi_i\left(\frac{M}{\Gamma(\beta_{ij}+1)} \right) \right) + \frac{|a_i||\Lambda_{i1}|}{|\Lambda_{i1} - \Lambda_{i2}|} < M,$$

which is a contradiction. Now by using Lemma 2, the operator T has at least one fixed point such x^*. One can check that x^* is a solution for problem (1). □

We shall use the Chebyshev and Legendre series expansion for finding approximate solution for problem (1). As you know, the well-known shifted Chebyshev polynomials in $[0, 1]$ have the recurrence relation

$$T^*_{n+1}(x) = 2(2x-1)T^*_n(x) - T^*_{n-1}(x)$$

for all $n \geqslant 1$, where $T^*_0(x) = 1$ and $T^*_1(x) = 2x-1$ [33]. The analytic form of the shifted Chebyshev polynomials $T^*_n(x)$ is given by $T^*_n(x) = n\sum_{i=0}^{n}(-1)^{n-i}(2^{2i}(n+i-1)!/(2i)!(n-i)!)x^i$ for all $n \geqslant 1$ [33]. We have the orthogonality condition $\int_0^1 T^*_n(x) \times T^*_m(x)/\sqrt{x-x^2}\,\mathrm{d}x = 0$ whenever $m \neq n$, $\int_0^1 T^*_n(x)T^*_m(x)/\sqrt{x-x^2}\,\mathrm{d}x = \pi/2$ whenever $m = n \neq 0$ and $\int_0^1 T^*_n(x)T^*_m(x)(x-x^2)^{-1/2}\,\mathrm{d}x = \pi$ whenever $m = n = 0$. Every function $u \in L^2([0,1])$ can be expressed by the shifted Chebyshev polynomials as $u(x) = \sum_{i=0}^{\infty} c_i T^*_i(x)$, where $c_0 = (1/\pi)\int_0^1 u(t)T^*_0(t)/\sqrt{t-t^2}\,\mathrm{d}t$ and $c_i = (2/\pi)\int_0^1 u(t)T^*_i(t)/\sqrt{t-t^2}\,\mathrm{d}t$ for all $i \geqslant 1$ [33]. One can consider the first $(m+1)$ terms of the shifted Chebyshev polynomials $u_m(x) = \sum_{i=0}^{m} c_i T^*_i(x)$ for all $m \geqslant 1$ [33].

Theorem 3. *Let $\alpha > 0$ be given. Then ${}^cD^\alpha(u_m(x)) = \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i w^{(\alpha)}_{i,k} x^{k-\alpha}$ and*

$$I^\alpha\big(u_m(x)\big) = \sum_{i=0}^{m}\sum_{k=0}^{i} c_i \Theta^{(\alpha)}_{i,k} x^{k+\alpha},$$

where

$$\Theta^{(\alpha)}_{0,0} = \frac{1}{\Gamma(\alpha+1)}, \qquad \Theta^{(\alpha)}_{i,k} = (-1)^{i-k}\frac{2^{2k} i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1+\alpha)}$$

and

$$w^{(\alpha)}_{i,k} = (-1)^{i-k}\frac{2^{2k} i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1-\alpha)}.$$

Proof. By using the linear properties of the Caputo fractional derivative, we get

$$\begin{aligned}{}^cD^\alpha\big(u_m(x)\big) &= {}^cD^\alpha\big(c_0T^*_0(x)\big) + \sum_{i=1}^{m} c_i {}^cD^\alpha(T^*_i)(x) \\ &= {}^cD^\alpha\big(c_0T^*_0(x)\big) + \sum_{i=1}^{m}\sum_{k=0}^{i} c_i(-1)^{i-k}\frac{2^{2k} i(i+k-1)!}{(i-k)!(2k)!}{}^cD^\alpha\big(x^k\big).\end{aligned}$$

Since ${}^{c}D^{\alpha}(x^k) = 0$ whenever $k = 0, 1, \dots, \lceil\alpha\rceil - 1$ and ${}^{c}D^{\alpha}(x^k) = (\Gamma(k+1)/\Gamma(k+1-\alpha))x^{k-\alpha}$ whenever $k \geqslant \lceil\alpha\rceil$, we have

$$\begin{aligned}{}^{c}D^{\alpha}\big(u_m(x)\big) &= \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i(-1)^{i-k}\frac{2^{2k}i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1+\alpha)}x^{k-\alpha} \\ &= \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} c_i w_{i,k}^{(\alpha)} x^{k-\alpha}.\end{aligned}$$

Also, by using the linear properties of the Riemann–Liouville fractional integral, we get

$$\begin{aligned}I^{\alpha}\big(u_m(x)\big) &= I^{\alpha}\big(c_0T_0^*(x)\big) + \sum_{i=1}^{m} c_i I^{\alpha}(T_i^*)(x) \\ &= I^{\alpha}\big(c_0T_0^*(x)\big) + \sum_{i=1}^{m}\sum_{k=0}^{i} c_i(-1)^{i-k}\frac{2^{2k}i(i+k-1)!}{(i-k)!(2k)!}I^{\alpha}\big(x^k\big).\end{aligned}$$

Since $I^{\alpha}x^k = (\Gamma(k+1)/\Gamma(k+1+\alpha))x^{k+\alpha}$, we obtain

$$\begin{aligned}I^{\alpha}\big(u_m(x)\big) &= \frac{c_0x^k}{\Gamma(\alpha+1)} + \sum_{i=1}^{m}\sum_{k=0}^{i} c_i(-1)^{i-k}\frac{2^{2k}i(i+k-1)!\Gamma(k+1)}{(i-k)!(2k)!\Gamma(k+1+\alpha)}x^{k+\alpha} \\ &= \sum_{i=0}^{m}\sum_{k=0}^{i} c_i\Theta_{i,k}^{(\alpha)}x^{k+\alpha}.\end{aligned}$$

This completes the proof. □

For solving problem (1) by using the Chebyshev method, we approximate $x_1(t), \dots, x_k(t)$ by

$$x_1(t) \cong \sum_{i=0}^{m} c_{1i}T_i^*(t), \quad x_2(t) \cong \sum_{i=0}^{m} c_{2i}T_i^*(t), \quad \dots, \quad x_k(t) \cong \sum_{i=0}^{m} c_{ki}T_i^*(t).$$

By substitution these relations in (1) and applying Theorem 3, we obtain

$$\begin{aligned}&\sum_{i=\lceil\alpha_1\rceil}^{m}\sum_{s=\lceil\alpha_1\rceil}^{i} c_{1i}w_{i,s}^{(\alpha_1)}t^{s-\alpha_1} \\ &\quad= f_1\Bigg(t, \sum_{i=0}^{m} c_{1i}T_i^*(t), \sum_{i=0}^{m} c_{2i}T_i^*(t), \dots, \sum_{i=0}^{m} c_{ki}T_i^*(t), \sum_{i=0}^{m}\sum_{s=0}^{i} c_{1i}\Theta_{i,s}^{(\beta_{11})}t^{s+\beta_{11}}, \\ &\qquad \sum_{i=0}^{m}\sum_{s=0}^{i} c_{2i}\Theta_{i,s}^{(\beta_{12})}t^{s+\beta_{12}}, \dots, \sum_{i=0}^{m}\sum_{s=0}^{i} c_{ki}\Theta_{i,s}^{(\beta_{1k})}t^{s+\beta_{1k}}\Bigg),\end{aligned}$$

$$\sum_{i=\lceil\alpha_2\rceil}^{m}\sum_{s=\lceil\alpha_2\rceil}^{i} c_{2i}w_{i,s}^{(\alpha_2)}t^{s-\alpha_2}
= f_2\Biggl(t,\sum_{i=0}^{m}c_{1i}T_i^*(t),\sum_{i=0}^{m}c_{2i}T_i^*(t),\dots,\sum_{i=0}^{m}c_{ki}T_i^*(t),\sum_{i=0}^{m}\sum_{s=0}^{i}c_{1i}\Theta_{i,s}^{(\beta_{21})}t^{s+\beta_{21}},
\sum_{i=0}^{m}\sum_{s=0}^{i}c_{2i}\Theta_{i,s}^{(\beta_{22})}t^{s+\beta_{22}},\dots,\sum_{i=0}^{m}\sum_{s=0}^{i}c_{ki}\Theta_{i,s}^{(\beta_{2k})}t^{s+\beta_{2k}}\Biggr),$$

$$\vdots$$

$$\sum_{i=\lceil\alpha_k\rceil}^{m}\sum_{s=\lceil\alpha_k\rceil}^{i} c_{ki}w_{i,s}^{(\alpha_k)}t^{s-\alpha_k}
= f_k\Biggl(t,\sum_{i=0}^{m}c_{1i}T_i^*(t),\sum_{i=0}^{m}c_{2i}T_i^*(t),\dots,\sum_{i=0}^{m}c_{ki}T_i^*(t),\sum_{i=0}^{m}\sum_{s=0}^{i}c_{1i}\Theta_{i,s}^{(\beta_{k1})}t^{s+\beta_{k1}},
\sum_{i=0}^{m}\sum_{s=0}^{i}c_{2i}\Theta_{i,s}^{(\beta_{k2})}t^{s+\beta_{k2}},\dots,\sum_{i=0}^{m}\sum_{s=0}^{i}c_{ki}\Theta_{i,s}^{(\beta_{kk})}t^{s+\beta_{kk}}\Biggr).$$

In the relation

$$\sum_{i=\lceil\alpha_j\rceil}^{m}\sum_{s=\lceil\alpha_j\rceil}^{i} c_{ji}w_{i,s}^{(\alpha_j)}t^{s-\alpha_j}
= f_j\Biggl(t,\sum_{i=0}^{m}c_{1i}T_i^*(t),\sum_{i=0}^{m}c_{2i}T_i^*(t),\dots,\sum_{i=0}^{m}c_{ki}T_i^*(t),\sum_{i=0}^{m}\sum_{s=0}^{i}c_{1i}\Theta_{i,s}^{(\beta_{j1})}t^{s+\beta_{j1}},
\sum_{i=0}^{m}\sum_{s=0}^{i}c_{2i}\Theta_{i,s}^{(\beta_{j2})}t^{s+\beta_{j2}},\dots,\sum_{i=0}^{m}\sum_{s=0}^{i}c_{ki}\Theta_{i,s}^{(\beta_{jk})}t^{s+\beta_{jk}}\Biggr),$$

we put $t = x_p$ for $p = 0,\dots,m+1-\lceil\alpha_j\rceil$ and $j = 1,\dots,k$. Then we obtain

$$\sum_{i=\lceil\alpha_j\rceil}^{m}\sum_{s=\lceil\alpha_j\rceil}^{i} c_{ji}w_{i,s}^{(\alpha_j)}x_p^{s-\alpha_j}
= f_j\Biggl(x_p,\sum_{i=0}^{m}c_{1i}T_i^*(x_p),\sum_{i=0}^{m}c_{2i}T_i^*(x_p),\dots,\sum_{i=0}^{m}c_{ki}T_i^*(x_p),\sum_{i=0}^{m}\sum_{s=0}^{i}c_{1i}\Theta_{i,s}^{(\beta_{j1})}x_p^{s+\beta_{j1}},
\sum_{i=0}^{m}\sum_{s=0}^{i}c_{2i}\Theta_{i,s}^{(\beta_{j2})}x_p^{s+\beta_{j2}},\dots,\sum_{i=0}^{m}\sum_{s=0}^{i}c_{ki}\Theta_{i,s}^{(\beta_{jk})}x_p^{s+\beta_{jk}}\Biggr) \tag{2}$$

for all $j = 1,\dots,k$.

For calculating the unknowns c_{ji} for $i = 0, \dots, m$ and $j = 1, \dots, k$, we consider the roots x_p of $T^*_{m+1-\lceil\alpha_j\rceil}(t)$ for all $j = 1, \dots, k$ and use the conditions

$$x_j(0) + x_j(1) = a_j, \qquad \sum_{t=1}^{k} I^{\beta_{jt}} x_j(\xi_t) + \sum_{t=1}^{k} I^{\beta_{jt}} x_j(\eta_t) = b_j \int_0^1 x_j(s)\,\mathrm{d}s,$$

$j = 1, 2, \dots, k$. Then we get

$$\begin{aligned} &\sum_{i=0}^{m} c_{ji}\bigl(T^*_i(0) + T^*_i(1)\bigr) = a_j, \\ &\sum_{t=1}^{k}\sum_{i=0}^{m}\sum_{s=0}^{i} c_{ji}\Theta^{(\beta_{jt})}_{i,s}\bigl(\xi_t^{s+\beta_{jt}} + \eta_t^{s+\beta_{jt}}\bigr) = b_j \sum_{i=0}^{m}\sum_{s=0}^{i} c_{ji}\Theta^{(1)}_{i,s} \end{aligned} \tag{3}$$

for all $j = 1, \dots, k$. Note that equations (2) and (3) generate $km + k$ nonlinear equations, which can be solved by using the Newton iterative method. Thus, we can find the unknowns c_{ij} for $i = 0, \dots, m$ and $j = 1, \dots, k$, and so one can calculate $x_1(x), \dots, x_k(x)$.

Similar to last case, the shifted Legendre polynomials in $[0, 1]$ have the recurrence relation

$$L^*_{n+1}(x) = \frac{(2n+1)(2x-1)}{n+1} L^*_n(x) - \frac{n}{n+1} L^*_{n-1}(x)$$

for all $n \geqslant 1$, where $L^*_0(x) = 1$ and $L^*_1(x) = 2x - 1$ [19]. In fact, $L^*_n(x) = \sum_{i=0}^{n}(-1)^{n+i} \times ((n+i)!/(n-i)!(i!)^2)x^i$ for all $n \geqslant 1$, $\int_0^1 L^*_n(x)L^*_m(x)\,\mathrm{d}x = 0$ whenever $m \neq n$, and $\int_0^1 L^*_n(x)L^*_m(x)\,\mathrm{d}x = 1/(2m+1)$ whenever $m = n$ [19]. Every function $u \in L^2([0,1])$ can be expressed by the shifted Legendre polynomials as $u(x) = \sum_{i=0}^{\infty} d_i L^*_i(x)$, where $d_i = (2i+1)\int_0^1 u(t)L^*_i(t)\,\mathrm{d}t$ for all $i \geqslant 1$. Again, we consider the first $(m+1)$-terms of the shifted Legendre polynomials $u_m(x) = \sum_{i=0}^{m} d_i L^*_i(x)$ for all $m \geqslant 1$. Similar to Theorem 3, we have next result.

Theorem 4. *Let $\alpha > 0$ be given. Then ${}^{c}D^{\alpha}(u_m(x)) = \sum_{i=\lceil\alpha\rceil}^{m}\sum_{k=\lceil\alpha\rceil}^{i} d_i \mathcal{A}^{(\alpha)}_{i,k} x^{k-\alpha}$ and*

$$I^{\alpha}\bigl(u_m(x)\bigr) = \sum_{i=0}^{m}\sum_{k=0}^{i} d_i \mathcal{B}^{(\alpha)}_{i,k} x^{k+\alpha},$$

where

$$\mathcal{A}^{(\alpha)}_{i,k} = \frac{(-1)^{i+k}(i+k)!}{(i-k)!(k)!\,\Gamma(k+1-\alpha)}, \qquad \mathcal{B}^{(\alpha)}_{i,k} = \frac{(-1)^{i-k}(i+k)!}{(i-k)!(k)!\,\Gamma(k+1+\alpha)}$$

for all i and k.

Again, for solving problem (1) by using the Legendre method, we approximate $x_1(t)$, $\dots, x_k(t)$ by

$$x_1(t) \cong \sum_{i=0}^{m} d_{1i} L^*_i(t), \quad x_2(t) \cong \sum_{i=0}^{m} d_{2i} L^*_i(t), \quad \dots, \quad x_k(t) \cong \sum_{i=0}^{m} d_{ki} L^*_i(t).$$

By substitution these relations in (1) and applying Theorem 4, we obtain

$$\sum_{i=\lceil\alpha_1\rceil}^{m}\sum_{s=\lceil\alpha_1\rceil}^{i} d_{1i}\mathcal{A}_{i,s}^{(\alpha_1)}t^{s-\alpha_1}$$
$$= f_1\Biggl(t,\sum_{i=0}^{m} d_{1i}L_i^*(t),\sum_{i=0}^{m} d_{2i}L_i^*(t),\dots,\sum_{i=0}^{m} d_{ki}L_i^*(t),\sum_{i=0}^{m}\sum_{s=0}^{i} d_{1i}\mathcal{B}_{i,s}^{(\beta_{11})}t^{s+\beta_{11}},$$
$$\sum_{i=0}^{m}\sum_{s=0}^{i} d_{2i}\mathcal{B}_{i,s}^{(\beta_{12})}t^{s+\beta_{12}},\dots,\sum_{i=0}^{m}\sum_{s=0}^{i} d_{ki}\mathcal{B}_{i,s}^{(\beta_{1k})}t^{s+\beta_{1k}}\Biggr),$$

$$\sum_{i=\lceil\alpha_2\rceil}^{m}\sum_{s=\lceil\alpha_2\rceil}^{i} d_{2i}\mathcal{A}_{i,s}^{(\alpha_2)}t^{s-\alpha_2}$$
$$= f_2\Biggl(t,\sum_{i=0}^{m} d_{1i}L_i^*(t),\sum_{i=0}^{m} d_{2i}L_i^*(t),\dots,\sum_{i=0}^{m} d_{ki}L_i^*(t),\sum_{i=0}^{m}\sum_{s=0}^{i} d_{1i}\mathcal{B}_{i,s}^{(\beta_{21})}t^{s+\beta_{21}},$$
$$\sum_{i=0}^{m}\sum_{s=0}^{i} d_{2i}\mathcal{B}_{i,s}^{(\beta_{22})}t^{s+\beta_{22}},\dots,\sum_{i=0}^{m}\sum_{s=0}^{i} d_{ki}\mathcal{B}_{i,s}^{(\beta_{2k})}t^{s+\beta_{2k}}\Biggr),$$
$$\vdots$$
$$\sum_{i=\lceil\alpha_k\rceil}^{m}\sum_{s=\lceil\alpha_k\rceil}^{i} d_{ki}\mathcal{A}_{i,s}^{(\alpha_k)}t^{s-\alpha_k}$$
$$= f_k\Biggl(t,\sum_{i=0}^{m} d_{1i}L_i^*(t),\sum_{i=0}^{m} d_{2i}L_i^*(t),\dots,\sum_{i=0}^{m} d_{ki}L_i^*(t),\sum_{i=0}^{m}\sum_{s=0}^{i} d_{1i}\mathcal{B}_{i,s}^{(\beta_{k1})}t^{s+\beta_{k1}},$$
$$\sum_{i=0}^{m}\sum_{s=0}^{i} d_{2i}\mathcal{B}_{i,s}^{(\beta_{k2})}t^{s+\beta_{k2}},\dots,\sum_{i=0}^{m}\sum_{s=0}^{i} d_{ki}\mathcal{B}_{i,s}^{(\beta_{kk})}t^{s+\beta_{kk}}\Biggr).$$

Again, in the relation

$$\sum_{i=\lceil\alpha_j\rceil}^{m}\sum_{s=\lceil\alpha_j\rceil}^{i} d_{ji}\mathcal{A}_{i,s}^{(\alpha_j)}t^{s-\alpha_j}$$
$$= f_j\Biggl(t,\sum_{i=0}^{m} d_{1i}L_i^*(t),\sum_{i=0}^{m} d_{2i}L_i^*(t),\dots,\sum_{i=0}^{m} d_{ki}L_i^*(t),\sum_{i=0}^{m}\sum_{s=0}^{i} d_{1i}\mathcal{B}_{i,s}^{(\beta_{j1})}t^{s+\beta_{j1}},$$
$$\sum_{i=0}^{m}\sum_{s=0}^{i} d_{2i}\mathcal{B}_{i,s}^{(\beta_{j2})}t^{s+\beta_{j2}},\dots,\sum_{i=0}^{m}\sum_{s=0}^{i} d_{ki}\mathcal{B}_{i,s}^{(\beta_{jk})}t^{s+\beta_{jk}}\Biggr),$$

we put $t = x_p$ for $p = 0, \dots, m + 1 - \lceil \alpha_j \rceil$ and $j = 1, \dots, k$. Then we obtain

$$\begin{aligned}
&\sum_{i=\lceil \alpha_j \rceil}^{m} \sum_{s=\lceil \alpha_j \rceil}^{i} d_{ji} \mathcal{A}_{i,s}^{(\alpha_j)} x_p^{s-\alpha_j} \\
&\quad = f_j \Biggl(x_p, \sum_{i=0}^{m} d_{1i} L_i^*(x_p), \sum_{i=0}^{m} d_{2i} L_i^*(x_p), \dots, \sum_{i=0}^{m} d_{ki} L_i^*(x_p), \\
&\qquad \sum_{i=0}^{m} \sum_{s=0}^{i} d_{1i} \mathcal{B}_{i,s}^{(\beta_{j1})} x_p^{s+\beta_{j1}}, \sum_{i=0}^{m} \sum_{s=0}^{i} d_{2i} \mathcal{B}_{i,s}^{(\beta_{j2})} x_p^{s+\beta_{j2}}, \dots, \\
&\qquad \sum_{i=0}^{m} \sum_{s=0}^{i} d_{ki} \mathcal{B}_{i,s}^{(\beta_{jk})} x_p^{s+\beta_{jk}} \Biggr)
\end{aligned} \tag{4}$$

for all $j = 1, \dots, k$. For calculating the unknowns d_{ji} for $i = 0, \dots, m$ and $j = 1, \dots, k$, we consider the roots x_p of $L^*_{m+1-\lceil \alpha_j \rceil}(t)$ for all $j = 1, \dots, k$ and use the conditions

$$x_j(0) + x_j(1) = a_j, \qquad \sum_{t=1}^{k} I^{\beta_{jt}} x_j(\xi_t) + \sum_{t=1}^{k} I^{\beta_{jt}} x_j(\eta_t) = b_j \int_0^1 x_j(s)\, \mathrm{d}s$$

for $j = 1, 2, \dots, k$. Then we get

$$\begin{aligned}
&\sum_{i=0}^{m} d_{ji} \bigl(L_i^*(0) + L_i^*(1) \bigr) = a_j, \\
&\sum_{t=1}^{k} \sum_{i=0}^{m} \sum_{s=0}^{i} d_{ji} \mathcal{B}_{i,s}^{(\beta_{jt})} \bigl(\xi_t^{s+\beta_{jt}} + \eta_t^{s+\beta_{jt}} \bigr) = b_j \sum_{i=0}^{m} \sum_{s=0}^{i} d_{ji} \mathcal{B}_{i,s}^{(1)}
\end{aligned} \tag{5}$$

for all $j = 1, \dots, k$. Note that, equations (4) and (5) generate $km+k$ nonlinear equations, which can be solved by using the Newton iterative method. Thus, we can find the unknowns c_{ij} for $i = 0, \dots, m$ and $j = 1, \dots, k$, and so one can calculate $x_1(x), \dots, x_k(x)$.

3 Numerical examples

In this section, we provide three examples for illustrating our results. In the first and second ones, we know the solution, and we provide these examples to demonstrate the validity of the presented methods. In the third example, by using the presented methods, we solve a 3-dimensional system of fractional integro-differential equations with unknown exact solution. In the first and second examples, we denote by $\tilde{x}_1(t)$ and $\tilde{x}_2(t)$ the Chebyshev approximations of $x_1(t)$ and $x_2(t)$. Also, we denote the Legendre approximations $\hat{x}_1(t)$ and $\hat{x}_2(t)$.

Example 1. Consider the 2-dimensional system of fractional integro-differential equations

$$
\begin{aligned}
{}^{c}D^{3/2}x_1(t) &= f(t) + \frac{4}{100}\left(x_1(t) + x_2(t) + \int_0^t\int_0^{\tau_1} x_1(\tau)\,\mathrm{d}\tau d\tau_1 + I^{1/3}x_2(t)\right),\\
{}^{c}D^{5/4}x_2(t) &= g(t) + \frac{4}{100}\left(x_1(t) + x_2(t) + \int_0^t\int_0^{\tau_1}\int_0^{\tau_2} x_1(\tau)\,\mathrm{d}\tau\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 + I^{1/2}x_2(t)\right)
\end{aligned}
\tag{6}
$$

with the boundary conditions

$$
\begin{aligned}
&I^2x_1(1/2) + I^{1/3}x_1(3/4) + I^2x_1(1/3) + I^{1/3}x_1(1/3)\\
&\quad = 1.286734952\int_0^1 x_1(s)\,\mathrm{d}s,\\
&I^3x_2(1/2) + I^{1/2}x_2(3/4) + I^3x_2(1/4) + I^{1/2}x_2(1/3)\\
&\quad = 0.7984935672\int_0^1 x_2(s)\,\mathrm{d}s,\\
&x_1(0) + x_1(1) = 1 \quad \text{and} \quad x_2(0) + x_2(1) = 1.
\end{aligned}
$$

Put $f(t) = 2/\Gamma(3/2)t^{1/2} - (4/100)(t^2 + t^3 + t^4/12 + 6t^{10/3}/\Gamma(13/3))$, $g(t) = 6/\Gamma(11/4)t^{7/4} - (4/100)(t^2 + t^3 + t^5/60 + 6t^{7/2}/\Gamma(9/2))$ for $t \in [0,1]$, $L = 4/100$, $k = 2$, $\alpha_1 = 3/2$, $\alpha_2 = 5/4$, $\beta_{11} = 2$, $\beta_{21} = 3$, $\beta_{12} = 1/3$, $\beta_{22} = 1/2$, $\xi_1 = 1/2$, $\xi_2 = 3/4$, $\eta_1 = 1/4$, $\eta_2 = 1/3$, $a_1 = a_2 = 1$, $b_1 = 1.286734952$, $b_2 = 0.7984935672$, $f_1(t, y_1, y_2, y_3, y_4) = f(t) + 4/100(y_1 + y_2 + y_3 + y_4)$ and $f_2(t, y_1, y_2, y_3, y_4) = g(t) + 4/100(y_1 + y_2 + y_3 + y_4)$ for $y_1, y_2, y_3, y_4 \in \mathbb{R}$. One can check problem (6) satisfy the conditions of Theorem 1. Thus, problem (6) has a unique solution in $C[0,1] \times C[0,1]$. We know that the exact solution for problem (6) is $x_1(t) = t^2$ and $x_2(t) = t^3$. We apply the presented methods with $m = 6$ for obtaining the numerical solution for the problem. One can see that the numerical solution coincides the exact solution as we show it in Figs. 1–4 and Tables 1 and 2.

Table 1

i	Coefficient value of Chebyshev method		Coefficient value of Legendre method	
	c_{1i}	c_{2i}	d_{1i}	d_{2i}
0	3.7500 e −01	3.1250 e −01	3.3333 e −01	2.5000 e −01
1	5.0000 e −01	4.6875 e −01	5.0000 e −01	4.5000 e −01
2	1.2500 e −01	1.8750 e −01	1.6667 e −01	2.5000 e −01
3	5.8364 e −13	3.1250 e −02	1.0066 e −12	5.0000 e −02
4	−2.5264 e −14	4.1786 e −13	−1.0651 e −13	2.1053 e −13
5	1.2623 e −13	2.7998 e −13	1.6516 e −13	3.2886 e −13
6	−8.1630 e −14	−3.9504 e −13	−1.1141 e −13	−4.8534 e −13

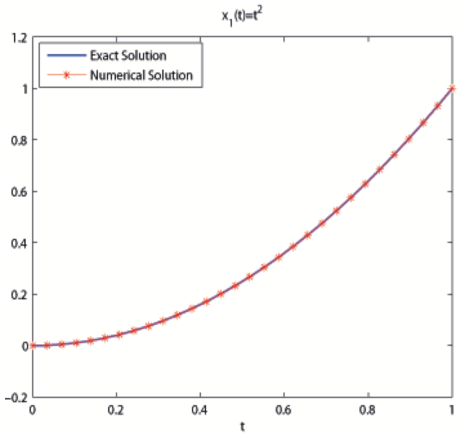

Figure 1. Comparison of t^2 with $\tilde{x}_1(t)$.

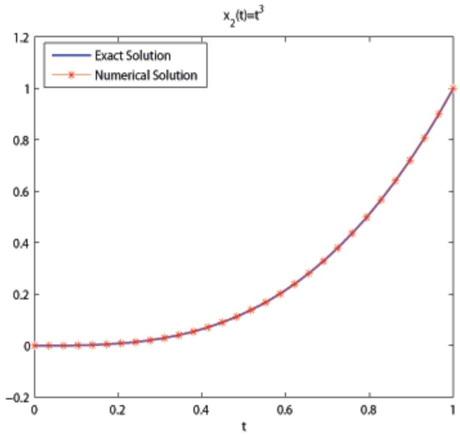

Figure 2. Comparison of t^3 with $\tilde{x}_2(t)$.

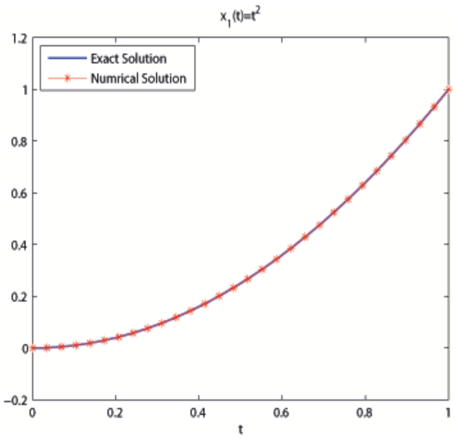

Figure 3. Comparison of t^2 with $\hat{x}_1(t)$.

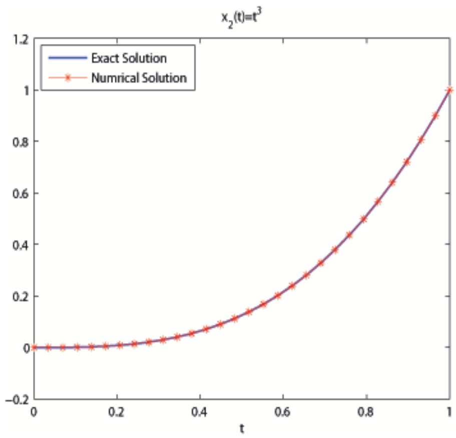

Figure 4. Comparison of t^3 with $\hat{x}_2(t)$.

Table 2

t_i	Absolute error of Chebyshev method		Absolute error of Legendre method	
	$\lvert x_1(t_i) - \tilde{x}_1(t_i)\rvert$	$\lvert x_2(t_i) - \tilde{x}_2(t_i)\rvert$	$\lvert x_1(t_i) - \hat{x}_1(t_i)\rvert$	$\lvert x_2(t_i) - \hat{x}_2(t_i)\rvert$
0.0	8.4621 e −10	1.8512 e −10	8.4695 e −10	1.8695 e −10
0.1	6.7698 e −10	1.4970 e −10	6.7746 e −10	1.5061 e −10
0.2	5.0864 e −10	1.1532 e −10	5.0881 e −10	1.1536 e −10
0.3	3.4030 e −10	7.9691 e −11	3.4030 e −10	7.9581 e −11
0.4	1.7171 e −10	4.2905 e −11	1.7168 e −10	4.3065 e −11
0.5	2.9101 e −12	5.8084 e −12	2.8692 e −12	6.1029 e −12
0.6	1.6604 e −10	3.1049 e −11	1.6617 e −10	3.1102 e −11
0.7	3.3522 e −10	6.7879 e −11	3.3552 e −10	6.8697 e −11
0.8	5.0484 e −10	1.0553 e −10	5.0535 e −10	1.0711 e −10
0.9	6.7515 e −10	1.4480 e −10	6.7581 e −10	1.4669 e −10
1.0	8.4621 e −10	1.8512 e −10	8.4695 e −10	1.8695 e −10

Example 2. Consider the following 2-dimensional system of fractional integro-differential equations

$$\begin{aligned} {}^{c}D^{7/4}x_1(t) &= f(t) + \frac{5}{100}\big(x_1(t) + x_2(t) + I^{1/5}x_1(t) + I^{1/4}x_2(t)\big), \\ {}^{c}D^{3/2}x_2(t) &= g(t) + \frac{5}{100}\big(x_1(t) + x_2(t) + I^{1/3}x_1(t) + I^{1/2}x_2(t)\big), \end{aligned} \tag{7}$$

with the boundary conditions

$$\begin{aligned} &I^{1/5}x_1(1/3) + I^{1/4}x_1(1/2) + I^{1/5}x_1(1/5) + I^{1/4}x_1(2/5) \\ &\quad = 2.5824 \int_0^1 x_1(s)\,\mathrm{d}s, \\ &I^{1/3}x_2(1/3) + I^{1/2}x_2(1/2) + I^{1/3}x_2(1/5) + I^{1/2}x_2(2/5) \\ &\quad = 1.502378 \int_0^1 x_2(s)\,\mathrm{d}s, \\ &x_1(0) + x_1(1) = (\mathrm{e}+2)/2 \quad \text{and} \quad x_2(0) + x_2(1) = \mathrm{e}^2 + 2. \end{aligned}$$

Put $f(t) = \sum_{k=0}^{\infty} t^{k+1/4}/(2\Gamma(k+5/4)) - (5/100)(\mathrm{e}^{2t} + \mathrm{e}^t + t^2 + t + t^{6/5}/(2\Gamma(11/5)) + 2t^{9/4}/\Gamma(13/4) + \sum_{k=0}^{\infty} t^{k+1/5}/(2\Gamma(k+6/5)) + \sum_{k=0}^{\infty} 2^k t^{k+1/4}/\Gamma(k+5/4)$, $g(t) = 2t^{1/2}/\Gamma(3/2) + \sum_{k=0}^{\infty} 2^{2+k} t^{k+1/2}/\Gamma(k+3/2) - (5/100)(\mathrm{e}^{2t} + \mathrm{e}^t + t^2 + t + t^{4/3}/2\Gamma(7/3) + 2t^{5/2}/\Gamma(7/2) + \sum_{k=0}^{\infty} t^{k+1/3}/(2\Gamma(k+4/3)) + \sum_{k=0}^{\infty} 2^k t^{k+1/2}/\Gamma(k+3/2))$ for $t \in [0,1]$, $L = 5/100$, $k = 2$, $\alpha_1 = 7/4$, $\alpha_2 = 3/2$, $\beta_{11} = 1/5$, $\beta_{12} = 1/4$, $\beta_{21} = 1/3$, $\beta_{22} = 12$, $\xi_1 = 1/3$, $\xi_2 = 1/2$, $\eta_1 = 1/5$, $\eta_2 = 2/5$, $a_1 = (\mathrm{e}+2)/2$, $a_2 = \mathrm{e}^2 + 2$, $b_1 = 2.5824$, $b_2 = 1.502378$, $f_1(t, y_1, y_2, y_3, y_4) = f(t) + (5/100)(y_1 + y_2 + y_3 + y_4)$ and $f_2(t, y_1, y_2, y_3, y_4) = g(t) + (5/100)(y_1 + y_2 + y_3 + y_4)$ for $y_1, y_2, y_3, y_4 \in \mathbb{R}$. It is easy to check that problem (7) satisfy the conditions of Theorem 1 and so has a unique solution in $C[0,1] \times C[0,1]$. We know that the exact solution for problem (7) is $x_1(t) = (\mathrm{e}^t + 1)/2$ and $x_2(t) = \mathrm{e}^{2t} + t^2$. We apply the presented methods with $m = 6$ for obtaining the numerical solution for the problem. One can see that the numerical solution coincides the exact solution as we show it Figs. 5–8 and Tables 3 and 4.

Table 3

i	Coefficient value of Chebyshev method		Coefficient value of Legendre method	
	c_{1i}	c_{2i}	d_{1i}	d_{2i}
0	1.1267 e +00	3.8165 e +00	1.1091 e +00	3.5279 e +00
1	6.7520 e −01	3.5725 e +00	6.7258 e −01	3.5000 e +00
2	5.2604 e −02	8.6299 e −01	6.9932 e −02	1.1393 e +00
3	4.3610 e −03	1.2051 e −01	6.9655 e −03	1.9151 e −01
4	2.7172 e −04	1.4909 e −02	4.9631 e −04	2.7129 e −02
5	1.3624 e −05	1.4951 e −03	2.7639 e −05	3.0209 e −03
6	5.6154 e −07	1.1232 e −04	1.2400 e −06	2.4796 e −04

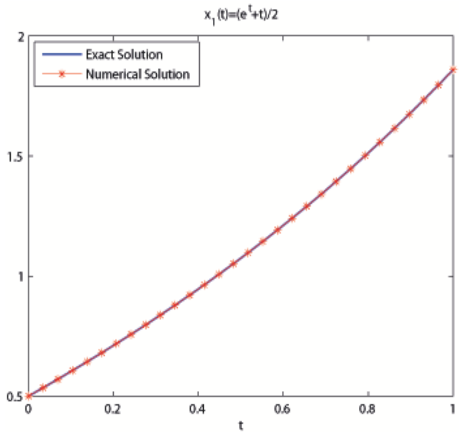

Figure 5. Comparison of $(\mathrm{e}^t + 1)/2$ with $\tilde{x}_1(t)$.

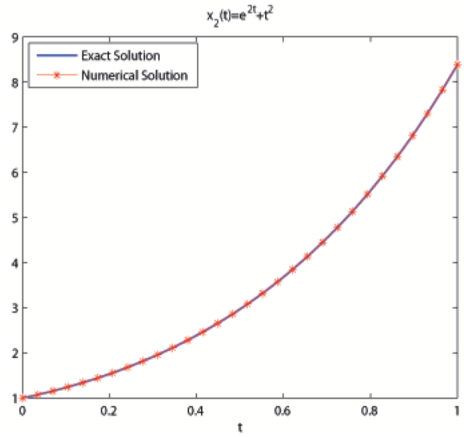

Figure 6. Comparison of $\mathrm{e}^{2t} + t^2$ with $\tilde{x}_2(t)$.

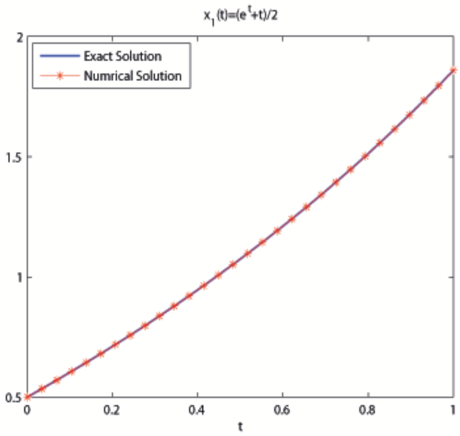

Figure 7. Comparison of $(\mathrm{e}^t + 1)/2$ with $\hat{x}_1(t)$.

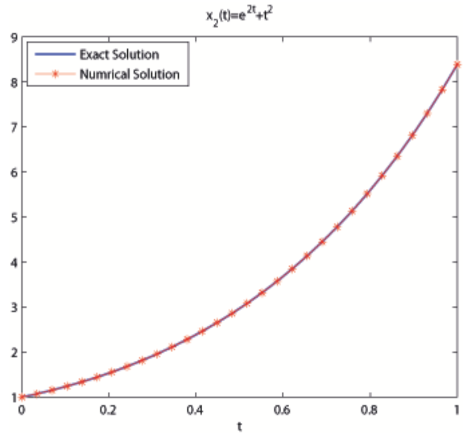

Figure 8. Comparison of $\mathrm{e}^{2t} + t^2$ with $\hat{x}_2(t)$.

Table 4

t_i	Absolute error of Chebyshev method		Absolute error of Legendre method	
	$\lvert x_1(t_i) - \tilde{x}_1(t_i)\rvert$	$\lvert x_2(t_i) - \tilde{x}_2(t_i)\rvert$	$\lvert x_1(t_i) - \hat{x}_1(t_i)\rvert$	$\lvert x_2(t_i) - \hat{x}_2(t_i)\rvert$
0.0	3.5844 e −07	1.8357 e −05	1.9321 e −08	2.0529 e −05
0.1	2.5476 e −07	1.0470 e −05	1.4957 e −08	1.6873 e −05
0.2	7.7343 e −08	1.2016 e −05	8.5637 e −09	2.8404 e −06
0.3	8.4717 e −08	1.8416 e −05	4.6269 e −08	1.6024 e −05
0.4	1.2544 e −07	9.0712 e −06	9.2186 e −09	3.4776 e −06
0.5	5.4960 e −08	4.7901 e −05	9.9099 e −08	1.9716 e −05
0.6	1.8931 e −08	5.5556 e −05	1.9331 e −07	2.0469 e −05
0.7	1.0541 e −08	1.1674 e −05	1.9383 e −07	1.4928 e −05
0.8	1.4883 e −07	4.9921 e −05	1.1307 e −07	5.4189 e −05
0.9	2.8506 e −07	5.8188 e −05	5.3172 e −08	4.2059 e −05
1.0	3.5844 e −07	1.8357 e −05	1.9321 e −08	2.0529 e −05

Example 3. Consider the 3-dimensional system of fractional integro-differential equations

$$\begin{aligned}
{}^{c}D^{\sqrt{2}}x_1(t) &= \frac{|x_1(t)|}{1+|x_1(t)|} + \mathrm{e}^{\cos t}\biggl[\frac{|x_2(t)|+|x_3(t)|}{1+|x_2(t)|+|x_3(t)|} \\
&\quad + \cos I^{1/2}x_1(t) + \sin\bigl(I^{1/3}x_2(t)+I^{1/4}x_3(t)\bigr)\biggr], \\
{}^{c}D^{\sqrt{3}}x_2(t) &= \frac{\mathrm{e}^{t}|x_1(t)|^3}{1+|x_1(t)|^3} + \frac{\mathrm{e}^{-|x_2(t)|}(t^2+1)}{t^2+2} + t\cos x_3(t) \\
&\quad + \frac{\mathrm{e}^{-\pi t}|I^{2/3}x_1(t)+I^{4/3}x_2(t)+I^{5/3}x_3(t)|}{10\sqrt{\pi}(1+|I^{2/3}x_1(t)+I^{4/3}x_2(t)+I^{5/3}x_3(t)|)}, \\
{}^{c}D^{7/4}x_3(t) &= t\sin x_1(t) + \cos x_2(t) \\
&\quad + \frac{t^2|x_3(t)+I^{1/4}x_1(t)+I^{1/6}x_3(t)|}{\sqrt{\pi}(1+|x_3(t)+I^{1/4}x_1(t)+I^{1/6}x_3(t)|)} + \frac{t|I^{1/5}x_2(t)|^3}{1+|I^{1/5}x_2(t)|^3},
\end{aligned} \tag{8}$$

with the boundary conditions

$$\begin{aligned}
&I^{1/2}x_1(1/4) + I^{1/3}x_1(1/3) + I^{1/4}x_1(1/2) \\
&\quad + I^{1/2}x_1(1/6) + I^{1/3}x_1(1/5) + I^{1/4}x_1(1/4) = \int_0^1 x_1(s)\,\mathrm{d}s, \\
&I^{2/3}x_2(1/4) + I^{4/3}x_2(1/3) + I^{5/3}x_2(1/2) \\
&\quad + I^{2/3}x_2(4/3) + I^{5/3}x_2(1/5) + I^{1/4}x_1(1/4) = \int_0^1 x_1(s)\,\mathrm{d}s, \\
&I^{1/4}x_3(1/4) + I^{1/5}x_3(1/3) + I^{1/6}x_3(1/2) \\
&\quad + I^{1/4}x_3(1/6) + I^{1/5}x_2(1/5) + I^{1/6}x_1(1/4) = \int_0^1 x_1(s)\,\mathrm{d}s, \\
&x_1(0)+x_1(1) = 1, \quad x_2(0)+x_2(1) = 1 \quad \text{and} \quad x_3(0)+x_3(1) = 2.
\end{aligned}$$

Put $k = 3$, $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt{3}$, $\alpha_3 = 7/4$, $\beta_{11} = 1/2$, $\beta_{12} = 1/3$, $\beta_{13} = 1/4$, $\beta_{21} = 2/3$, $\beta_{22} = 4/3$, $\beta_{23} = 5/3$, $\beta_{31} = 1/4$, $\beta_{32} = 1/5$, $\beta_{33} = 1/6$, $\xi_1 = 1/4$, $\xi_2 = 1/3$, $\xi_3 = 1/2$, $\eta_1 = 1/6$, $\eta_2 = 1/5$, $\eta_3 = 1/4$, $a_1 = 1$, $a_2 = -1$, $a_3 = 2$, $b_1 = 1$, $b_2 = -1$, $b_3 = 2$,$f_1(t, y_1, y_2, y_3, y_4, y_5, y_6) = |y_1|/(1 + |y_1|) + \mathrm{e}^{\cos t}[(|y_2| + |y_3|)/(1 + |y_2| + |y_3|) + \cos y_4 + \sin(y_5 + y_6)]$, $f_2(t, y_1, y_2, y_3, y_4, y_5, y_6) = \mathrm{e}^{t}|y_1|^3/(1 + |y_1|^3) + \mathrm{e}^{-|y_2|}(t^2 + 1)/(t^2 + 2) + t\cos y_3 + \mathrm{e}^{-\pi t}|y_4 + y_5 + y_6|/(10\sqrt{\pi} \times (1 + |y_4 + y_5 + y_6|))$, $f_3(t, y_1, y_2, y_3, y_4, y_5, y_6) = t\sin y_1 + \cos y_2 + t^2|y_3 + y_4 + y_6|/(\sqrt{\pi}(1 + |y_3 + y_4 + y_6|)) + t|y_5|^3/(1 + |y_5|^3)$ for $t \in [0, 1]$, $y_1, y_2, y_3, y_4, y_5, y_6 \in \mathbb{R}$. One can check that problem (8) satisfy the conditions of Theorem 2 with $\psi_1 = \psi_2 =$

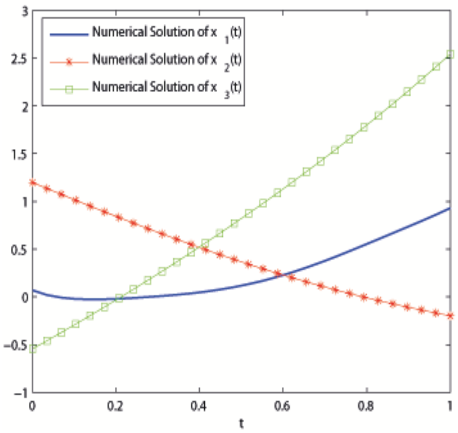

Figure 9. Chebyshev method.

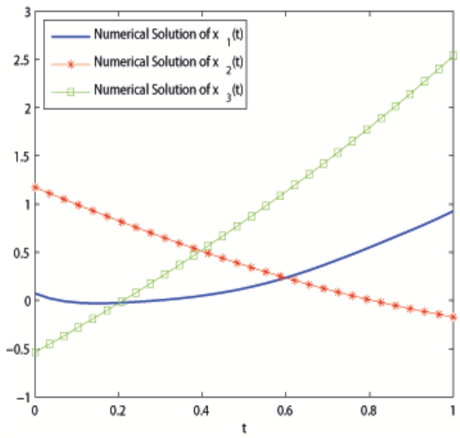

Figure 10. Legendre method.

Table 5

i	Coeficient values of Chebyshev method			Coeficient values of Legendre method		
	c_{1i}	c_{2i}	c_{3i}	d_{1i}	d_{2i}	d_{3i}
0	3.1086 e −01	4.3468 e −01	9.0987 e −01	2.5006 e −01	4.1217 e −01	8.7998 e −01
1	4.5506 e −01	−6.7931 e −01	1.5371 e +00	4.6415 e −01	−6.7931 e −01	1.5266 e +00
2	1.8361 e −01	6.6594 e −02	8.9756 e −02	2.3807 e −01	9.0421 e −02	1.1801 e −01
3	−1.3902 e −02	5.1344 e −03	6.7440 e −03	−1.9944 e −02	8.1119 e −03	1.0342 e −02
4	−2.0399 e −03	−1.2617 e −03	1.0886 e −03	−3.0047 e −03	−2.4608 e −03	2.0997 e −03
5	−1.0226 e −02	−6.4247 e −04	1.5461 e −04	−1.7063 e −02	−1.0749 e −03	4.0757 e −04
6	7.5645 e −03	−1.6293 e −05	−2.4232 e −05	1.4873 e −02	−3.2794 e −04	−8.5696 e −05

Table 6

t	$\lvert\tilde{x}_1(t) - \hat{x}_1(t)\rvert$	$\lvert\tilde{x}_2(t) - \hat{x}_2(t)\rvert$	$\lvert\tilde{x}_3(t) - \hat{x}_3(t)\rvert$
0.0	3.7905 e − 03	2.6263 e − 02	6.6857 e − 03
0.1	1.2589 e − 03	2.1241 e − 02	5.1368 e − 03
0.2	4.9394 e − 03	1.6554 e − 02	3.6195 e − 03
0.3	2.9277 e − 03	1.1625 e − 02	2.2606 e − 03
0.4	2.6737 e − 03	6.3281 e − 03	1.0412 e − 03
0.5	7.6077 e − 03	7.6994 e − 04	1.2149 e − 04
0.6	8.4750 e − 03	4.8740 e − 03	1.3145 e − 03
0.7	4.4869 e − 03	1.0459 e − 02	2.5899 e − 03
0.8	2.0412 e − 03	1.5909 e − 02	3.9472 e − 03
0.9	6.4560 e − 03	2.1198 e − 02	5.3369 e − 03
1.0	3.7905 e − 03	2.6263 e − 02	6.6857 e − 03

$\psi_3 = 1$, $h_1(t) = 1 + 3\mathrm{e}^{\cos t}$, $h_2(t) = \mathrm{e}^t + t(t_2 + 1)/(t^2 + 2) + (\mathrm{e}^{-\pi t})/(10\sqrt{\pi})$, $h_3(t) = t_2/\sqrt{\pi} + 2t + 1$ and $M > 195$. Hence, problem (8) has a solution in $C[0,1] \times C[0,1] \times C[0,1]$. Similar to Examples 1 and 2, we can approximate the solution by the Chebyshev and Legendre methods. We denote numerical solution of the Chebyshev and Legendre methods by $\tilde{x}_1$, $\tilde{x}_2$, $\tilde{x}_3$, $\hat{x}_1$, $\hat{x}_2$ and $\hat{x}_3$. One can see that the numerical solution coincides the exact solution as we show it in Figs. 9, 10 and Tables 5 and 6.

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