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Modeling the impact of mitigation options on abatement of methane emission from livestock

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Abstract. Mitigation of methane emission from livestock sector is crucial to combat the menace of global warming. In the present paper, a nonlinear mathematical model is proposed to investigate the impact of mitigation options for curtailing livestock methane emission on the reduction of atmospheric concentration of methane. In modeling process, it is assumed that the mitigation options are applied at a rate proportion to the livestock population. The conditions for reduction and stabilization of atmospheric methane have been obtained. Numerical simulation has been performed to verify the analytical findings by taking the secondary data of atmospheric concentration of methane, human and livestock populations. Sensitivity analysis is carried out to explore the impact of the key parameters of the model system.

Keywords: mathematical model, CH_4 gas, livestock farming, mitigation options, stability.

1 Introduction

Global warming is presently one of the most gravest threats to human environment. Global temperatures are at much higher levels than they are in past millions of years and are expected to rise in the future. This warming is expected to have many adverse consequences including melting of glaciers and ice caps, increase in frequency and intensity of extreme weather events, increase in incidence of vector-borne infections, change in rain fall pattern, etc. The main culprit behind global warming is the excessive emission of greenhouse gases by human activities. The prime human-influenced greenhouse gases are carbon dioxide (CO_2), methane (CH_4), nitrous oxide (N_2O) and chlorofluorocarbon (CFC). Methane is the most prevalent anthropogenic greenhouse gas after carbon dioxide.

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It accounts for nearly 16% of the global anthropogenic greenhouse gas emissions [10]. While methane is less abundant than carbon dioxide, it contributes significantly to the warming of Earth's atmosphere due to its high global warming potential. The 100 years global warming potential of methane is 25, i.e., it absorbs 25 times more heat per mass unit than CO_2 over a period of 100 years [21]. This makes it crucial to control the rising level of atmospheric methane for addressing the goal of mitigation of global warming. The major anthropogenic sources of methane include livestock farms, rice fields, biomass burning, coal mines, landfills, natural gas and oil systems, etc.

Livestock farming is the largest anthropogenic source of methane. Methane emission from livestock accounts for nearly 33% of global anthropogenic methane emissions [8]. Livestock, specially ruminants like cattle (*Bos taurus*), buffalo (*Bubalus bubalis*), sheep (*Ovis aris*) and goat (*Capra hircus*), produce significant amount of CH_4 directly via enteric fermentation and indirectly via manure management. Enteric fermentation is a part of the digestive process in livestock, where anaerobic microbes, called methanogens, residing in animal digestive systems ferment the feed consumed. Methane is produced as a waste product of this process. Enteric fermentation is stronger in ruminant animals due to the presence of a large fore-stomach or rumen; while, it is weak in non-ruminant animals, such as pigs and horse. CH_4 production in livestock is affected by various factors including the physical and chemical characteristics of the feed, the level of intake, the use of feed additives, health of the animal, etc. There are a variety of mitigation options available for mitigation of methane emission from livestock [2, 8, 19, and references therein]. One of the promising options is to change the forage type and quality. It is found that methane emission from enteric fermentation can be reduced effectively by increasing dry matter intake, increasing the proportion of concentrate in the diet, replacing fibrous concentrate with starchy one, replacing rapidly degraded starch with slowly degraded starch, using legume rather than grass forage, using grinded and pelleted hay rather than long hay, changing forage preservation method (ensiled rather than dried), and upgrading and supplementation of poor quality forages [1]. Apart from dietary interventions, methane emission can also be reduced by manipulations of rumen fermentation by addition of fats, ionosphere, elimination of protozoa from the rumen by dietary or chemical agents, etc. [6, 8]. Methane emission can also be suppressed directly through the use of chemical compounds, like bromoethanesulfonate, halogenated methane analogues such as bromochloromethane (BCM) and chloroform, etc. [7,19]. Use of probiotics, essential oils and acetogens, immunization and genetic selection of animal are also effective strategies for inhibition of methane production [2].

Implementation of the above mentioned mitigation options reduces the methane emission from livestock. But, the amount of methane emission from livestock depends in large on the number of livestock. The global livestock population trends show that livestock population has grown considerably over last few decades. This rise is expected to continue in future due to the vital role of livestock in global food system. Livestock products, like milk, meat, etc., are good sources of high quality protein and micronutrients. The global milk and meat consumption is increasing day-by-day with population growth, rising incomes and urbanization. Worldwide, between 1964–1966 to 1997–1999, the per capita meat consumption has increased from 24.2 kg per year to 36.4 kg per year and is anticipated to rise by 45.3 kg per year till 2030. The per capita milk consumption has also increased from 73.9 kg per year in 1964–1966 to 78.1 kg per year in 1997–1999 and is anticipated to rise by 89.5 kg per year till 2030 [29]. Due to the high consumption and demand of livestock products, livestock farming has increased around the globe and taken the form of livestock revolution. This livestock revolution is also important for rural development. A major portion of rural population in developing countries depends on livestock for their livelihood and nutritional requirements. Thus, expansion in livestock framing aids in poverty alleviation in developing countries. Government in many countries have taken steps to promote the livestock farming. But, the expansion in livestock sector must be accomplished with substantial reduction in livestock's environmental impact. To devise strategies for attaining such scenario, it is crucial to investigate the effectiveness of the available mitigation options for livestock methane emission on the reduction of atmospheric methane. Mathematical modeling using differential equations may be an effective tool for such investigations. In recent years, differential equation models have played a crucial role in exploring the impact of various factors on the reduction and stabilization of greenhouse gases and pollutants in the atmosphere [4, 11–15, 17, 18, 20, 22, 23]. The objective of the present study is to formulate a nonlinear mathematical model to study the impact of mitigation options for livestock methane emission on the atmospheric level of methane in a scenario, where effort are made to increase the livestock population to meet food demand.

2 Model formulation

Let $N(t)$ and $P(t)$ be the global human and livestock population, respectively. Also, let $C(t)$ be the atmospheric concentration of CH_4 and $S(t)$ be a measure of mitigation options, which are applied to reduce the livestock methane emission at the time t. These mitigation options can be measured in terms of the cost involved in their implementation. It is assumed that concentration of atmospheric methane is increasing due to its emission from livestock farming as well as other human activities. Since the amount of methane emission from livestock depends in large on the number of livestock, the livestock methane emission is assumed to be proportional to livestock population. Methane emission from other human activities is assumed to be proportional to human population. The emission rate of methane from non-anthropogenic (i.e., natural) sources is assumed to be constant. Since mitigation options reduce the methane emission from livestock, therefore we have taken the emission rate coefficient of methane from livestock as a decreasing function of mitigation options. Under these assumptions, the dynamics of atmospheric methane can be modeled as

$$\dot{C} = -\alpha_0(C - C_0) + \lambda_1 N + \left(\lambda_2 - \frac{\eta_2 S}{k_2 + S}\right) P,$$

where $\dot{C}$ stands for $\mathrm{d}C/\mathrm{d}t$. In the above equation, C_0 is the constant input methane concentration in the atmosphere from natural sources. The constant α_0 is the natural depletion rate coefficient of atmospheric methane. The constants λ_1 is the emission rate coefficient of methane from human activities other than livestock farming, whereas λ_2

is emission rate coefficient of methane from livestock. The constant η_2 represents the efficiency of mitigation options to reduce the livestock methane emission, and k_2 is a half saturation constant, which represents the mitigation options at which the reduction in livestock methane emission is half of its maximum possible reduction, which can be ever achieved through mitigation options. The constant k_2 limits the effect of mitigation options in reducing the livestock methane emissions. As the reduction in methane emission rate cannot exceed the emission rate itself, so $\eta_2 < \lambda_2$.

It is assumed that human and livestock population both grow logistically. The demand of livestock product increases with the increase in human population and this leads to expansion in livestock sector [24,31]. Also, the increase in livestock population boosts the growth of human population directly via consumption of livestock products and indirectly via promoting the economical state of the people involved in livestock farming [9, 26]. Thus, human and livestock share a mutualistic relationship. In this view, it is assumed that human population and livestock population both facilitate each other's growth. Since global warming due to the increased concentration of methane has adverse impact on human as well as livestock population, we assume that human and livestock populations decrease due to elevated level of atmospheric methane. Under these assumptions, the differential equations describing the dynamics of human and livestock populations are

$$\dot{N} = sN\left(1 - \frac{N}{L}\right) + \beta_1 NP - \theta_1(C - C_0)N$$

and

$$\dot{P} = s_1 P\left(1 - \frac{P}{L_1}\right) + \beta_2 PN - \theta_2(C - C_0)P,$$

respectively. In the above equations, the constants s and L are the intrinsic growth rate and carrying capacity of human population in the absence of livestock population and the adverse impacts of elevated level of methane. The constants s_1 and L_1 are the intrinsic growth rate and carrying capacity of livestock population in absence of human population and the adverse impacts of elevated level of methane. The constants θ_1 and θ_2 are respectively the decline rate coefficients of human and livestock populations due to the adverse effects of elevated level of atmospheric methane. The constant β_1 is the growth rate coefficient of human population due to increase in livestock population. The constant β_2 is the growth rate coefficient of livestock population due to human efforts.

The implementation rate of mitigation options is assumed to be proportional to the livestock population. Some of these mitigation efforts diminish with the passage of time due to their inefficacy in reducing livestock methane emission or economical barriers. Thus, the differential equation governing dynamics of mitigation options is

$$\dot{S} = \nu P - \delta S.$$

In the above equation, the constants ν and δ are the implementation and depletion rate coefficients of mitigation options, respectively.

Thus, we have the following system of nonlinear differential equations depicting the dynamics of the problem:

$$\begin{aligned}\dot{C} &= -\alpha_0(C - C_0) + \lambda_1 N + \left(\lambda_2 - \frac{\eta_2 S}{k_2 + S}\right)P,\\ \dot{N} &= sN\left(1 - \frac{N}{L}\right) + \beta_1 NP - \theta_1(C - C_0)N,\\ \dot{P} &= s_1 P\left(1 - \frac{P}{L_1}\right) + \beta_2 PN - \theta_2(C - C_0)P,\\ \dot{S} &= \nu P - \delta S,\end{aligned} \tag{1}$$

where $C(0) > C_0$, $N(0) \geqslant 0$, $P(0) \geqslant 0$, $S(0) \geqslant 0$.

Boundedness of model solutions

Let $x(t) = C(t) + N(t) + P(t) + S(t)$, then

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} + \zeta x &\leqslant \alpha_0 C_0 + (\lambda_1 + s + \zeta)N + (\lambda_2 + s_1 + \nu + \zeta)P\\ &\quad - \frac{sN^2}{L} - \frac{s_1 P^2}{L_1} + (\beta_1 + \beta_2)NP\\ &= \alpha_0 C_0 + f(N, P),\end{aligned}$$

where $\zeta = \min\{\alpha_0, \delta\}$ and $f(N, P) = (\lambda_1 + \zeta)N + (\lambda_2 + \nu + \zeta)P + sN(1 - N/L) + s_1 P(1 - P/L_1) + (\beta_1 + \beta_2)NP$. The function $f(N, P)$ has only one stationary point $(((2s_1/L_1)(\lambda_1 + s + \zeta) + (\beta_1 + \beta_2)(\lambda_2 + s_1 + \nu + \zeta))/(4ss_1/(LL_1) - (\beta_1 + \beta_2)^2),$ $((\beta_1 + \beta_2)(\lambda_1 + s + \zeta) + (2s/L)(\lambda_2 + s_1 + \nu + \zeta))/(4ss_1/(LL_1) - (\beta_1 + \beta_2)^2))$, which is a maxima provided the following condition holds:

$$\frac{4ss_1}{LL_1} - (\beta_1 + \beta_2)^2 > 0. \tag{2}$$

The maximum value of $f(N, P)$ is obtained as

$$\begin{aligned}M = \frac{1}{\frac{4ss_1}{LL_1} - (\beta_1 + \beta_2)^2}\biggl(&\frac{s_1}{L_1}(\lambda_1 + s + \zeta)^2 + \frac{s}{L}(\lambda_2 + s_1 + \nu + \zeta)^2\\ &+ (\beta_1 + \beta_2)(\lambda_1 + s + \zeta)(\lambda_2 + s_1 + \nu + \zeta)\biggr).\end{aligned}$$

Thus, under the condition (2), we have

$$\frac{\mathrm{d}x}{\mathrm{d}t} + \zeta x \leqslant \alpha_0 C_0 + M = K \quad \text{(say)}.$$

Using Gronwall's inequality, we have

$$x(t) \leqslant \frac{K}{\zeta} + \left(x(0) - \frac{K}{\zeta}\right)\mathrm{e}^{-\zeta t}.$$

As $\zeta > 0$, so for large values of t, we can write

$$x(t) \leqslant \frac{K}{\zeta} + \epsilon,$$

where ϵ is an arbitrary positive constant. This shows the boundedness of model solutions. Here it may be noted that condition (2) implies $ss_1/(LL_1) - \beta_1\beta_2 > 0$, which is the condition for boundedness of solutions of the classical two-species models of mutualism [16].

3 Equilibria and their stability

Since it is not possible to find out the exact solutions to the nonlinear model system (1), we determine the long-term behavior of the system by using stability theory of differential equations. In the following, we determine the equilibrium points of the model system and then perform stability analysis of the equilibria.

3.1 Equilibria

The model system (1) has four non-negative equilibria, which are listed as below:

(i) $E_1(C_0, 0, 0, 0)$ always exists. This equilibrium implies that human and livestock populations both are absent and thus not contributing to the methane emission, and in this case, atmospheric methane is at its natural level C_0.

(ii) $E_2(C_2, N_2, 0, 0)$, where $C_2 = C_0 + \lambda_1 sL/(s\alpha_0 + \theta_1\lambda_1 L)$ and $N_2 = s\alpha_0 L/(s\alpha_0 + \theta_1\lambda_1 L)$, always exists. This equilibrium states that only human population is present however livestock population is absent. In this case, the concentration of methane will be more than its natural level due to its emission from human activities. As livestock population is absent, so the mitigation options to control the emission of methane from livestock are zero.

(iii) The equilibrium $E_3(C_3, 0, P_3, S_3)$ always exists. This equilibrium states that livestock population is present and human population is absent. In this case, also the concentration of methane will be more than its natural level due to its emission from livestock. As the mitigation options are applied to control the emission of methane from livestock population, which is present, and so in this case, mitigation options are also present.

(iv) The interior equilibrium $E^*(C^*, N^*, P^*, S^*)$ exists if the following conditions hold:

$$\left(\frac{s}{L} + \frac{\theta_1\lambda_1}{\alpha_0}\right)\left(\frac{s_1}{L_1} + \frac{\theta_2(\lambda_2 - \eta_2)}{\alpha_0}\right) - \left(\beta_1 - \frac{\theta_1(\lambda_2 - \eta_2)}{\alpha_0}\right)\left(\beta_2 - \frac{\theta_2\lambda_1}{\alpha_0}\right) > 0, \quad (3)$$

$$s\left(s_1 + \beta_2 L - \frac{\theta_2\lambda_1 L}{\alpha_0}\right) + \frac{s_1\theta_1\lambda_1 L}{\alpha_0} > 0, \quad (4)$$

$$s + \beta_1 P^* - \frac{\theta_1}{\alpha_0}\left(\lambda_2 - \frac{\eta_2 S^*}{k_2 + S^*}\right)P^* > 0. \quad (5)$$

The existence of equilibria E_1 and E_2 is obvious. In the following, we show the existence of equilibria E_3 and E^*. In the equilibrium $E_3(C_3, 0, P_3, S_3)$, the values C_3, P_3 and S_3 may be obtained by solving the following set of algebraic equations:

$$-\alpha_0(C - C_0) + \left(\lambda_2 - \frac{\eta_2 S}{k_2 + S}\right)P = 0, \tag{6}$$

$$s_1\left(1 - \frac{P}{L_1}\right) - \theta_2(C - C_0) = 0, \tag{7}$$

$$\nu P - \delta S = 0. \tag{8}$$

From equation (8), we have

$$S = \frac{\nu P}{\delta}. \tag{9}$$

Using equation (9) in equation (6), we get

$$C = C_0 + \left(\lambda_2 - \frac{\eta_2 \nu P}{k_2\delta + \nu P}\right)\frac{P}{\alpha_0}. \tag{10}$$

Now, using equation (10) in equation (7), we get following quadratic equation in P:

$$aP^2 + bP + c = 0, \tag{11}$$

where $a = \nu(s_1/L_1 + \theta_2(\lambda_2 - \eta_2)/\alpha_0)$, $b = k_2\delta(s_1/L_1 + \theta_2\lambda_2/\alpha_0) - s_1\nu$ and $c = -s_1k_2\delta$. Here $a > 0$ and $c < 0$, this implies that there exists a unique positive root, say P_3, of equation (11). Using this value of P in (9) and (10), we get positive values of $S = S_3$ and $C = C_3$, respectively.

The values of C^*, N^*, P^* and S^* in the equilibrium E^* may be obtained by solving the following set of algebraic equations:

$$-\alpha_0(C - C_0) + \lambda_1 N + \left(\lambda_2 - \frac{\eta_2 S}{k_2 + S}\right)P = 0, \tag{12}$$

$$s\left(1 - \frac{N}{L}\right) + \beta_1 P - \theta_1(C - C_0) = 0, \tag{13}$$

$$s_1\left(1 - \frac{P}{L_1}\right) + \beta_2 N - \theta_2(C - C_0) = 0, \tag{14}$$

$$\nu P - \delta S = 0. \tag{15}$$

From equation (15), we have

$$S = \frac{\nu P}{\delta}. \tag{16}$$

From equation (12), we have

$$C = C_0 + \frac{\lambda_1 N}{\alpha_0} + \left(\lambda_2 - \frac{\eta_2 S}{k_2 + S}\right)\frac{P}{\alpha_0}. \tag{17}$$

Using equation (17) in equation (13), we get

$$N = \frac{\alpha_0 L}{s\alpha_0 + \theta_1\lambda_1 L}\left[s + \beta_1 P - \frac{\theta_1}{\alpha_0}\left(\lambda_2 - \frac{\eta_2 S}{k_2 + S}\right)P\right]. \tag{18}$$

Using equations (16), (17) and (18) in equation (14), we get the following quadratic equation in P:

$$\tilde{a}P^2 + \tilde{b}P + \tilde{c} = 0, \tag{19}$$

where

$$\begin{aligned}
\tilde{a} &= \nu\left[\frac{ss_1}{LL_1} - \beta_1\beta_2 + \left(\frac{s_1\theta_1}{L_1} + \beta_1\theta_2\right)\frac{\lambda_1}{\alpha_0} + \left(\frac{s\theta_2}{L} + \beta_2\theta_1\right)\frac{(\lambda_2 - \eta_2)}{\alpha_0}\right],\\
\tilde{b} &= k_2\delta\left[\frac{ss_1}{LL_1} - \beta_1\beta_2 + \left(\frac{s_1\theta_1}{L_1} + \beta_1\theta_2\right)\frac{\lambda_1}{\alpha_0} + \left(\frac{s\theta_2}{L} + \beta_2\theta_1\right)\frac{\lambda_2}{\alpha_0}\right]\\
&\quad - \nu\left[s_1\left(\frac{s}{L} + \frac{\theta_1\lambda_1}{\alpha_0}\right) + s\left(\beta_2 - \frac{\theta_2\lambda_1}{\alpha_0}\right)\right],\\
\tilde{c} &= -\frac{k_2\delta}{L}\left[s\left(s_1 + \beta_2 L - \frac{\theta_2\lambda_1 L}{\alpha_0}\right) + \frac{s_1 L\theta_1\lambda_1}{\alpha_0}\right].
\end{aligned}$$

Now equation (19) has a unique positive root, say P^*, if $\tilde{a}$ and $\tilde{c}$ are of opposite signs. Thus, here arises two possibilities for unique positive root of (19):

$$\text{(i)}\quad \tilde{a} > 0 \quad\text{and}\quad \tilde{c} < 0, \qquad \text{(ii)}\quad \tilde{a} < 0 \quad\text{and}\quad \tilde{c} > 0.$$

Possibility (ii) is dropped out as it is not biologically feasible. The term $s_1 + \beta_2 L - \theta_2\lambda_1 L/\alpha_0$ represents the net intrinsic growth rate in livestock population and hence should be positive. This gives $\tilde{c} < 0$. Thus, equation (19) has a unique positive root if $\tilde{a} > 0$ and $\tilde{c} < 0$. Now, $\tilde{a} > 0$ if condition (3) holds, and $\tilde{c} < 0$ if condition (4) holds. This implies that equation (19) has a unique positive root if conditions (3) and (4) are satisfied. Condition (3) is analogous to the condition of existence of interior equilibrium in basic two-species model of mutualism. It says that the self effects in human and livestock population must be greater than the effect of their interactions.

Using the value of $P = P^*$ in (16), we get the positive value of $S = S^*$. Using $P = P^*$ and $S = S^*$ in (18), we get the positive value of $N = N^*$ provided condition (5) is satisfied. Finally, using the above positive values of N, P and S in equation (17), we get positive value of $C = C^*$.

Remark 1. From equations (12) to (15), we find that

$$\frac{\mathrm{d}C^*}{\mathrm{d}\nu} = -\frac{(\frac{ss_1}{LL_1} - \beta_1\beta_2)\eta_2 k_2 {P^*}^2}{\alpha_0\delta(k_2 + S^*)^2 A} \quad\text{and}\quad \frac{\mathrm{d}C^*}{\mathrm{d}\eta_2} = -\frac{(\frac{ss_1}{LL_1} - \beta_1\beta_2)S^* P^*}{\alpha_0(k_2 + S^*)A},$$

where

$$\begin{aligned}
A &= \frac{ss_1}{LL_1} - \beta_1\beta_2 + \left(\frac{s_1\theta_1}{L_1} + \beta_1\theta_2\right)\frac{\lambda_1}{\alpha_0}\\
&\quad + \frac{1}{\alpha_0}\left(\frac{s\theta_2}{L} + \beta_2\theta_1\right)\left(\lambda_2 - \frac{\eta_2 S^*}{k_2 + S^*} - \frac{\eta_2 k_2 S^*}{(k_2 + S^*)^2}\right).
\end{aligned}$$

Since $A > 0$ (from condition (3)), $\mathrm{d}C^*/\mathrm{d}\nu < 0$ and $\mathrm{d}C^*/\mathrm{d}\eta_2 < 0$ if

$$\frac{ss_1}{LL_1} - \beta_1\beta_2 > 0. \tag{20}$$

This shows that an increase in the implementation rate of mitigation option or efficiency of mitigation options leads to reduction in atmospheric level of methane under condition (20).

3.2 Stability of equilibria

In this section, we study the stability behavior of the obtained equilibria. The results regarding the stability of equilibria are stated in the following theorems.

Theorem 1. *The equilibrium E_1 is always unstable. The equilibrium E_2 is unstable whenever E^* exists. The equilibrium E_3 is unstable if $s + \beta_1 P_3 - \theta_1(C_3 - C_0) > 0$. The interior equilibrium E^*, if exists, is locally asymptotically stable if and only if the following conditions are satisfied*:

$$A_3 > 0 \quad \textit{and} \quad A_3(A_1A_2 - A_3) - A_1^2A_4 > 0, \tag{21}$$

where $A_i's$ $(i = 1, 2, 3, 4)$ are the coefficients of characteristic equation of Jacobian matrix evaluated at E^* and are defined in the proof.

Proof. The Jacobian matrix for model system (1) is given as follows:

$$J = \begin{pmatrix} -\alpha_0 & \lambda_1 & \lambda_2 - \frac{\eta_2 S}{k_2+S} & -\frac{\eta_2 k_2 P}{(k_2+S)^2} \\ -\theta_1 N & a_{22} & \beta_1 N & 0 \\ -\theta_1 P & \beta_2 P & a_{33} & 0 \\ 0 & 0 & \nu & -\delta \end{pmatrix},$$

where $a_{22} = s(1 - 2N/L) + \beta_1 P - \theta_1(C - C_0)$ and $a_{33} = s_1(1 - 2P/L_1) + \beta_2 N - \theta_2(C - C_0)$.

Let J_i be the Jacobian matrix J evaluated at E_i $(i = 1, 2, 3)$.

From the matrix J_1, it is found that its eigenvalues are $-\alpha_0$, s, s_1 and $-\delta$. Thus, E_1 always has a stable manifold locally in C,S-plane, whereas it has unstable manifold locally in N,P-plane. Thus, E_1 is always unstable.

From the Jacobian matrix J_2, it is found that one of the eigenvalues of the matrix J_2 is $s_1 + \beta_2 N_2 - \theta_2(C_2 - C_0)$. This eigenvalue can be written as

$$\begin{aligned} s_1 + \beta_2 N_2 - \theta_2(C_2 - C_0) &= s_1 + \frac{\beta_2 s \alpha_0 L}{\lambda_1\theta_1 L + s\alpha_0} - \frac{\theta_2\lambda_1 s L}{\lambda_1\theta_1 L + s\alpha_0} \\ &= \frac{\alpha_0}{\lambda_1\theta_1 L + s\alpha_0}\left[s\left(s_1 + \beta_2 L - \frac{\theta_2\lambda_1 L}{\alpha_0}\right) + \frac{s_1\lambda_1\theta_1 L}{\alpha_0}\right]. \end{aligned}$$

Since $s(s_1 + \beta_2 L - \theta_2\lambda_1 L/\alpha_0) + s_1\lambda_1\theta_1 L/\alpha_0 > 0$ if E^* exists (condition (4)), therefore $s_1 + \beta_2 N_2 - \theta_2(C_2 - C_0)$ is positive whenever E^* exists. The other three eigenvalues of J_{E_2} are either negative or with negative real part. Thus, E_2 has stable manifold locally in C, N, S-space whereas it has an unstable manifold locally in P-direction whenever E^* exists. From the Jacobian matrix J_3, it is found that one of its eigenvalues is $s + \beta_1 P_3 - \theta_1(C_3 - C_0)$ and the other three eigenvalues are either negative or with negative real part. Thus, the local stability of equilibrium E_3 depends on the sign of this eigenvalue. The equilibrium E_3 is stable or unstable according to the sign of $s + \beta_1 P_3 - \theta_1(C_3 - C_0)$ is negative or positive.

To determine the sign of eigenvalues of the Jacobian matrix J evaluated at the equilibrium E^*, we make use of Routh–Hurwitz criterion. The characteristic equation for Jacobian matrix J evaluated at the equilibrium E^* is given as

$$\psi^4 + A_1\psi^3 + A_2\psi^2 + A_3\psi + A_4 = 0, \tag{22}$$

where

$$\begin{aligned}
A_1 &= \alpha_0 + \delta + \frac{sN^*}{L} + \frac{s_1P^*}{L_1},\\
A_2 &= \alpha_0\delta + \left(\frac{ss_1}{LL_1} - \beta_1\beta_2\right)N^*P^* + (\alpha_0 + \delta)\left(\frac{sN^*}{L} + \frac{s_1P^*}{L_1}\right)\\
&\quad + \lambda_1\theta_1N^* + \left(\lambda_2 - \frac{\eta_2S^*}{k_2 + S^*}\right)\theta_2P^*,\\
A_3 &= (\alpha_0 + \delta)\left(\frac{ss_1}{LL_1} - \beta_1\beta_2\right)N^*P^* + \alpha_0\delta\left(\frac{sN^*}{L} + \frac{s_1P^*}{L_1}\right)\\
&\quad + \lambda_1\left(\frac{s_1\theta_1}{L_1} + \beta_1\theta_2\right)N^*P^* + \delta\left(\lambda_2 - \frac{\eta_2S^*}{k_2 + S^*} - \frac{\eta_2k_2S^*}{(k_2 + S^*)^2}\right)\theta_2P^*\\
&\quad + \lambda_1\theta_1\delta N^* + \left(\lambda_2 - \frac{\eta_2S^*}{k_2 + S^*}\right)\left(\frac{s\theta_2}{L} + \beta_2\theta_1\right)N^*P^*,\\
A_4 &= \alpha_0\delta\left(\frac{ss_1}{LL_1} - \beta_1\beta_2\right)N^*P^* + \delta\lambda_1\left(\frac{s_1\theta_1}{L_1} + \beta_1\theta_2\right)N^*P^*\\
&\quad + \delta\left(\lambda_2 - \frac{\eta_2S^*}{k_2 + S^*} - \frac{\eta_2k_2S^*}{(k_2 + S^*)^2}\right)\left(\frac{s\theta_2}{L} + \beta_2\theta_1\right)N^*P^*.
\end{aligned}$$

Here it is apparent that A_1 and A_4 are positive. Using Routh–Hurwitz criterion, it is inferred that all the eigenvalues of the Jacobian matrix J_{E^*} will be either negative or with negative real part iff the conditions stated in (21) are satisfied. ☐

Theorem 2. *The equilibrium E^*, if exists, is globally asymptotically stable provided the following inequalities are satisfied*:

$$\beta_1^2 < \frac{\lambda_2ss_1\theta_1}{2\lambda_1LL_1\theta_2}, \tag{23}$$

$$\beta_2^2 < \frac{\lambda_1ss_1\theta_2}{2\lambda_2LL_1\theta_1}, \tag{24}$$

$$\eta_2^2 < \frac{\lambda_2s_1\alpha_0}{2\theta_2L_1}\min\left\{1, \frac{(k_2\delta + \nu P^*)^2}{\nu^2{P^*}^2}\right\}. \tag{25}$$

Proof. Consider the following positive definite function:

$$V = \frac{1}{2}(C - C^*)^2 + m_1\left(N - N^* - N^* \ln \frac{N}{N^*}\right) + m_2\left(P - P^* - P^* \ln \frac{P}{P^*}\right) + \frac{m_3}{2}(S - S^*)^2, \tag{26}$$

where m_1, m_2 and m_3 are positive constants to be chosen appropriately.

Now differentiating V with respect to t along the solution of model system (1), we get

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} = {}& (C - C^*)\biggl[-\alpha_0(C - C^*) + \lambda_1(N - N^*) + \lambda_2(P - P^*) \\ & - \eta_2\left(\frac{SP}{k_2 + S} - \frac{S^*P^*}{k_2 + S^*}\right)\biggr] \\ & + m_1(N - N^*)\left[-\frac{s}{L}(N - N^*) + \beta_1(P - P^*) - \theta_1(C - C^*)\right] \\ & + m_2(P - P^*)\left[-\frac{s_1}{L_1}(P - P^*) + \beta_2(N - N^*) - \theta_2(C - C^*)\right] \\ & + m_3(S - S^*)\bigl[\nu(P - P^*) - \delta(S - S^*)\bigr].\end{aligned} \tag{27}$$

Since $SP/(k_2 + S) - S^*P^*/(k_2 + S^*) = S(P - P^*)/(k_2 + S) + k_2P^*(S - S^*)/((k_2 + S)(k_2 + S^*))$, we get

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} = {}& -\alpha_0(C - C^*)^2 - \frac{m_1 s}{L}(N - N^*)^2 - \frac{m_2 s_1}{L_1}(P - P^*)^2 - m_3\delta(S - S^*)^2 \\ & + (\lambda_1 - m_1\theta_1)(C - C^*)(N - N^*) + (\lambda_2 - m_2\theta_2)(C - C^*)(P - P^*) \\ & - \frac{\eta_2 S}{k_2 + S}(C - C^*)(P - P^*) - \frac{\eta_2 k_2 P^*}{(k_2 + S)(k_2 + S^*)}(C - C^*)(S - S^*) \\ & + m_1\beta_1(N - N^*)(P - P^*) + m_2\beta_2(N - N^*)(P - P^*) \\ & + m_3\nu(P - P^*)(S - S^*).\end{aligned} \tag{28}$$

Choosing $m_1 = \lambda_1/\theta_1$ and $m_2 = \lambda_2/\theta_2$, we get

$$\begin{aligned}\frac{\mathrm{d}V}{\mathrm{d}t} = {}& -\alpha_0(C - C^*)^2 - \frac{\lambda_1 s}{\theta_1 L}(N - N^*)^2 - \frac{\lambda_2 s_1}{\theta_2 L_1}(P - P^*)^2 - m_3\delta(S - S^*)^2 \\ & - \frac{\eta_2 S}{k_2 + S}(C - C^*)(P - P^*) - \frac{\eta_2 k_2 P^*}{(k_2 + S)(k_2 + S^*)}(C - C^*)(S - S^*) \\ & + \frac{\lambda_1\beta_1}{\theta_1}(N - N^*)(P - P^*) + \frac{\lambda_2\beta_2}{\theta_2}(N - N^*)(P - P^*) \\ & + m_3\nu(P - P^*)(S - S^*).\end{aligned}$$

Now, $\mathrm{d}V/\mathrm{d}t$ can be made negative definite provided

$$\eta_2^2 < \frac{\lambda_2 s_1 \alpha_0}{2\theta_2 L_1}, \tag{29}$$

$$\frac{\eta_2^2 k_2^2 P^{*2}}{(k_2+S)^2(k_2+S^*)^2} < m_3 \alpha_0 \delta, \tag{30}$$

$$\beta_1^2 < \frac{\lambda_2 s s_1 \theta_1}{2\lambda_1 L L_1 \theta_2}, \tag{31}$$

$$\beta_2^2 < \frac{\lambda_1 s s_1 \theta_2}{2\lambda_2 L L_1 \theta_1}, \tag{32}$$

$$m_3 \nu^2 < \frac{\lambda_2 s_1 \delta}{2\theta_2 L_1}. \tag{33}$$

Now, we can choose m_3 from inequalities (30) and (33) provided the following condition holds:

$$\frac{\eta_2^2 P^{*2}}{(k_2+S^*)^2} < \frac{\lambda_2 s_1 \alpha_0 \delta^2}{2\theta_2 L_1 \nu^2}.$$

Since $S^* = \nu P^*/\delta$, the above inequality can be rewritten as

$$\frac{\eta_2^2 P^{*2}}{(k_2\delta+\nu P^*)^2} < \frac{\lambda_2 s_1 \alpha_0}{2\theta_2 L_1 \nu^2}.$$

Combining the above inequality with (29), we get inequality (25). Thus, $\mathrm{d}V/\mathrm{d}t$ can be made negative definite provided inequalities (23)–(25) are satisfied. □

Remark 2. From the above theorem, it may be noted that the parameters β_1 and β_2 have destabilizing effect on the dynamics of system (1). This implies that if the growth in human population due to livestock product consumption or growth in livestock population due to human efforts is large, the atmospheric concentration of methane may not get stabilized.

4 Numerical simulation

4.1 Parameter estimation

To estimate the model parameters, the secondary data of atmospheric concentration of methane, human population and livestock population is used. The initial time is taken to be the year 1961. The annual time series data for average atmospheric concentration of methane for the period 1961–2011 is taken from European Environment Agency [28]. The data for world population for the period 1961–2011 is obtained from United nations population division [27] and world's livestock population data for the period of 1961–2006 is taken from [32]. Since the life time of methane in the atmosphere is 12.5 years [25], the value of α_0 is taken as $1/12.5 = 0.08$ year^{-1}. The value of C_0 is taken to be 700 ppb (parts per billion), the pre-industrial time methane concentration [5]. While fitting the data, it is assumed that there are no mitigation options in the system (i.e., $\eta_2 = 0$, $k_2 = 0$, $\nu = 0$ and $\delta = 0$). In the absence of appropriate data regarding other parameters, the model system (1) is calibrated for different values of parameters λ_1, λ_2, s, L, s_1, L_1, θ_1, β_1, β_2 and find the best fit for $\lambda_1 = 0.0095$, $\lambda_2 = 0.01$, $s = 0.03$, $L = 10000$, $s_1 = 0.01$, $L_1 = 6000$, $\theta_1 = 10^{-7}$, $\beta_1 = 4 \times 10^{-7}$, $\beta_2 = 5 \times 10^{-7}$. Thus, we have the following estimated parameter values:

$$\begin{aligned} &\alpha_0 = 0.08, && C_0 = 700, && \lambda_1 = 0.0095, && \lambda_2 = 0.01, \\ &L = 10000, && \theta_1 = 10^{-7}, && \beta_1 = 4 \times 10^{-7}, && s = 0.03, \\ &L_1 = 6000, && \theta_2 = 10^{-7}, && \beta_2 = 5 \times 10^{-7}, && s_1 = 0.01. \end{aligned} \tag{34}$$

Since in 1961, the average atmospheric concentration of methane was 1247 ppb [28]; human population was 3082 million [27]; and livestock population was 2292 million [32]; the model system (1) is simulated with initial starts

$$C(0) = 1247 \text{ ppb}, \qquad N(0) = 3082 \text{ million}, \qquad P(0) = 2292 \text{ million}.$$

The values of R-squared for the actual and fitted data of atmospheric CH_4, human population and livestock population are calculated as 0.8673, 0.9998 and 0.9524, respectively. This shows the strong correlation between the actual data and the data projected by the proposed model. The actual data and the model projections of these variables are plotted in Fig. 1. It is evident from this figure that the concentration of CH_4, human population and livestock population projected by the model resemble closely to the actual data.

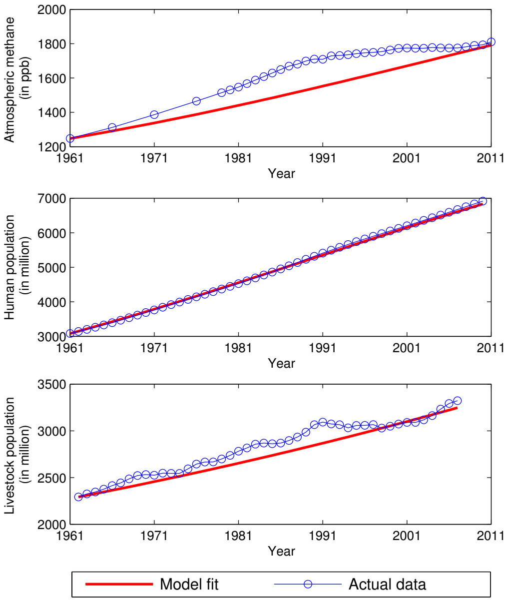

Figure 1. Comparison between actual data and model fit data of atmospheric methane, human population and livestock population.

4.2 Validation of analytical results

To validate the analytical findings and to illustrate the impact of implementation of mitigation options over the reduction of atmospheric level of methane, we have taken the values of η_2, k_2, ν and δ as $\eta_2 = 0.005$, $k_2 = 1000$, $\nu = 0.002$ and $\delta = 0.01$ along with the estimated set of parameter values (34). It is considered that mitigation options are implemented from the year 2006. Now, the initial time is fixed at the year 2006 and the initial conditions are taken to be $C(0) = 1775.56$ ppb, $N(0) = 6593.228$ millions, $P(0) = 3322$ millions and S(0) = 300 million dollars (since in 2006, the average atmospheric concentration of methane was 1775.56 ppb [28]; human population was 6593.228 million [27]; and livestock population was 3322 million [32]). For this data, conditions (3)–(5) for the existence of interior equilibrium E^* are satisfied. The component of interior equilibrium E^* are obtained as:

$$C^* = 2804.1592 \text{ ppb}, \qquad N^* = 11159.4043 \text{ million}, \qquad P^* = 9221.5717 \text{ million}$$
$$\text{and} \quad S^* = 1844.3143 \text{ million dollars}.$$

The eigenvalues of the Jacobian matrix corresponding to the equilibrium E^* for the model system (1) are -0.07969, -0.0347, -0.01449 and -0.00996. Since all the eigenvalues are negative, this implies that the interior equilibrium E^* is locally asymptotically stable. The nonlinear stability conditions (23)–(25) are also satisfied. To demonstrate the nonlinear stability of E^* in C, P, S-space, the solution trajectories of the model system (1) are plotted in C, P, S-space with different initial starts as shown in Fig. 2. It is evident from this figure that the solution trajectories starting from different points are approaching to the equilibrium values (C^*, P^*, S^*). This depicts the nonlinear stability of E^* in C, P, S-space. The effect of increase in parameters ν and η_2 over the atmospheric level of methane is demonstrated in Fig. 3. This figure shows that on increasing the implementation rate and efficiency of mitigation options, the atmospheric concentration of methane decreases. The impact of increase in the growth rate coefficient of livestock population due to human efforts (i.e., β_2) is depicted in Fig. 4. From this figure, it is clear that as the value of β_2 increases, atmospheric level of methane increases. Moreover, β_2 has destabilizing effect over the dynamics of system. It can be noted from Fig. 5 that when the values of β_2 exceeds 1.5411×10^{-6}, stability condition (24) violates.

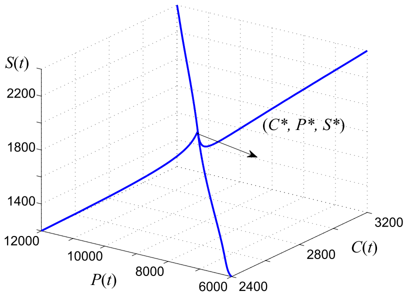

Figure 2. Nonlinear stability of E^* in C, P, S-space.

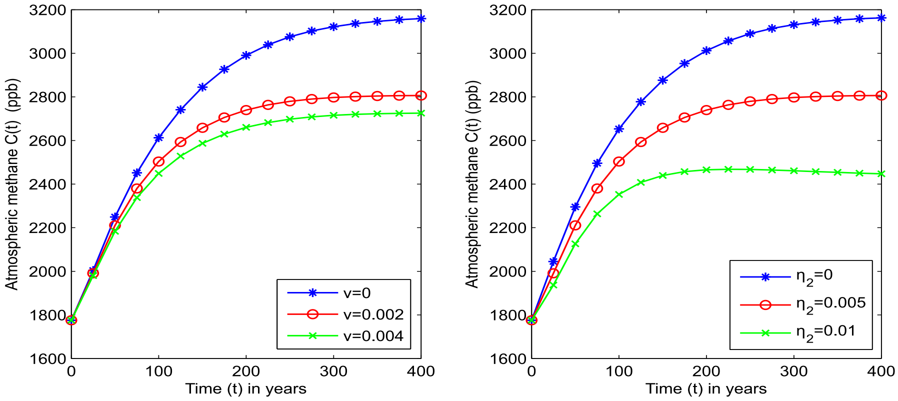

Figure 3. Variation in atmospheric concentration of methane with respect to time for different values of ν and η_2.

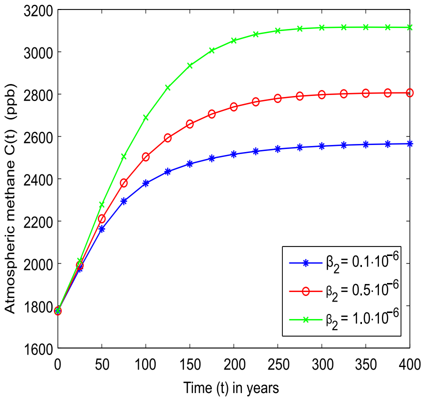

Figure 4. Variation in atmospheric concentration of methane with respect to time for different values of β_2.

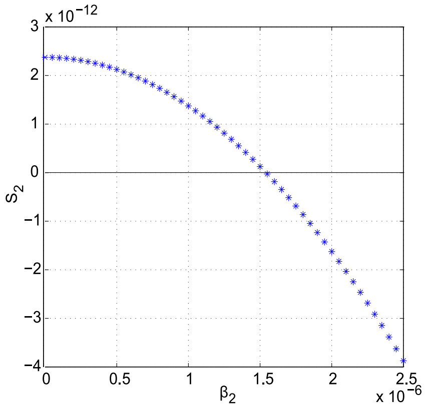

Figure 5. The impact of parameter β_2 on the stability condition (24). Here $S_2 = \lambda_1 s s_1 \theta_2 / (2\lambda_2 L L_1 \theta_1) - \beta_2^2$.

4.3 Sensitivity analysis

Sensitivity analysis assess the response of model variables to change in parameter values. To determine the effect of changes in the values of parameters β_2, η_2 and ν on the state of the model system (1), the basic sensitivity analysis of the model system (1) is performed with respect to these parameters following Bortz and Nelson [3]. The sensitivity systems with respect to parameters β_2, η_2 and ν are given by

$$\begin{aligned}\dot{C}_{\beta_2}(t,\beta_2) &= -\alpha_0 C_{\beta_2}(t,\beta_2) + \lambda_1 N_{\beta_2}(t,\beta_2) + \left(\lambda_2 - \frac{\eta_2 S(t,\beta_2)}{k_2 + S(t,\beta_2)}\right) P_{\beta_2}(t,\beta_2)\\ &\quad - \frac{\eta_2 k_2 P(t,\beta_2)}{(k_2 + S(t,\beta_2))^2} S_{\beta_2}(t,\beta_2),\end{aligned}$$

$$\begin{aligned}
\dot N_{\beta_2}(t,\beta_2) &= s\left(1-\frac{2N(t,\beta_2)}{L}\right)N_{\beta_2}(t,\beta_2)+\beta_1 N_{\beta_2}(t,\beta_2)P(t,\beta_2)\\
&\quad+\beta_1 N(t,\beta_2)P_{\beta_2}(t,\beta_2)-\theta_1(C(t,\beta_2)-C_0)N_{\beta_2}(t,\beta_2)\\
&\quad-\theta_1 C_{\beta_2}(t,\beta_2)N(t,\beta_2),\\
\dot P_{\beta_2}(t,\beta_2) &= s_1\left(1-\frac{2P(t,\beta_2)}{L_1}\right)P_{\beta_2}(t,\beta_2)+\beta_2 N_{\beta_2}(t,\beta_2)P(t,\beta_2)\\
&\quad+\beta_2 N(t,\beta_2)P_{\beta_2}(t,\beta_2)+P(t,\beta_2)N(t,\beta_2)\\
&\quad-\theta_2(C(t,\beta_2)-C_0)P_{\beta_2}(t,\beta_2)-\theta_2 C_{\beta_2}(t,\beta_2)P(t,\beta_2),\\
\dot S_{\beta_2}(t,\beta_2) &= \nu P_{\beta_2}(t,\beta_2)-\delta S_{\beta_2}(t,\beta_2),\\
\dot C_{\eta_2}(t,\eta_2) &= -\alpha_0 C_{\eta_2}(t,\eta_2)+\lambda_1 N_{\eta_2}(t,\eta_2)+\left(\lambda_2-\frac{\eta_2 S(t,\eta_2)}{k_2+S(t,\eta_2)}\right)P_{\eta_2}(t,\eta_2)\\
&\quad-\frac{\eta_2 k_2 P(t,\eta_2)}{(k_2+S(t,\eta_2))^2}S_{\eta_2}(t,\eta_2)-\frac{S(t,\eta_2)P(t,\eta_2)}{k_2+S(t,\eta_2)},\\
\dot N_{\eta_2}(t,\eta_2) &= s\left(1-\frac{2N(t,\eta_2)}{L}\right)N_{\eta_2}(t,\eta_2)+\beta_1 N_{\eta_2}(t,\eta_2)P(t,\eta_2)\\
&\quad+\beta_1 N(t,\eta_2)P_{\eta_2}(t,\eta_2)-\theta_1(C(t,\eta_2)-C_0)N_{\eta_2}(t,\eta_2)\\
&\quad-\theta_1 C_{\eta_2}(t,\eta_2)N(t,\eta_2),\\
\dot P_{\eta_2}(t,\eta_2) &= s_1\left(1-\frac{2P(t,\eta_2)}{L_1}\right)P_{\eta_2}(t,\eta_2)+\beta_2 N_{\eta_2}(t,\eta_2)P(t,\eta_2)\\
&\quad+\beta_2 N(t,\eta_2)P_{\eta_2}(t,\eta_2)-\theta_2\big(C(t,\eta_2)-C_0\big)P_{\eta_2}(t,\eta_2)\\
&\quad-\theta_2 C_{\eta_2}(t,\eta_2)P(t,\eta_2),\\
\dot S_{\eta_2}(t,\eta_2) &= \nu P_{\eta_2}(t,\eta_2)-\delta S_{\eta_2}(t,\eta_2),
\end{aligned}$$

and

$$\begin{aligned}
\dot C_{\nu}(t,\nu) &= -\alpha_0 C_{\nu}(t,\nu)+\lambda_1 N_{\nu}(t,\nu)+\left(\lambda_2-\frac{\eta_2 S(t,\nu)}{k_2+S(t,\nu)}\right)P_{\nu}(t,\nu)\\
&\quad-\frac{\eta_2 k_2 P(t,\nu)}{(k_2+S(t,\nu))^2}S_{\nu}(t,\nu),\\
\dot N_{\nu}(t,\nu) &= s\left(1-\frac{2N(t,\nu)}{L}\right)N_{\nu}(t,\nu)+\beta_1 N_{\nu}(t,\nu)P(t,\nu)+\beta_1 N(t,\nu)P_{\nu}(t,\nu)\\
&\quad-\theta_1(C(t,\nu)-C_0)N_{\nu}(t,\nu)-\theta_1 C_{\nu}(t,\nu)N(t,\nu),\\
\dot P_{\nu}(t,\nu) &= s_1\left(1-\frac{2P(t,\nu)}{L_1}\right)P_{\nu}(t,\nu)+\beta_2 N_{\nu}(t,\nu)P(t,\nu)+\beta_2 N(t,\nu)P_{\nu}(t,\nu)\\
&\quad-\theta_2(C(t,\nu)-C_0)P_{\nu}(t,\nu)-\theta_2 C_{\nu}(t,\nu)P(t,\nu),\\
\dot S_{\nu}(t,\nu) &= \nu P_{\nu}(t,\nu)-\delta S_{\nu}(t,\nu)+P(t,\nu),
\end{aligned}$$

respectively. Here $C_{\beta_2}(t,\beta_2)$ denote the rate of change of state variable C with respect to parameter β_2 and is called sensitivity function of C with respect to the parameter β_2.

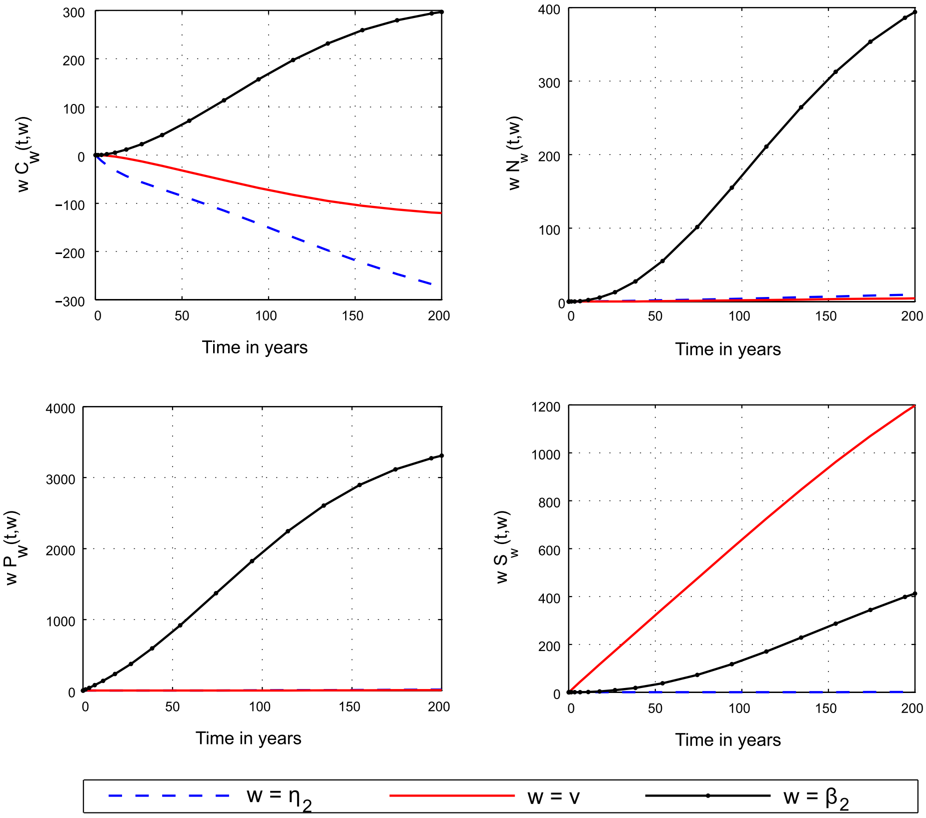

Figure 6. Semi-relative sensitivity solutions for the state variables with respect to parameters η_2, ν and β_2.

The semi-relative sensitivity solutions have been calculated with respect to each of the three parameters of interest β_2, η_2 and ν. The semi-relative sensitivity solution is obtained by multiplying the sensitivity solution with the parameter. These solutions provides information about the change in the state of a variable when a parameter value is doubled. These solutions are depicted in Fig. 6. The first plot of this figure clearly shows that on doubling the parameter η_2, the atmospheric level of methane is decreased by 150.2 ppb in a period of 100 years and 273.5 ppb in 200 years. On doubling the parameter ν, the atmospheric level of methane is decreased by 72.6 ppb in 100 years and 120.3 ppb in 200 years. The doubling of the parameter β_2 brings a rise of 169 ppb and 297 ppb in atmospheric concentration of methane over the periods of 100 years and 200 years, respectively. Thus, the atmospheric concentration of CH_4 is highly affected by the changes in values of parameters η_2, ν and β_2. It can be noted that doubling of the efficiency of mitigation options η_2 brings more reduction in atmospheric concentration of methane than that of the implementation rate of mitigation options ν. From the second and third plots of Fig. 6, it can be seen that the doubling of the growth rate coefficient of livestock population due to human efforts β_2 brings large increase in livestock and human populations; while the impact of doubling of the efficiency and the implementation rate of mitigation options on livestock and human populations is very small.

5 Conclusion

Enhanced level of atmospheric methane is one of the main culprits for global warming. Livestock farming is the largest source of methane, thus, it is crucial to mitigate methane emission from livestock sector. But, the livestock farming is increasing around the globe to meet the food demand of the growing population. In this scenario, it is important to investigate the effectiveness of mitigation options for methane emission from livestock over the reduction of atmospheric level of methane. For this purpose, we have proposed a nonlinear mathematical model. In the modeling process, it is assumed that human population makes efforts to increase the livestock population and the increase in livestock population facilitates the growth of human population. Further, it is assumed that the mitigation options are implemented to reduce the methane emission from livestock farming at a rate proportional to livestock population. The proposed nonlinear model is analyzed by using stability theory of differential equations. The model system exhibits four non-negative equilibria. The conditions for local and global stability of interior equilibrium have been derived. In the analysis of model, it is found that increase in the implementation rate of mitigation options ν and efficiency of mitigation options η_2 reduces the atmospheric level of methane under condition (20). Sensitivity analysis is performed with respect to the key parameters η_2, ν and β_2, which clearly demonstrates the impact of these parameters on the atmospheric level of methane and other model variables.

The findings of this paper suggest various strategies for the reduction of methane emission from livestock farming. Sensitivity analysis shows that η_2 is more influential parameter than ν towards the reduction of methane emission. This suggest that the implementation of high efficient options with low implementation rate would be a better policy than the implementation of low efficient options with high rate. For instance, use of propionate precursors is more effective option than use of probiotics to reduce methane emission from enteric fermentation in dairy cows [30]. The parameter β_2, which represents the growth in livestock population due to human efforts, is found to have large impact on the dynamics of the system. For very high values of β_2, condition (20) for reduction of atmospheric level of methane via mitigation options will not satisfy. Also, the parameter β_2 has destabilizing effect on the dynamics of the system. Thus, for attaining sustainable scenario, the livestock farming should not be promoted beyond a critical level. This critical level can be obtained with help of condition (20). Overall, the present paper provides a basic framework to access the impact of mitigation options for livestock methane emission on the reduction of atmospheric concentration of methane and addresses the constraints that exist in achieving the dual goal of reduction in atmospheric level of methane and increase in livestock production.

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