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Rational g-ω-weak contractions and fixed point theorems in 0-σ-complete metric-like spaces

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Abstract. We bring in the notion of rational g-ω-weak contractions in metric-like spaces and demonstrate common fixed point results for such mappings in 0-σ-complete metric-like spaces. Examples are given to support the usability of our results and to show that they are improvements of some known ones. An application to second-order differential equations is presented in the final section.

Keywords: common fixed point, metric-like space, partial metric space, 0-σ-complete space.

1 Introduction

Matthews [7] introduced the notion of partial metric space as a part of the study of denotational semantics of dataflow network. In such spaces, the usual metric is replaced by a partial metric with the property that the self distance for a point of space may not be zero. Further, Matthews showed that the Banach contraction principle is valid in partial metric spaces and can be applied in program verification. Later, several authors generalized the Matthews's result.

Romaguera [9] introduced the notion of 0-Cauchy sequence and 0-complete partial metric space, and proved some characterizations of partial metric spaces in terms of completeness and 0-completeness.

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Recently, Amini-Harandi [5] has introduced the notion of metric-like space, which is a new generalization of partial metric space. Amini-Harandi defined σ-completeness of metric-like spaces. Further, Shukla et al. have introduced in [10] the notion of 0-σ-complete metric-like space and proved some fixed point theorems in such spaces, as improvements of Amini-Harandi's results.

Alber and Guerre-Delabriere in [4] suggested a generalization of the Banach contraction mapping principle by introducing the concept of a weak contraction in Hilbert spaces. Rhoades [8] extended their result to complete metric spaces. Very recently, Abbas and Đorić [3], as well as Abbas and Ali Khan [2] have obtained common fixed points for four and two mappings, respectively, which satisfy generalized weak contractive conditions.

The purpose of this paper is to present some fixed point theorems involving weakly contractive mappings in the context of metric-like spaces. The presented theorems improve the results of papers [5] and [10]. We introduce the notion of rational g-ω-weak contractions in metric-like spaces and prove some fixed point results for such mappings in 0-σ-complete metric-like spaces. Examples are given to support the usability of our results and to show that the mentioned improvements are proper.

An application to the study of existence and uniqueness of solutions for a class of two-point boundary value problems for second-order differential equations is discussed by using the obtained fixed point results.

2 Preliminaries

A selfmap f of a metric space X is weakly contractive (or ψ-weakly contractive) if for all $x, y \in X$,

$$d(fx, fy) \leqslant d(x, y) - \psi\big(d(x, y)\big),$$

where $\psi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\psi(0) = 0$, $\psi(t) > 0$ for all $t \in (0, \infty)$, and $\lim_{t\to\infty} \psi(t) = \infty$. It is clear that weakly contractive maps are continuous and include contraction maps as a special case for the choice $\psi(t) = (1 - k)t$, $k \in [0, 1)$.

Let f and g be self maps on a set X. Recall [1] that if $w = fx = gx$ for some $x \in X$, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g. The pair $\{f, g\}$ of self maps is weakly compatible if they commute at their coincidence points. It is easy to show [1] that if weakly compatible self maps f and g on a set X have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

Definition 1. (See [5].) A metric-like on a nonempty set X is a mapping $\sigma : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$,

(σ1) $\sigma(x, y) = 0$ implies $x = y$,
(σ2) $\sigma(x, y) = \sigma(y, x)$,
(σ3) $\sigma(x, y) \leqslant \sigma(x, z) + \sigma(z, y)$.

The pair (X, σ) is called a metric-like space.

Definition 2. (See [5, 10].) Let (X, σ) be a metric-like space. A sequence $\{x_n\}$ is said to σ-converge to a point $x \in X$ if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$. A sequence $\{x_n\}$ in X is called a 0-σ-sequence if there exists a point $x \in X$ such that $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = 0$. A subset $A \subset X$ is called σ-closed if every convergent sequence in A has all of its limits in A. The subset A is called 0-closed if every 0-σ-convergent sequence in A has a limit in A.

Remark 1. Every σ-closed subset of X is necessarily 0-closed, but the converse is not necessarily true. For instance, let $X = \mathbb{R}^+$, $A = [0, 1) \subset X$ and the metric-like on X be defined by $\sigma(x, y) = \max\{x, y\}$ for all $x, y \in X$. Then A is not a σ-closed subset of X. Indeed, for any sequence $\{x_n\} \subset A$, we have $\lim_{n\to\infty} \sigma(x_n, 1) = \sigma(1, 1) = 1$, i.e., $x_n \to 1 \notin A$ as $n \to \infty$. Whereas, it is easy to see that A is 0-closed.

Definition 3. (See [10].) Let (X, σ) be a metric-like space. A sequence $\{x_n\}$ is called a 0-σ-Cauchy sequence if $\lim_{m,n\to\infty} \sigma(x_n, x_m) = 0$. The space (X, σ) is said to be 0-σ-complete if every 0-σ-Cauchy sequence in X σ-converges to a point $x \in X$ such that $\sigma(x, x) = 0$.

Lemma 1. (*See* [10].) *Let (X, σ) be a metric-like space, and let $\{x_n\}$ be a sequence in X such that $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$. If $\{x_n\}$ is not a 0-σ-Cauchy sequence in (X, σ), then there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$, and the following four sequences tend to ε when $k \to \infty$:*

$$\bigl\{\sigma(x_{m_k}, x_{n_k})\bigr\},\ \bigl\{\sigma(x_{m_k}, x_{n_k+1})\bigr\},\ \bigl\{\sigma(x_{m_k-1}, x_{n_k})\bigr\},\ \bigl\{\sigma(x_{m_k-1}, x_{n_k+1})\bigr\}.$$

Remark 2. Notice that if the condition of the above lemma is satisfied, then the sequences $\{\sigma(x_{m_k-1}, x_{n_k-1})\}$ and $\{\sigma(x_{m_k+1}, x_{n_k+1})\}$ also converge to ε when $k \to \infty$.

Proof. By (σ3) we have

$$\begin{aligned}
&\sigma(x_{m_k-1}, x_{n_k-1}) \leqslant \sigma(x_{m_k-1}, x_{m_k}) + \sigma(x_{m_k}, x_{n_k}) + \sigma(x_{n_k}, x_{n_k-1}),\\
&\sigma(x_{m_k}, x_{n_k}) \leqslant \sigma(x_{m_k}, x_{m_k-1}) + \sigma(x_{m_k-1}, x_{n_k-1}) + \sigma(x_{n_k-1}, x_{n_k}).
\end{aligned}$$

Passing to the limit when $k \to \infty$ and using Lemma 1 in the above inequalities, we obtain $\lim_{k\to\infty} \sigma(x_{m_k-1}, x_{n_k-1}) = \varepsilon$. Similarly, one can obtain that the sequence $\{\sigma(x_{m_k+1}, x_{n_k+1})\}$ also converges to ε when $k \to \infty$. $\square$

As an improvement of [5, Thm. 2.4], the following result was proved in [10].

Theorem 1. (*See* [10].) *Let (X, σ) be a 0-σ-complete metric-like space. Suppose that mappings $f, g : X \to X$ satisfy*

$$\sigma(fx, fy) \leqslant \psi\bigl(M(x, y)\bigr)$$

for all $x, y \in X$, where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $\psi(t) < t$ for all $t > 0$, $\lim_{s\to t^+} \psi(s) < t$ for all $t > 0$, $\lim_{t\to\infty}(t - \psi(t)) = \infty$ and

$$\begin{aligned}
M(x, y) = \max\bigl\{&\sigma(gx, gy),\ \sigma(gx, fx),\ \sigma(gy, fy),\ \sigma(gx, fy),\ \sigma(gy, fx),\\
&\sigma(gx, gx), \sigma(gy, gy)\bigr\}.
\end{aligned}$$

If the range of g contains the range of f and $f(X)$ or $g(X)$ is a 0-closed subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point v and $\sigma(v,v)=0$.

3 Main results

In further discussion, we denote by Ω, the class of all functions $\omega:[0,\infty)\to[0,\infty)$ such that ω is lower semi-continuous with $\omega(t)=0$ if and only if $t=0$.

Definition 4. Let (X,d) be a metric-like space and $f,g:X\to X$ be two mappings. The mapping f is called a rational g-ω-weak contraction if there exists $\omega\in\Omega$ such that the condition

$$\sigma(fx,fy)\leqslant R_{f,g}(x,y)-\omega\bigl(R_{f,g}(x,y)\bigr) \tag{1}$$

is satisfied for all $x,y\in X$, where

$$R_{f,g}(x,y)=\max\left\{\sigma(gx,gy),\ \sigma(gx,fx),\ \sigma(gy,fy),\ \frac{\sigma(gx,fx)\sigma(gy,fy)}{1+\sigma(gx,gy)}\right\}. \tag{2}$$

Lemma 2. *Let (X,σ) be a metric-like space and $f,g:X\to X$ be two mappings such that f is a rational g-ω-weak contraction. If f and g have a point of coincidence $v\in X$, then $\sigma(v,v)=0$.*

Proof. Let $v\in X$ be the point of coincidence of f and g and u be the corresponding coincidence point, that is, $gu=fu=v$. Notice that

$$\begin{aligned}R_{f,g}(u,u)&=\max\left\{\sigma(gu,gu),\ \sigma(gu,fu),\ \sigma(gu,fu),\ \frac{\sigma(gu,fu)\sigma(gu,fu)}{1+\sigma(gu,gu)}\right\}\\&=\max\left\{\sigma(v,v),\ \sigma(v,v),\ \sigma(v,v),\ \frac{\sigma(v,v)\sigma(v,v)}{1+\sigma(v,v)}\right\}=\sigma(v,v).\end{aligned}$$

Using (1), we obtain

$$\begin{aligned}\sigma(v,v)=\sigma(fu,fu)&\leqslant R_{f,g}(u,u)-\omega\bigl(R_{f,g}(u,u)\bigr)\\&=\sigma(v,v)-\omega\bigl(\sigma(v,v)\bigr).\end{aligned}$$

The above inequality shows that $\omega(\sigma(v,v))=0$, that is, $\sigma(v,v)=0$, which completes the proof. □

The next theorem gives a sufficient condition for the existence of a unique common fixed point of two mappings on a 0-σ-complete metric-like space.

Theorem 2. *Let (X,σ) be a 0-σ-complete metric-like space and $f,g:X\to X$ be two mappings such that f is a rational g-ω-weak contraction. If the range of g contains the range of f and $f(X)$ or $g(X)$ is a 0-closed subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point v and $\sigma(v,v)=0$.*

Proof. Let x_0 be an arbitrary point in X and choose a $x_1 \in X$ such that $fx_0 = y_0 = gx_1$ (say). This can be done, since the range of g contains the range of f. Similarly, choose $x_2 \in X$ such that $fx_1 = y_1 = gx_2$ (say). Continuing this process, having chosen $x_n \in X$, we obtain $x_{n+1} \in X$ such that $fx_n = y_n = gx_{n+1}$ (say). Thus, we obtain a Jungck sequence $\{y_n\}_{n\in\mathbb{N}} = \{gx_{n+1}\}_{n\in\mathbb{N}}$.

First, we show the existence of point of coincidence of f and g. If $y_{n-1} = y_n$ for some $n \in \mathbb{N}$, then $gx_n = fx_n = y_n$ is a point of coincidence. Therefore, in further calculations, we assume that $y_{n-1} \neq y_n$ for all $n \geqslant 1$. We shall show that $\{y_n\}$ is a 0-σ-Cauchy sequence in X.

Since $\sigma(gx_n, gx_{n+1}) = \sigma(y_{n-1}, y_n) > 0$, for every $n \in \mathbb{N}$, we have

$$\begin{aligned} R_{f,g}(x_n, x_{n+1}) &= \max\biggl\{\sigma(gx_n, gx_{n+1}),\ \sigma(gx_n, fx_n),\ \sigma(gx_{n+1}, fx_{n+1}), \\ &\qquad \frac{\sigma(gx_n, fx_n)\sigma(gx_{n+1}, fx_{n+1})}{1+\sigma(gx_n, gx_{n+1})}\biggr\} \\ &= \max\biggl\{\sigma(y_{n-1}, y_n),\ \sigma(y_{n-1}, y_n),\ \sigma(y_n, y_{n+1}), \\ &\qquad \frac{\sigma(y_{n-1}, y_n)\sigma(y_n, y_{n+1})}{1+\sigma(y_{n-1}, y_n)}\biggr\}. \end{aligned}$$

Therefore, using (1), we obtain

$$\begin{aligned} \sigma(y_n, y_{n+1}) &= \sigma(fx_n, fx_{n+1}) \leqslant R_{f,g}(x_n, x_{n+1}) - \omega\bigl(R_{f,g}(x_n, x_{n+1})\bigr) \\ &< R_{f,g}(x_n, x_{n+1}) \\ &= \max\biggl\{\sigma(y_{n-1}, y_n),\ \sigma(y_n, y_{n+1}),\ \frac{\sigma(y_{n-1}, y_n)\sigma(y_n, y_{n+1})}{1+\sigma(y_{n-1}, y_n)}\biggr\} \\ &\leqslant \max\bigl\{\sigma(y_{n-1}, y_n),\ \sigma(y_n, y_{n+1})\bigr\}. \end{aligned}$$

Therefore,

$$\sigma(y_n, y_{n+1}) < \max\bigl\{\sigma(y_{n-1}, y_n), \sigma(y_n, y_{n+1})\bigr\}.$$

If $\max\{\sigma(y_{n-1}, y_n), \sigma(y_n, y_{n+1})\} = \sigma(y_n, y_{n+1})$ for any $n \in \mathbb{N}$, then the above inequality yields a contradiction. Therefore, we must have $\sigma(y_n, y_{n+1}) < \sigma(y_{n-1}, y_n)$ for all $n \in \mathbb{N}$. Thus, the sequence $\{\sigma(y_n, y_{n+1})\}$ is a strictly decreasing sequence of positive numbers. Let $\lim_{n\to\infty} \sigma(y_n, y_{n+1}) = \delta \geqslant 0$. If $\delta > 0$, then we have

$$\begin{aligned} \sigma(y_n, y_{n+1}) &= \sigma(fx_n, fx_{n+1}) \leqslant R_{f,g}(x_n, x_{n+1}) - \omega\bigl(R_{f,g}(x_n, x_{n+1})\bigr) \\ &= \max\biggl\{\sigma(y_{n-1}, y_n),\ \sigma(y_{n-1}, y_n),\ \sigma(y_n, y_{n+1}),\ \frac{\sigma(y_{n-1}, y_n)\sigma(y_n, y_{n+1})}{1+\sigma(y_{n-1}, y_n)}\biggr\} \\ &\quad - \omega\biggl(\max\biggl\{\sigma(y_{n-1}, y_n),\ \sigma(y_{n-1}, y_n),\ \sigma(y_n, y_{n+1}), \\ &\quad \frac{\sigma(y_{n-1}, y_n)\sigma(y_n, y_{n+1})}{1+\sigma(y_{n-1}, y_n)}\biggr\}\biggr). \end{aligned}$$

Since $\omega \in \Omega$, taking the upper limit as $n \to \infty$, we obtain

$$\begin{aligned}
\delta &\leqslant \max\left\{\delta, \frac{\delta^2}{1+\delta}\right\} \\
&\quad - \liminf_{n\to\infty} \omega\left(\max\left\{\sigma(y_{n-1}, y_n),\ \sigma(y_n, y_{n+1}), \frac{\sigma(y_{n-1}, y_n)\sigma(y_n, y_{n+1})}{1+\sigma(y_{n-1}, y_n)}\right\}\right) \\
&\leqslant \delta - \omega(\delta) < \delta.
\end{aligned}$$

This contradiction shows that

$$\lim_{n\to\infty} \sigma(y_n, y_{n+1}) = \delta = 0. \tag{3}$$

We shall show that the sequence $\{y_n\}$ is a 0-σ-Cauchy sequence. Suppose to the contrary that $\{y_n\}$ is not a 0-σ-Cauchy sequence. Then by Lemma 1 there exist $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers such that $n_k > m_k > k$ and

$$\lim_{k\to\infty} \sigma(y_{m_k}, y_{n_k}) = \lim_{k\to\infty} \sigma(y_{m_k-1}, y_{n_k-1}) = \lim_{k\to\infty} \sigma(y_{m_k+1}, y_{n_k+1}) = \varepsilon. \tag{4}$$

For any $k \in \mathbb{N}$, by definition

$$\begin{aligned}
R_{f,g}(x_{m_k}, x_{n_k}) &= \max\biggl\{\sigma(gx_{m_k}, gx_{n_k}),\ \sigma(gx_{m_k}, fx_{m_k}),\ \sigma(gx_{n_k}, fx_{n_k}), \\
&\qquad \frac{\sigma(gx_{m_k}, fx_{m_k})\sigma(gx_{n_k}, fx_{n_k})}{1+\sigma(gx_{m_k}, gx_{n_k})}\biggr\} \\
&= \max\biggl\{\sigma(y_{m_k-1}, y_{n_k-1}),\ \sigma(y_{m_k-1}, y_{m_k}),\ \sigma(y_{n_k-1}, y_{n_k}), \\
&\qquad \frac{\sigma(y_{m_k-1}, y_{m_k})\sigma(y_{n_k-1}, y_{n_k})}{1+\sigma(y_{m_k-1}, y_{n_k-1})}\biggr\}.
\end{aligned}$$

Therefore, it follows from (1) that

$$\begin{aligned}
\sigma(y_{m_k}, y_{n_k}) &= \sigma(fx_{m_k}, fx_{n_k}) \\
&\leqslant R_{f,g}(x_{m_k}, x_{n_k}) - \omega\bigl(R_{f,g}(x_{m_k}, x_{n_k})\bigr) \\
&= \max\biggl\{\sigma(y_{m_k-1}, y_{n_k-1}),\ \sigma(y_{m_k-1}, y_{m_k}),\ \sigma(y_{n_k-1}, y_{n_k}), \\
&\qquad \frac{\sigma(y_{m_k-1}, y_{m_k})\sigma(y_{n_k-1}, y_{n_k})}{1+\sigma(y_{m_k-1}, y_{n_k-1})}\biggr\} \\
&\quad - \omega\biggl(\max\biggl\{\sigma(y_{m_k-1}, y_{n_k-1}),\ \sigma(y_{m_k-1}, y_{m_k}),\ \sigma(y_{n_k-1}, y_{n_k}), \\
&\qquad \frac{\sigma(y_{m_k-1}, y_{m_k})\sigma(y_{n_k-1}, y_{n_k})}{1+\sigma(y_{m_k-1}, y_{n_k-1})}\biggr\}\biggr).
\end{aligned}$$

Taking the upper limit as $k \to \infty$ in the above inequality and using (4), we obtain

$$\varepsilon \leqslant \max\{0, \varepsilon\} - \omega\left(\max\left\{0, \varepsilon, \frac{\varepsilon^2}{1+\varepsilon}\right\}\right) = \varepsilon - \omega(\varepsilon) < \varepsilon.$$

This contradiction shows that $\{y_n\} = \{gx_{n+1}\} = \{fx_n\}$ is a 0-σ-Cauchy sequence.

Suppose that $g(X)$ is 0-closed (the proof for the case when $f(X)$ is 0-closed is similar). Since X is 0-σ complete, there exists $v = gu \in X$ such that

$$\lim_{n,m\to\infty} \sigma(y_n, y_m) = \lim_{n\to\infty} \sigma(y_n, v) = \sigma(v, v) = 0. \tag{5}$$

We shall show that u is a coincidence point of f and g.

Suppose that $\sigma(v, fu) > 0$. By definition we have

$$\begin{aligned} R_{f,g}(x_n, u) &= \max\left\{\sigma(gx_n, gu),\, \sigma(gx_n, fx_n),\, \sigma(gu, fu),\, \frac{\sigma(gx_n, fx_n)\sigma(gu, fu)}{1+\sigma(gx_n, gu)}\right\} \\ &= \max\left\{\sigma(y_{n-1}, v),\, \sigma(y_{n-1}, y_n),\, \sigma(v, fu),\, \frac{\sigma(y_{n-1}, y_n)\sigma(v, fu)}{1+\sigma(y_{n-1}, v)}\right\}. \end{aligned}$$

In view of (5), there exists $n_0 \in \mathbb{N}$ such that

$$R_{f,g}(x_n, u) = \sigma(v, fu) \quad \text{for all } n > n_0.$$

Therefore, by (1) we have for all $n > n_0$

$$\begin{aligned} \sigma(v, fu) &\leqslant \sigma(v, y_n) + \sigma(y_n, fu) \\ &= \sigma(v, y_n) + \sigma(fx_n, fu) \\ &\leqslant \sigma(v, y_n) + R_{f,g}(x_n, u) - \omega\big(R_{f,g}(x_n, u)\big) \\ &= \sigma(v, y_n) + \sigma(v, fu) - \omega\big(\sigma(v, fu)\big). \end{aligned}$$

Letting $n \to \infty$, we get

$$\sigma(v, fu) \leqslant \sigma(v, fu) - \omega\big(\sigma(v, fu)\big) < \sigma(v, fu).$$

This contradiction shows that $\sigma(v, fu) = 0$, i.e., $fu = v = gu$. Thus, u is a coincidence point and v is the corresponding point of coincidence of f and g. We shall show that the point of coincidence is unique. If possible, let v' is another point of coincidence of f and g and $\sigma(v, v') > 0$. Then there exists $u' \in X$ such that $fu' = gu' = v'$. By Lemma 2 we have $\sigma(v', v') = 0$. Then by definition

$$\begin{aligned} R_{f,g}(u, u') &= \max\left\{\sigma(gu, gu'),\, \sigma(gu, fu),\, \sigma(gu', fu'),\, \frac{\sigma(gu, fu)\sigma(gu', fu')}{1+\sigma(gu, gu')}\right\} \\ &= \max\left\{\sigma(v, v'),\, \sigma(v, v),\, \sigma(v', v'),\, \frac{\sigma(v, v)\sigma(v', v')}{1+\sigma(v, v')}\right\} = \sigma(v, v'). \end{aligned}$$

Therefore, it follows from (1) that

$$\begin{aligned}\sigma(v,v') &= \sigma(fu,fu') \leqslant R_{f,g}(u,u') - \omega\big(R_{f,g}(u,u')\big)\\ &= \sigma(v,v') - \omega\big(\sigma(v,v')\big) < \sigma(v,v').\end{aligned}$$

This contradiction shows that $\sigma(v,v') = 0$, i.e., $v = v'$. Thus, v is the unique point of coincidence of f and g and v is the unique common fixed point of f and g. □

Let (X,d) be a metric-like space and $f : X \to X$ be a mapping. The mapping f will be called a rational ω-weak contraction if there exists $\omega \in \Omega$ such that the condition

$$\sigma(fx,fy) \leqslant R_f(x,y) - \omega\big(R_f(x,y)\big) \tag{6}$$

is satisfied for all $x,y \in X$, where

$$R_f(x,y) = \max\left\{\sigma(x,y),\, \sigma(x,fx),\, \sigma(y,fy), \frac{\sigma(x,fx)\sigma(y,fy)}{1+\sigma(x,y)}\right\}.$$

Taking $g = I_X$ (the identity mapping on X) in Theorem 2, we obtain the following corollary.

Corollary 1. *Let (X,σ) be a 0-σ-complete metric-like space and $f : X \to X$ be a rational ω-weak contraction. Then f has a unique fixed point v and $\sigma(v,v) = 0$.*

If we take $\omega(t) = (1-k)t$ for $k \in (0,1)$ in contraction condition (1), we have the following corollary.

Corollary 2. *Let (X,σ) be a 0-σ-complete metric-like space and $f,g : X \to X$ be two mappings such that*

$$\sigma(fx,fy) \leqslant kR_{f,g}(x,y)$$

holds for all $x,y \in X$, where $0 < k < 1$ and $R_{f,g}(x,y)$ is defined by (2)*. Then f and g have a unique point of coincidence in X, and if they are weakly compatible, they have a unique common fixed point.*

Now, we present an example to support the usability of our results (more precisely, of Corollary 1).

Example 1. Let $X = \{a,b,c\}$. Define $\sigma : X \times X \to R^+$ as follows:

$$\sigma(a,a) = 0, \qquad \sigma(b,b) = 3, \qquad \sigma(c,c) = 5, \quad \sigma(a,b) = \sigma(b,a) = 9,$$
$$\sigma(a,c) = \sigma(c,a) = 4, \qquad \sigma(b,c) = \sigma(c,b) = 5.$$

Then (X,σ) is a 0-σ-complete metric-like space, which is neither a metric space (since, e.g., $\sigma(b,b) > 0$) nor a partial metric space (since, e.g., $\sigma(a,b) = 9 \nleqslant 4 = \sigma(a,c) + \sigma(c,b) - \sigma(c,c)$). Let $f,g : X \to X$ be defined by

$$fa = a, \qquad fb = c, \qquad fc = a$$

and

$$ga = a, \quad gb = b, \quad gc = c.$$

We next verify that the pair $\{f, g\}$ satisfies the inequality (1) (i.e., inequality (6)) with $\omega(t) = t/5$. Let us consider the following possible cases:

1. If $\{x, y\} \subseteq \{a, c\}$, then $\sigma(fx, fy) = \sigma(a, a) = 0$ and (6) trivially holds.
2. If $x = b$, $y = a$, then $\sigma(fx, fy) = \sigma(c, a) = 4$ and
$$R_{f,g}(x, y) = \max\left\{\sigma(b, a),\, \sigma(b, c),\, \sigma(a, a),\, \frac{\sigma(b, c)\sigma(a, a)}{1 + \sigma(b, a)}\right\} = 9$$
and $4 \leqslant 9 - (1/5) \cdot 9$ holds.
3. If $x = b$, $y = b$, then $\sigma(fx, fy) = \sigma(c, c) = 5$ and
$$R_{f,g}(x, y) = \max\left\{\sigma(b, b),\, \sigma(b, c),\, \sigma(b, c),\, \frac{\sigma(b, c)\sigma(b, c)}{1 + \sigma(b, b)}\right\} = \frac{25}{4}$$
and $5 \leqslant (25/4) - (1/5) \cdot (25/4)$ holds.
4. If $x = b$, $y = c$, then $\sigma(fx, fy) = \sigma(c, a) = 4$ and
$$R_{f,g}(x, y) = \max\left\{\sigma(b, c),\, \sigma(b, c),\, \sigma(c, a),\, \frac{\sigma(b, c)\sigma(c, a)}{1 + \sigma(b, c)}\right\} = 5$$
and $4 \leqslant 5 - (1/5) \cdot 5$ holds.

Thus, all the required hypotheses of condition (6) are satisfied. Then f and g have a unique point of coincidence in X and a unique common fixed point. Here a is the unique fixed point of f, g.

Note that Theorem 2.4 of [5], as well as Theorem 1 of [10] (i.e., Theorem 1), cannot be used to reach this conclusion since $\sigma(fb, fb) = 5 = M(b, b)$ and no function ψ can be chosen such that $\sigma(fb, fb) \leqslant \psi(M(b, b))$.

Without essential changes, the following version of Theorem 2 can be proved.

Theorem 3. *Let (X, σ) be a 0-σ-complete metric-like space and $f, g : X \to X$ be two mappings such that*

$$\psi\bigl(\sigma(fx, fy)\bigr) \leqslant \psi\bigl(R_{f,g}(x, y)\bigr) - \omega\bigl(R_{f,g}(x, y)\bigr)$$

for all $x, y \in X$, where $\omega \in \Omega$, $\psi : [0, \infty) \to [0, \infty)$ is continuous, nondecreasing and $\psi^{-1}(0) = \{0\}$ and $R_{f,g}(x, y)$ is given by (2). *If the range of g contains the range of f and $f(X)$ or $g(X)$ is a 0-closed subset of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point v and $\sigma(v, v) = 0$.*

The following example can be used to illustrate the usage of previous theorem.

Example 2. Let $X = [0,1] \cap \mathbb{Q}$ and $\sigma : X \times X \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2x & \text{if } x = y, \\ \max\{x,y\} & \text{otherwise} \end{cases}$$

for all $x, y \in X$. Then (X, σ) is a 0-σ-complete metric-like space, which is not σ-complete [10, Ex. 5]. Let $f, g : X \to X$ be mappings given by $fx = x/3$ and $gx = x/2$. Moreover, let $\psi(t) = 4t/3$ and $\omega(t) = t/3$. Then, for $x > y$,

$$\psi\bigl(\sigma(fx,fy)\bigr) = \frac{4}{3}\sigma\left(\frac{x}{3}, \frac{y}{3}\right) = \frac{4x}{9}$$

and

$$\begin{aligned} R_{f,g}(x,y) &= \max\left\{\sigma\left(\frac{x}{2}, \frac{y}{2}\right), \sigma\left(\frac{x}{2}, \frac{x}{3}\right), \sigma\left(\frac{y}{2}, \frac{y}{3}\right), \frac{\sigma(\frac{x}{2}, \frac{x}{3})\sigma(\frac{y}{2}, \frac{y}{3})}{1 + \sigma(\frac{x}{2}, \frac{y}{2})}\right\} \\ &= \max\left\{\frac{x}{2}, \frac{x}{2}, \frac{y}{2}, \frac{\frac{xy}{4}}{1 + \frac{x}{2}}\right\} = \frac{x}{2}. \end{aligned}$$

Hence,

$$\begin{aligned} \psi\bigl(\sigma(fx,fy)\bigr) = \frac{4x}{9} \leqslant \frac{x}{2} &= \frac{4}{3} \cdot \frac{x}{2} - \frac{1}{3} \cdot \frac{x}{2} \\ &= \psi\bigl(R_{f,g}(x,y)\bigr) - \omega\bigl(R_{f,g}(x,y)\bigr). \end{aligned}$$

Similarly, for $x = y$, one gets that

$$\psi\bigl(\sigma(fx,fy)\bigr) = \frac{8x}{9} \leqslant x = \psi\bigl(R_{f,g}(x,y)\bigr) - \omega\bigl(R_{f,g}(x,y)\bigr).$$

Thus, all the conditions of Theorem 3 are satisfied. Then f and g have a unique point of coincidence in X and a unique common fixed point (which is 0).

4 An application to second-order differential equations

In this section, we are going to apply Corollary 1 to the study of existence and uniqueness of solutions for a type of second-order differential equations. Our approach is inspired by Section 5 of [6].

Denote $I = [0,1]$ and consider the following boundary value problem for second-order differential equation:

$$\begin{aligned} x''(t) &= -F\bigl(t, x(t)\bigr), \quad t \in I, \\ x(0) &= x(1) = 0, \end{aligned} \tag{7}$$

where $F \in C(I \times \mathbb{R}, \mathbb{R})$.

It is known and easy to check that problem (7) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s)F\big(s, x(s)\big)\,\mathrm{d}s \quad \text{for } t \in I, \tag{8}$$

where G is the Green function defined by

$$G(t,s) = \begin{cases} (1-t)s & \text{if } 0 \leqslant s \leqslant t \leqslant 1, \\ (1-s)t & \text{if } 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

That is, if $x \in C^2(I, \mathbb{R})$, then x is a solution of problem (7) if and only if it is a solution of the integral equation (8).

Let $X = C(I)$ be the space of all continuous functions defined on I and $\|u\|_\infty = \max_{t\in I} |u(t)|$ for each $u \in X$. Consider the metric-like σ on X given by

$$\sigma(x,y) = \|x-y\|_\infty + \|x\|_\infty + \|y\|_\infty \quad \text{for all } x, y \in X.$$

Note that σ is also a partial metric on X and that

$$d_\sigma(x,y) := 2\sigma(x,y) - \sigma(x,x) - \sigma(y,y) = 2\|x-y\|_\infty.$$

Hence, (X, σ) is complete as the metric space $(X, \|\cdot\|_\infty)$ is complete.

Theorem 4. *Assume the following conditions*:

(i) *there exist continuous functions* $\alpha : I \to \mathbb{R}_0^+$ *and* $\beta : I \to \mathbb{R}_0^+$ *such that*

$$\big|F(s,a) - F(s,b)\big| \leqslant 8\alpha(s)|a-b| \quad \textit{for } s \in I \textit{ and } a, b \in \mathbb{R},$$
$$\big|F(s,a)\big| \leqslant 8\beta(s)|a| \quad \textit{for } s \in I \textit{ and } a \in \mathbb{R};$$

(ii) $\max_{s\in I} \alpha(s) = \lambda_1 < 1/3$ *and* $\max_{s\in I} \beta(s) = \lambda_2 < 1/3$.

Then problem (7) *has a unique solution* $u \in X = C(I, \mathbb{R})$.

Proof. Define the self-map $f : X \to X$ by

$$fx(t) = \int_0^1 G(t,s)F\big(s, x(s)\big)\,\mathrm{d}s$$

for all $x \in X$ and $t \in I$. Then problem (7) is equivalent to finding a fixed point u of f in X. Let $x, y \in X$. We have

$$\begin{aligned} \big|fx(t) - fy(t)\big| &= \left| \int_0^1 G(t,s)F\big(s, x(s)\big)\,\mathrm{d}s - \int_0^1 G(t,s)F\big(s, y(s)\big)\,\mathrm{d}s \right| \\ &\leqslant \int_0^1 G(t,s)\big|F\big(s, x(s)\big) - F\big(s, y(s)\big)\big|\,\mathrm{d}s \end{aligned}$$

$$\leqslant 8\int_0^1 G(t,s)\alpha(s)\bigl|x(s)-y(s)\bigr|\,\mathrm{d}s$$

$$\leqslant 8\lambda_1\|x-y\|_\infty \sup_{t\in I}\int_0^1 G(t,s)\,\mathrm{d}s = \lambda_1\|x-y\|_\infty.$$

Next, we recall that for each $t \in I$, one has $\int_0^1 G(t,s)\,\mathrm{d}s = t(1-t)/2$, and then

$$\sup_{t\in I}\int_0^1 G(t,s)\,\mathrm{d}s = \frac{1}{8}.$$

Therefore,

$$\|fx - fy\|_\infty \leqslant \lambda_1\|x-y\|_\infty. \tag{9}$$

Moreover, we have

$$\begin{aligned}
\bigl|fx(t)\bigr| &= \left|\int_0^1 G(t,s)F\bigl(s,x(s)\bigr)\,\mathrm{d}s\right| \leqslant \int_0^1 G(t,s)\bigl|F\bigl(s,x(s)\bigr)\bigr|\,\mathrm{d}s\\
&\leqslant 8\int_0^1 G(t,s)\beta(s)\bigl|x(s)\bigr|\,\mathrm{d}s \leqslant 8\lambda_2\|x\|_\infty \sup_{t\in I}\int_0^1 G(t,s)\,\mathrm{d}s\\
&\leqslant \lambda_2\|x\|_\infty.
\end{aligned}$$

Thus

$$\|fx\|_\infty \leqslant \lambda_2\|x\|_\infty, \tag{10}$$

and also

$$\|fy\|_\infty \leqslant \lambda_2\|y\|_\infty. \tag{11}$$

Put now $\lambda = \lambda_1 + 2\lambda_2 < 1$. Summing up (9)–(11), we obtain

$$\begin{aligned}
\sigma(fx,fy) &= \|fx-fy\|_\infty + \|fx\|_\infty + \|fy\|_\infty\\
&\leqslant \lambda_1\|x-y\|_\infty + \lambda_2\|x\|_\infty + \lambda_2\|y\|_\infty\\
&\leqslant (\lambda_1+2\lambda_2)\bigl(\|x-y\|_\infty + \|x\|_\infty + \|y\|_\infty\bigr)\\
&= \lambda\sigma(x,y) \leqslant \lambda R_f(x,y).
\end{aligned}$$

Now, by considering the control function $\omega : [0,\infty) \to [0,\infty)$ defined by $\omega(t) = (1-\lambda)t$ for $\lambda \in (0,1)$, we get

$$\sigma(fx,fy) \leqslant R_f(x,y) - \omega\bigl(R_f(x,y)\bigr).$$

Therefore, all hypotheses of Corollary 1 are satisfied, and so f has a unique fixed point $u \in X$, that is, problem (7) has a unique solution $u \in C^2(I)$. □

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