Best proximity points of p-cyclic orbital Meir–Keeler contraction maps

Saravanan Karpagam[a], **Boyan Zlatanov**[b,1]

[a]Department of Mathematics, VelTech University, Avadi, Chennai, India
karpagam.saravanan@gmail.com

[b]Faculty of Mathematics and Informatics, Plovdiv University "Paisii Hilendarski",
24 Tzar Assen str., Plovdiv 4000, Bulgaria
bzlatanov@gmail.com

Abstract. Let (X, d) be a metric space, and $A_1, A_2, \ldots, A_p$ be nonempty subsets of X. We introduce a self map T on X, called p-cyclic orbital contraction map on the union of $A_1, A_2, \ldots, A_p$, and obtain a unique best proximity point of T, that is, a point $x \in \cup_{i=1}^{p} A_i$ such that $d(x, Tx) = \mathrm{dist}(A_i, A_{i+1})$, $1 \leqslant i \leqslant p$, where $\mathrm{dist}(A_i, A_{i+1}) = \inf\{d(x, y): x \in A_i, y \in A_{i+1}\}$.

Keywords: uniformly convex Banach space, best proximity points, p-cyclic maps, orbital contractions.

1 Introduction

The importance of Mathematics lies in solving equations of the form $f(x) = 0$. This equation can also be written as $f(x) = g(x) - x$ for some suitable function g. Finding the solution of the equation $f(x) = 0$ is equivalent to finding the solution of the equation $g(x) = x$. Theorems, which provide a theory by enhancing the possibilities for the existence of a solution to the given equation $g(x) = x$, are called fixed point theorems. One such theorem is the famous Banach contraction theorem. It stats that "if (X, d) is a complete metric space and T is a self map on X such that there exists a k, $0 < k < 1$, such that $d(Tx, Ty) \leqslant kd(x, y)$ for all $x, y \in X$, then, for any $\xi \in X$, $\{T^n\xi\}$ converges to a unique fixed point.

One of the interesting extensions of the classical Banach contraction theorem is Meir–Keeler contraction introduced by Meir and Keeler in [18].

Later, the authors of [16] introduced a class of mappings called cyclic contractive mappings. If (X, d) is a metric space and $A_1, A_2, \ldots, A_p$ $(p > 2)$ are the nonempty

---

subsets of X, then a cyclic contraction mapping is defined on the union of $A_1, A_2, \ldots, A_p$ with some contraction type of condition imposed on this map. Some fixed point results were obtained for this map in [16]. In [5], the authors introduced a notion of best proximity points as an extension of fixed points in the following manner.

Let A and B be non empty subsets of a metric space, and $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. The map T is called a cyclic map. A point $x \in A \cup B$ is said to be a best proximity point if $d(x, Tx) = \text{dist}(A, B)$, where $\text{dist}(A, B) = \inf\{d(x, y) = x \in A, y \in B\}$. Hence, best proximity point theorems are direct extensions of fixed point theorems.

In [15], a notion of cyclic orbital contraction map is introduced and defined as follows:

Definition 1. Let A and B be non empty subsets of a metric space, and $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. If for some $x \in A$, there exists a $k_x \in (0, 1)$ such that
$$d\big(T^{2n}x, Ty\big) \leqslant k_x d\big(T^{2n-1}x, y\big), \quad n \in \mathbb{N}, \ y \in A,$$
then T is called a cyclic orbital contraction map.

Also, in [15], the following best proximity theorem is obtained in which the map is of Meir–Keeler type and the underlying space need to be a uniformly convex Banach space.

Theorem 1. *Let X be a uniformly convex Banach space. Let A and B be non empty, closed and convex subsets of X. Suppose that $T : A \cup B \to A \cup B$ satisfy the following conditions*:

(i) *$T(A) \subseteq B$ and $T(B) \subseteq A$;*
(ii) *For every $\epsilon > 0$, there exists a $\delta > 0$ such that for some $x \in A$,*

$$\big\|T^{2n-1}x - y\big\| < \text{dist}(A, B) + \epsilon + \delta$$
$$\implies \big\|T^{2n}x - Ty\big\| < \text{dist}(A, B) + \epsilon, \quad n \in \mathbb{N}, \ y \in A. \tag{1}$$

Then there exists a best proximity point $z \in A$ such that for every $x \in A$ satisfying condition (1)*, the sequence $\{T^{2n}x\}$ converges to $z \in A$.*

For more on best proximity point theorems, one may refer to [1, 2, 4, 9, 10, 11, 12, 19, 22, 24].

So far, the authors generalized best proximity points of cyclic orbital contractions, which are defined on the union of two sets only. But in [24], the author considered p-cyclic map with Meir–Keeler orbital type in different direction.

In this article, the map which we consider is a p-cyclic map (Definition 2) on which a Meir–Keeler type of contraction is imposed. That is, a notion of p-cyclic orbital Meir–Keeler contraction is introduced. Sufficient conditions are given for the existence of a best proximity point of this map, which is also a unique periodic point of the map in a given set. The main result of this article generalizes the main results of the theorems given in the literature.

In Theorem 1, there is no question of the distance between the sets. But in this article, we consider a p-cyclic map in which the distance between the adjacent sets play an

important role in obtaining a best proximity point. The condition, under which a best proximity point is obtained for cyclic maps, need not be the same for obtaining best proximity point for p-cyclic sets, $p \geqslant 2$. Therefore, the main result of this article is not a direct generalization of Theorem 1.

2 Preliminaries

The following notion of p-cyclic maps was first introduced by Kirk et al. in [16].

Definition 2. Let $A_1, A_2, \ldots, A_p$ $(p \geqslant 2)$ be non empty subsets of a metric space X. Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$, and if $T(A_i) \subseteq A_{i+1}$ for all $i = 1, 2, \ldots, p$ $(A_{p+i} = A_i)$, then T is said to be p-cyclic map.

Eldred and Veeramani in [5] introduced the concept of best proximity point for a cyclic map defined on union of two sets, which is an approximation of fixed point defined as follows:

Definition 3. Let (X, d) be a metric space. Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic map. A point $x \in A_i$ is said to be a best proximity point of T in A_i if $d(x, Tx) = \mathrm{dist}(A_i, A_{i+1})$, where $\mathrm{dist}(A_i, A_{i+1}) = \inf\{d(x, y) \colon x \in A_i,\ y \in A_{i+1}\}$.

Remark 1. Let (X, d) be a metric space. Let A_i, $i = 1, \ldots, p$, be subsets of X. Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic map. A best proximity point $x \in \cup_{i=1}^p A_i$ is a fixed point of T if and only if $\mathrm{dist}(A_i, A_{i+1}) = 0$. A fixed point of a p-cyclic map, if it exists, it exists only in the intersection $\cap_{i=1}^p A_i$.

Lim in [17] introduced the following notion of L-function, which is an useful tool to study the Meir–Keeler contraction maps.

Definition 4. (See [17].) A function $\phi : [0, \infty) \to [0, \infty)$ is called an L-function if $\phi(0) = 0$, $\phi(s) > 0$ for $s \in (0, \infty)$, and for every $s \in (0, \infty)$, there exists a $\delta > 0$ such that $\phi(t) \leqslant s$ for every $t \in [s, s + \delta]$.

Also, Lim also gave a set of equivalent conditions for L-functions [17]. Suzuki generalize Lim's results [23]. We will need the following result for the proof of the main result.

Lemma 1. (*See* [23].) *Let Y be a non empty set and let $f, g : Y \to [0, \infty)$. Then the following are equivalent*:

(i) *For each $\epsilon > 0$, there exists a $\delta > 0$ such that $f(x) < \epsilon + \delta \Rightarrow g(x) < \epsilon$.*

(ii) *There exists an L-function ϕ (which may be chosen non decreasing and continuous) such that $f(x) > 0 \Rightarrow g(x) < \phi(f(x))$, $x \in Y$ and $f(x) = 0 \Rightarrow g(x) = 0$, $x \in Y$.*

Eldred and Veeramani in [5] proved the following lemma, which is an important property of a uniformly convex Banach space. It is used to prove the main results.

Lemma 2. (*See* [5].) *Let X be a uniformly convex Banach space. Let A and B be non empty and closed subsets of X. Let A be convex. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A, and $\{y_n\}$ be a sequence in B such that $\lim_n \|x_n - y_n\| = \operatorname{dist}(A, B)$ and $\lim_n \|z_n - y_n\| = \operatorname{dist}(A, B)$. Then $\lim_n \|x_n - z_n\| = 0$.*

Definition 5. (See [3, p. 42].) We say that the Banach space $(X, \|\cdot\|)$ is strictly convex if $x = y$ whenever $x, y \in X$ are such that $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$.

The next theorem is stated for Banach spaces in [24], but it holds true for any normed space. We omit the proof because it is similar to that done in [24]. The theorems, which give characterization for the strict convexity in Banach spaces [3] and [7], holds true for normed spaces too [8].

Lemma 3. (*See* [24].) *Let A, B be closed subsets of a strictly convex normed space $(X, \|\cdot\|)$ such that $\operatorname{dist}(A, B) > 0$, and let A be convex. If $x, z \in A$ and $y \in B$ be such that $\|x - y\| = \|z - y\| = \operatorname{dist}(A, B)$, then $x = z$.*

An excellent overview of the development of the geometry of Banach spaces may be found in [3]. Basic concepts about geometry of Banach spaces can be found in two other excellent books [6] and [7].

3 Main results

We introduce a notion of p-cyclic orbital non expansive map, which is defined as follows:

Definition 6. Let (X, d) be a metric space. Let A_i, $i = 1, \ldots, p$, be non empty subsets of X. Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic map such that for some $x \in A_i$ $(1 \leqslant i \leqslant p)$ and for each $k = 1, 2, \ldots, p$, the following condition is satisfied:

$$d\big(T^{pn+k}x, T^{k+1}y\big) \leqslant d\big(T^{pn+k-1}x, T^k y\big), \quad n \in \mathbb{N}, \ y \in A_i. \tag{2}$$

Then T is called a p-cyclic orbital non expansive map.

The conditions, for which $\operatorname{dist}(A_{i-1}, A_i) = \operatorname{dist}(A_{i-1}, A_{i-2}) = \operatorname{dist}(A_1, A_2)$, is given in the following proposition.

Proposition 1. *Let X be strictly convex normed linear space. Let A_i, $i = 1, \ldots, p$, be non empty subsets of X. Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic orbital non expansive map with an $x \in A_i$ satisfying (2). Suppose that for each $k = 0, 1, 2 \ldots, (p-1)$ and $y \in A_i$, $\lim_n d(T^{pn+k-1}x, T^{pn+k}y) = \operatorname{dist}(A_{i+k-1}, A_{i+k})$ and $\{T^{pn+k}x\}$ converges to $z_k \in A_{i+k}$. Then:*

    (a) $\operatorname{dist}(A_1, A_2) = \operatorname{dist}(A_2, A_3) = \cdots = \operatorname{dist}(A_{p-1}, A_p) = \operatorname{dist}(A_p, A_1)$;

    (b) z_k *is a best proximity point of T in A_{i+k} and $z_k = T^k z_0$ for $i = 1, 2, \ldots, p$;*

    (c) z_k *is the unique periodic point of T with period p in A_{i+k}.*

Proof. (a) For any $k \in \{1, 2, \ldots, p\}$, there hold the inequalities

$$\operatorname{dist}(A_{i+k}, A_{i+k+1}) \leqslant d(z_k, Tz_k) = \lim_n d\big(T^{pn+k}x, Tz_k\big) \leqslant \lim_n d\big(T^{pn+k-1}x, z_k\big)$$
$$= \lim_n d\big(T^{pn+k-1}x, T^{pn+k}x\big) = \operatorname{dist}(A_{i+k-1}, A_{i+k}).$$

Consequently, we get the chain of inequalities

$$\operatorname{dist}(A_{i+1}, A_{i+p}) \leqslant \operatorname{dist}(A_{i+p}, A_{i+p-1}) \leqslant \cdots \leqslant \operatorname{dist}(A_{i+k}, A_{i+k-1})$$
$$\leqslant \cdots \leqslant \operatorname{dist}(A_{i+2}, A_{i+1}) \leqslant \operatorname{dist}(A_{i+1}, A_i) = \operatorname{dist}(A_{i+1}, A_{i+p}).$$

Thus, (a) holds true.

(b) For each $k = 0, 1, 2, \ldots, (p-1)$, consider

$$\operatorname{dist}(A_{i+k}, A_{i+k+1}) \leqslant d(z_k, Tz_k) = \lim_n d\big(T^{pn+k}x, Tz_k\big) \leqslant \lim_n d\big(T^{pn+k-1}x, z_k\big)$$
$$= \lim_n d\big(T^{pn+k-1}x, T^{pn+k}x\big) = \operatorname{dist}(A_{i+k-1}, A_{i+k})$$
$$= \operatorname{dist}(A_{i+k}, A_{i+k+1}).$$

Hence, $d(z_k, Tz_k) = \operatorname{dist}(A_{i+k}, A_{i+k+1})$.

Consider

$$\big\| z_1 - T^2 z_0 \big\| = \lim_n \big\| T^{pn+1}x - T^2 z_0 \big\| \leqslant \lim_n \big\| T^{pn}x - Tz_0 \big\|$$
$$\leqslant \lim_n \big\| T^{pn-1}x - z_0 \big\| = \lim \big\| T^{pn-1}x - T^{pn}x \big\|$$
$$= \operatorname{dist}(A_{i-1}, A_i) = \operatorname{dist}(A_{i+1}, A_{i+2}).$$

It is obvious that $\| Tz_0 - T^2 z_0 \| = \operatorname{dist}(A_{i+1}, A_{i+2})$, and since the underlying space is strictly convex, it follows from Lemma 3 that $z_1 = Tz_0$. Hence, $z_1 = Tz_0$. Similarly, we can prove that if $x \in A_1$ and $T^{pn}x \to z_0$, then $T^k z_0 = z_k$ for $k = 1, 2, \ldots, p$.

(c) To prove that each z_k is a periodic point in A_{i+k}, consider,

$$\big\| T^p z_0 - Tz_0 \big\| = \lim_n \big\| T^p z_0 - T^{pn+1}x \big\| \leqslant \lim_n \big\| z_0 - T^{p(n-1)+1}x \big\|$$
$$= \lim_n \big\| T^{pn}x - T^{p(n-1)+1}x \big\| = \operatorname{dist}(A_i, A_{i+1}).$$

Hence, $\| T^p z_0 - Tz_0 \| = \operatorname{dist}(A_i, A_{i+1})$. Since $\| z_0 - Tz_0 \| = \operatorname{dist}(A_i, A_{i+1})$, it follows that $T^p z_0 = z_0$. Since $T^k z_0 = z_k$, each $z_k = T^{p+k} z_0 = T^p z_k$.

To prove the uniqueness of z_0 as the periodic point, suppose that $\xi \in A_i$ is a periodic point of T in A_i. Then $T^{pn}\xi = \xi$ for all $n \in \mathbb{N}$. Since $z_1 = Tz_0$ and $\lim_n T^{pn+1}x = z_1 = Tz_0$, we have

$$\| \xi - Tz_0 \| = \lim_n \big\| T^{pn}\xi - T^{pn+1}x \big\| = \operatorname{dist}(A_i, A_{i+1}).$$

Since $\| z_0 - Tz_0 \| = \operatorname{dist}(A_i, A_{i+1})$ and from the above, $\| \xi - Tz_0 \| = \operatorname{dist}(A_i, A_{i+1})$, and since X is strictly convex, it follows that $\xi = z_0$. Hence, z_0 is unique. From $z_k = T^k z_0$ the uniqueness of each z_k as a periodic point follows. $\square$

We would like to point out that all known results about cyclic maps, where the conditions are symmetric as like as (2); see, for example, [13,15,20] etc. The distances between the consecutive sets appear to be equal. It is in contrast to the conditions investigated in [21] and [24], where the conditions, which ensure existence and uniqueness of the best proximity points, hold true for sets with different distances between them.

Now we introduce a notion of p-cyclic orbital Meir–Keeler contraction map.

Definition 7. Let (X, d) be a metric space. Let A_i, $i = 1, \ldots, p$, be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map. Then T is called a p-cyclic orbital Meir–Keeler contraction map if for some $x \in A_i$ $(1 \leqslant i \leqslant p)$ and for each $k = 0, 1, 2, \ldots, (p - 1)$, the following holds: for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$d\big(T^{pn+k-1}x, T^k y\big) < \operatorname{dist}(A_{i+k-1}, A_{i+k}) + \epsilon + \delta$$
$$\implies \quad d\big(T^{pn+k}x, T^{k+1}y\big) < \operatorname{dist}(A_{i+k}, A_{i+k+1}) + \epsilon, \quad n \in \mathbb{N}, \ y \in A_i. \quad (3)$$

In the above definition, if we omit the distances between the sets in condition (3), then we get the following condition (4). In this case, a unique fixed point is obtained.

Theorem 2. *Let (X, d) be a complete metric space. Let A_i, $i = 1, \ldots, p$, be non empty subsets of X. Let $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be a p-cyclic map such that for some $x \in A_i$ and for each $k = 1, 2, \ldots, p$, the following condition is satisfied: for every $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$d\big(T^{pn+k-1}x, T^k y\big) < \epsilon + \delta \quad \implies \quad d\big(T^{pn+k}x, T^{k+1}y\big) < \epsilon, \quad n \in \mathbb{N}, \ y \in A_i. \quad (4)$$

Then $\{T^{pn}x\}$ converges to a limit say, $z_0 \in A_i$, which is the unique fixed point of T in $\cap_{i=1}^{p} A_i$.

Proof. Let $x \in A_i$ satisfy (4). Define, for each $k = 1, 2, \ldots, p$, the following sets:

$$C_k = \big\{T^{pn+k-1}x \colon n \in \mathbb{N}\big\} \quad \text{and} \quad B_k = \big\{T^k y \colon y \in A_i\big\}.$$

Define $f_k, g_k : C_k \times B_k \to [0, \infty)$ as follows:

$$f_k(a_k, b_k) = d\big(T^{pn+k-1}x, T^k y\big) \quad \text{and} \quad g_k(a_k, b_k) = d\big(T^{pn+k}x, T^{k+1}y\big).$$

Then each f_k and g_k satisfies condition (i) of Lemma 1. Hence, there exists an L-function ϕ such that if $d(T^{pn+k-1}x, T^k y) > 0$, then

$$d\big(T^{pn+k}x, T^{k+1}y\big) < \phi\big\{d\big(T^{pn+k-1}x, T^k y\big)\big\}. \quad (5)$$

From the definition of L-function it follows that

$$d\big(T^{pn+k}x, T^{k+1}y\big) < d\big(T^{pn+k-1}x, T^k y\big) \quad \text{when} \quad d\big(T^{pn+k-1}x, T^k y\big) > 0, \quad (6)$$
$$d\big(T^{pn+k}x, T^{k+1}y\big) = d\big(T^{pn+k-1}x, T^k y\big) \quad \text{when} \quad d\big(T^{pn+k-1}x, T^k y\big) = 0 \quad (7)$$

for all $n \in \mathbb{N}$, $y \in A_i$ and for each $k = 1, 2, \dots, p$. From (6) and (7) it follows that

$$d\big(T^{pn+k}x, T^{k+1}y\big) \leqslant d\big(T^{pn+k-1}x, T^k y\big). \tag{8}$$

Let $s_n = d(T^{pn}x, T^{pn+1}y)$, where $y \in A_i$, $n \in \mathbb{N}$. If $s_n = 0$ for some $n \in \mathbb{N}$, then from (8) it follows that $s_n \to 0$ as $n \to \infty$. Assume that $s_n > 0$ for all $n \in \mathbb{N}$. Then by (6) $s_{n+1} < s_n$ and hence converges to an $r \geqslant 0$. If $r > 0$, then by (4) there exists a $\delta > 0$ such that

$$r \leqslant d\big(T^{pn}x, T^{pn+1}y\big) < r + \delta, \quad n \in \mathbb{N}.$$

Then there exists an L-function ϕ such that

$$d\big(T^{pn+1}x, T^{pn+2}y\big) < \phi\big\{d\big(T^{pn}x, T^{pn+1}y\big)\big\} \leqslant r,$$

which is a contradiction. Hence, $r = 0$. Therefore, $d(T^{pn}x, T^{pn+1}y) \to 0$ as $n \to \infty$. Let $k \in \{1, 2, \dots, p\}$. Then

$$\mathrm{dist}(A_i, A_{i+k})$$
$$\leqslant d\big(T^{pn}x, T^{pn+k}y\big) \leqslant \sum_{i=0}^{k-1} d\big(T^{pn+i}x, T^{pn+i+1}y\big) \to 0 \quad \text{as } n \to \infty.$$

Thus,

$$d\big(T^{pn+k-1}x, T^{pn+k}y\big) \to 0 \quad \text{as } n \to \infty \tag{9}$$

for $k = 0, 1, 2, \dots, (p-1)$. Therefore, $\cap_{i=1}^p A_i$ is nonempty. Let us prove that for $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that $d(T^{pn}x, T^{pm}x) < \epsilon$ for all $n, m \geqslant n_0$. Let $\epsilon > 0$ be given. From (9) it follows that there exists $n_0 \in \mathbb{N}$ such that the inequality $d(T^{pm+k}x, T^{pm+k-1}x) < \delta/p$ holds for every $m \geqslant n_0$ and $k \in \mathbb{N}$. Let us prove that $d(T^{pm+1}x, T^{pn}x) < \epsilon/2$ by induction on n. This is true for $n = m$. Let us assume that this inequality is true for some $n \geqslant n_0$. We need to prove that the inequality holds for $n + 1$. By the inductive assumption we obtain the inequalities

$$
\begin{aligned}
S_1 &= d\big(T^{pm}x, T^{p(n+1)-1}x\big) \\
&\leqslant d\big(T^{p(m+1)}x, T^{p(n+1)-1}x\big) + d\big(T^{pm}x, T^{p(m+1)}x\big) \\
&\leqslant d\big(T^{p(n+1)-1}x, T^{p(m+1)}x\big) + \sum_{j=1}^p d\big(T^{pm+j-1}x, T^{pm+j}x\big) \\
&\leqslant d\big(T^{pn}x, T^{pm+1}x\big) + \sum_{j=1}^p d\big(T^{pm+j-2}x, T^{pm+j-1}x\big) \\
&< \frac{\varepsilon}{2} + p\frac{\delta}{p} = \frac{\varepsilon}{2} + \delta.
\end{aligned} \tag{10}
$$

The map T is a p-cyclic orbital Meir–Keeler contraction, and thus, it follows from (10) that $d(T^{pm+1}x, T^{p(n+1)}x) < \varepsilon/2$. Therefore, from the inequality

$$d\big(T^{pm}x, T^{pn}x\big) \leqslant d\big(T^{pm}x, T^{pm+1}x\big) + d\big(T^{pm+1}x, T^{pn}x\big) \tag{11}$$

we get that $d(T^{pn}x, T^{pm}x) < \epsilon$ for all $n, m \geqslant n_0$. Hence, $\{T^{pn}x\}$ is a Cauchy sequence, and thus, it converges to a $z \in A_i$. Now using (8) and (9), we get

$$d(z, Tz) = \lim_n d\big(T^{pn}x, Tz\big) \leqslant \lim_n d\big(T^{pn-1}x, z\big) = \lim_n d\big(T^{pn-1}x, T^{pn}x\big) = 0.$$

Hence, $z \in A_i$ is a fixed point of T in A_i. By Remark 1, $z \in \cap_{i=1}^p A_i$.

To prove the uniqueness, let $\xi \in \cap_{i=1}^p A_i$ be such that $\xi = T\xi$. Hence, $T^{pn}\xi = \xi$, $n \in \mathbb{N}$. Now $d(z, \xi) = \lim_n d(T^{pn}x, T^{pn+1}\xi) = 0$. Hence, $\xi = z$. $\qquad\square$

The notion of L-function, given in Definition 4 and Lemma 1, is used to obtain the following result for a p-cyclic orbital Meir–Keeler contraction map.

Lemma 4. *Let (X, d) be a metric space. Let A_i, $i = 1, \ldots, p$, be non empty subsets of X. Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic orbital Meir–Keeler contraction map. Then there exists an L-function ϕ such that for an $x \in A_i$ satisfying (3), the following hold*:

$$d\big(T^{pn+k-1}x, T^k y\big) > \mathrm{dist}(A_{i+k-1}, A_{i+k})$$
$$\implies \quad \lambda_{p,n,k}(x, y) < \phi\big(\lambda_{p,n,k-1}(x, y)\big), \quad n \in \mathbb{N}, \ y \in A_i, \tag{12}$$

where we use the notation $\lambda_{p,n,k}(x, y) = d(T^{pn+k}x, T^{k+1}y) - \mathrm{dist}(A_{i+k}, A_{i+k+1})$, and

$$d\big(T^{pn+k-1}x, T^k y\big) = \mathrm{dist}(A_{i+k-1}, A_{i+k})$$
$$\implies \quad d\big(T^{pn+k}x, T^{k+1}y\big) = \mathrm{dist}(A_{i+k}, A_{i+k+1}) \quad n \in \mathbb{N}, \ y \in A_i, \tag{13}$$

for each $k = 1, 2, \ldots, p$.

Proof. Let $x \in A_i$ satisfy (3). For each $k = 1, 2, \ldots, p$, define the following sets:

$$C_k = \big\{T^{pn+k-1}x\colon n \in \mathbb{N}\big\} \quad \text{and} \quad B_k = \big\{T^k y\colon y \in A_i\big\}.$$

Let $f_k, g_k : C_k \times B_k \to [0, \infty)$ be defined as follows:

$$f_k(a_k, b_k) = d\big(T^{pn+k-1}x, T^k y\big) - \mathrm{dist}(A_{i+k-1}, A_{i+k}),$$

and

$$g_k(a_k, b_k) = d\big(T^{pn+k}x, T^{k+1}y\big) - \mathrm{dist}(A_{i+k}, A_{i+k+1}).$$

Since T is a p-cyclic orbital Meir–Keeler contraction map, each f_k and g_k satisfy condition (i) of Lemma 1, and hence, (12) and (13) hold. $\qquad\square$

Remark 2. From Lemma 4 it follows that a p-cyclic orbital Meir–Keeler contraction map is p-cyclic orbital non expansive map.

Lemma 5. *Let (X, d) be a metric space. Let A_i, $i = 1, \ldots, p$, be non empty subsets of X. Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic orbital Meir–Keeler contraction map with an $x \in A_i$ satisfying (3). Then, for all $y \in A_i$ and for each $k \in \{0, 1, 2, \ldots, (p-1)\}$, the sequence $\{d(T^{pn+k}x, T^{pn+k+1}y)\}_{n=1}^\infty$ converges to $\mathrm{dist}(A_{i+k}, A_{i+k+1})$.*

Proof. Let $s_n = d(T^{pn+k}x, T^{pn+k+1}y) - \text{dist}(A_{i+k}, A_{i+k+1})$. Then $s_n \geqslant 0$ for all $n \in \mathbb{N}$. By Remark 4, $s_{n+1} \leqslant s_n$ for all $n \in \mathbb{N}$. If $s_n = 0$ for some n, then the lemma follows. Suppose $s_n > 0$ for every $n \in \mathbb{N}$. Then by Lemma 4 there exists an L-function ϕ satisfying (12) and (13). Since $s_{n+1} \leqslant s_n$, $\{s_n\}$ converges to an $r \geqslant 0$. Suppose $r > 0$. Then, for this $r > 0$, by (3) there exists a $\delta > 0$ such that $r \leqslant s_n < r + \delta$ and such that $s_{n+1} < \phi(s_n) \leqslant r$. That is, $s_{n+1} < r$, which is a contradiction.

Hence, $r = 0$. Thus, $d(T^{pn+k}x, T^{pn+k+1}y) \to \text{dist}(A_{i+k}, A_{i+k+1})$ as $n \to \infty$. $\square$

Theorem 3. *Let X be a uniformly convex Banach space. Let $A_1, A_2, \ldots, A_p$ be non empty, closed and convex subsets of X. Let $T : \cup_{i=1}^p A_i \to \cup_{i=1}^p A_i$ be a p-cyclic orbital Meir–Keeler contraction map. Then, for every $x \in A_i$ satisfying (3), the sequence $\{T^{pn}x\}$ converges to a unique point $z \in A_i$, which is the best proximity point as well as the unique periodic point of T in A_i. Also, $T^k z$ is a best proximity point of T in A_{i+k}, which is also a unique periodic point of T in A_{i+k} for each $k = 1, 2, \ldots, (p-1)$.*

Proof. Let $\epsilon > 0$ be given. Since T is a p-cyclic orbital Meir–Keeler contraction map, there exists an $x \in A_i$ and a $\delta > 0$ satisfying (3). Without loss of generality, let $\delta < \epsilon$. By Lemma 5,

$$\left\| T^{pn+1}x - T^{pn+2}x \right\| \to \text{dist}(A_{i+1}, A_{i+2}) \quad \text{as } n \to \infty$$

and

$$\left\| T^{p(n+1)+1}x - T^{pn+2}x \right\| \to \text{dist}(A_{i+1}, A_{i+2}) \quad \text{as } n \to \infty.$$

Hence, by Lemma 2, $\left\| T^{p(n+1)+1}x - T^{pn+1}x \right\| \to 0$ as $n \to \infty$. Therefore, it is possible to choose an $n_1 \in \mathbb{N}$ such that

$$\left\| T^{pn+1}x - T^{p(n+1)+1}x \right\| < \frac{\delta}{2} \quad \text{for all } n \geqslant n_1, \tag{14}$$

and, by Lemma 5,

$$\left\| T^{pn}x - T^{pn+1}x \right\| < \text{dist}(A_i, A_{i+1}) + 2\epsilon \quad \text{for all } n \geqslant n_1. \tag{15}$$

Fix $n \geqslant n_1$. We show that

$$\left\| T^{pm}x - T^{pn+1}x \right\| < \text{dist}(A_i, A_{i+1}) + \epsilon + \delta \quad \text{for all } m, n \geqslant n_1 \tag{16}$$

by the method of induction. It is obvious that condition (16) is true for $m = n$. Assume that condition (16) is true for an $m > n$. To prove this condition for $m + 1$, consider

$$\begin{aligned} \mu_{p,m,n}(x) &= \left\| T^{p(m+1)}x - T^{pn+1}x \right\| \\ &\leqslant \left\| T^{pn+1}x - T^{p(n+1)+1}x \right\| + \left\| T^{p(n+1)+1}x - T^{p(m+1)}x \right\|. \end{aligned} \tag{17}$$

Now

$$\begin{aligned} \left\| T^{p(n+1)+1}x - T^{p(m+1)}x \right\| &\leqslant \left\| T^{p(n+1)}x - T^{p(m+1)-1}x \right\| \\ &\leqslant \left\| T^{pn+1}x - T^{pm}x \right\| \leqslant \text{dist}(A_i, A_{i+1}) + \epsilon + \delta. \end{aligned}$$

Using (14) in (17), we obtain

$$\left\| T^{p(m+1)}x - T^{pn+1}x \right\| < \operatorname{dist}(A_i, A_{i+1}) + \frac{\delta}{2} + \epsilon + \delta.$$

Hence, (16) holds for $(m+1)$ in place m.

Consider (16) and (15). By Lemma 5 the following holds: for $m > n \geqslant n_1$,

$$\left\| T^{pn}x - T^{pm}x \right\| < 2\epsilon.$$

Hence, $\{T^{pn}x\}$ is a Cauchy sequence and converges to a $z \in A_i$. By Proposition 1 z is a best proximity point of T in A_i, and z is a unique periodic point of T in A_i. Let $\xi \in A_i$ satisfy (3). Then by what we have proved, $\{T^{pn}\xi\}$ converges to an $\eta \in A_i$ such that $\|\eta - T\eta\| = \operatorname{dist}(A_i, A_{i+1})$ and $T^p\eta = \eta$. But z is the unique periodic point of T in A_i. Hence, $\eta = z$. By proposition 1 $T^k z$ is a best proximity point of T in A_{i+k} for each $k = 1, 2, \ldots, p$. □

It is obvious that if condition (3) is satisfied for all $x \in A_i$, then the obtained best proximity point is unique. Theorem 3 is a generalization of Theorem 1, and the following theorem proved in [13].

Theorem 4. (*See* [13].) *Let X be a uniformly convex Banach space, and let $A_1, A_2, \ldots, A_p$ ($p \geqslant 2$) be non empty, closed and convex subsets of X. Let T be a p - cyclic map such that for every $x \in A_i$ and $y \in A_{i+1}$, the following condition is satisfied: for every $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d(x, y) < \operatorname{dist}(A_i, A_{i+1}) + \epsilon + \delta \quad \implies \quad d(Tx, Ty) < \operatorname{dist}(A_{i+1}, A_{i+2}) + \epsilon. \quad (18)$$

Then, for any $x \in A_i$, the sequence $\{T^{pn}x\}$ converges to a unique $z \in A_i$, which is a best proximity point of T in A_i. Moreover, this point is a unique periodic point of T in A_i. Further, $T^k z = z_k$ is a best proximity point of T in A_{i+k} for each $k = 1, 2, \ldots, p$.

From Theorem 4 we observe that a best proximity point is obtained if condition (18) is satisfied for all $x \in A_i$ and $y \in A_{i+1}$ and for all $i = 1, 2, \ldots, p$. From Theorem 3 we observe that a best proximity point of T is obtained even if condition (3) is satisfied for an $x \in A_i$, for all $y \in A_i$ and for some i, $1 \leqslant i \leqslant p$.

Remark 3. From Proposition 1 and Theorem 3 we observe that if X is a uniformly convex Banach space and A_i, $i = 1, \ldots, p$, are closed convex subsets of X and if $T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ is a p-cyclic orbital Meir–Keeler contraction map, then $\{T^{pn}x\}$ converges if and only if $\operatorname{dist}(A_i, A_{i+1}) = \operatorname{dist}(A_{i+1}, A_{i+2}) = \operatorname{dist}(A_1, A_2)$.

4 Examples and applications

We will illustrate the above results with some examples, and we will give an application for integral operators. We will define a map T, and we will prove that T satisfies

800   *S. Karpagam, B. Zlatanov*

condition (3) for $k = 1$. The proofs that the map T satisfies conditions (3) for $k = 0, 2, 3, \ldots, p-1$ can be done a similar fashion.

We will show with the first example the difference between p-cyclic Meir–Keeler and p-cyclic orbital (Meir–Keeler) contraction maps.

Example 1. Let us consider the space $\mathbb{R}^2 = \{(u,v)\colon u, v \in \mathbb{R}\}$ endowed with the Euclidian norm $\|(u,v)\|_2 = \sqrt{u^2 + v^2}$. Let $\alpha \geqslant 0$, $\lambda \in (0,1)$, and let us denote the sets

$$A_1 = \big\{(u,v) \in \mathbb{R}^2\colon \alpha \leqslant u \leqslant \alpha+1,\ \alpha \leqslant v \leqslant u\big\},$$
$$A_2 = \big\{(u,v) \in \mathbb{R}^2\colon -\alpha-1 \leqslant u \leqslant -\alpha,\ \alpha \leqslant v \leqslant |u|\big\},$$
$$A_3 = \big\{(u,v) \in \mathbb{R}^2\colon -\alpha-1 \leqslant u \leqslant \alpha+1,\ -|u| \leqslant v \leqslant -\alpha\big\},$$
$$A_4 = \big\{(u,v) \in \mathbb{R}^2\colon \alpha \leqslant u \leqslant \alpha+1,\ -|u| \leqslant v \leqslant -\alpha\big\}.$$

We define the function $f\colon \mathbb{R} \to \mathbb{R}^+$ by $f(t) = \mathrm{sign}(t)(\lambda|t| + (1-\lambda)\alpha)$, and we define a map $T\colon \cup_{i=1}^4 A_i \to \cup_{i=1}^4 A_i$ by $T((u,v)) = ((-1)^i f(u), (-1)^{i+1} f(v))$ for $(u,v) \in A_i$.

It is easy to see that $T(A_i) \subset A_{i+1}$, $i = 1, 2, 3, 4$. The distance between the consecutive sets is equal to 2α.

We will show that T is a p-cyclic orbital Meir–Keeler contraction map. Let us choose $x = (\alpha, \alpha) \in A_1$. Let $\varepsilon > 0$ be arbitrary chosen. Let us put

$$\delta = \frac{\sqrt{4\alpha\varepsilon + \varepsilon^2 + 4\alpha^2\lambda^2}}{\lambda} - (2\alpha + \varepsilon).$$

Let $y \in A_1$ be such that $\|T^{4n}x - Ty\|_2 < 2\alpha + \varepsilon + \delta$. Let us put $Ty = (-|y_1|, |y_2|) \in A_2$. Then there holds the inequality $(|y_1| + \alpha)^2 + (|y_2| - \alpha)^2 < (2\alpha + \varepsilon + \delta)^2$. By the definition of the sets A_i it follows that $0 \leqslant \alpha \leqslant |y_2| \leqslant |y_1|$ and, consequently, we get the chain of inequalities

$$\begin{aligned}
\big\|T^{4n}x - T^2 y\colon\big\|_2^2 &= \lambda^2\big(|y_1| - \alpha\big)^2 + \big(2\alpha + \lambda(|y_2| - \alpha)\big)^2 \\
&= \lambda^2\big((|y_1| + \alpha)^2 + (|y_2| - \alpha)^2\big) + 4\alpha^2 + 4\lambda\alpha\big((1-\lambda)|y_1| - \alpha\big) \\
&\leqslant \lambda^2(2\alpha + \varepsilon + \delta)^2 + 4\alpha^2 - 4\alpha^2\lambda^2 < (2\alpha + \varepsilon)^2.
\end{aligned}$$

Consequently, T is 4-cyclic orbital Meir–Keeler contraction map. It is easy to observe that x is a best proximity point of T in A_1.

We would like to point out that T is not a 4-cyclic Meir–Keeler map defined in [13]. This can be observed by taking $U = (\alpha+1, (1+\mu)\alpha)$ and $V = (\alpha+1, -(1+\mu)\alpha)$ for $\mu > 0$ small enough. The map T is not a p-cyclic orbital contraction in the sense of [14]. It can be observed by taking $z = (-\alpha, \alpha)$, $w = (\alpha, (1-\mu)\alpha)$ for $\mu > 0$ converging to zero.

If $\alpha = 0$, then we get that T satisfies (4), and by Theorem 2 there is a unique fixed point $z_0 \in \cap_{i=1}^4 A_i$. It can be observed in a similar fashion that T is not 4-cyclic Meir–Keeler map and is not a p-cyclic orbital contraction.

http://www.mii.lt/NA

We will present an example in infinite dimensional Banach space, where the map T is defined as an integral operator. We will need the fact that the inequality

$$g(\varepsilon) = \sqrt{\left(\frac{\sqrt{2}}{\sqrt{3}} + \varepsilon\right)^2 - \frac{1}{3}} - \frac{1}{3}\left(\frac{5}{6} + \frac{1}{6}\left(2(1 + 2\varepsilon)^2 - 1\right)^{1/2}\right)^2 > 0 \qquad (19)$$

holds for every $\varepsilon \in (0, +\infty)$. First, we will show that $g'(\varepsilon) > 0$ for every $\varepsilon \in (0, +\infty)$. Indeed, using the inequalities $\sqrt{2(1 + 2\varepsilon)^2 - 1} > 1 + 2\varepsilon$ and $\sqrt{2}/\sqrt{3} + \varepsilon > \sqrt{(\sqrt{2}/\sqrt{3} + \varepsilon)^2 - 1/3}$, we get that the inequality

$$g'(\varepsilon) = \frac{9\sqrt{2(1 + 2\varepsilon)^2 - 1}\left(\frac{\sqrt{2}}{\sqrt{3}} + \varepsilon\right) - 4(1 + 2\varepsilon)\sqrt{\left(\frac{\sqrt{2}}{\sqrt{3}} + \varepsilon\right)^2 - \frac{1}{3}}}{9\left(\sqrt{\left(\frac{\sqrt{2}}{\sqrt{3}} + \varepsilon\right)^2 - \frac{1}{3}}\right)\sqrt{2(1 + 2\varepsilon)^2 - 1}} > 0$$

holds for every $\varepsilon \in (0, +\infty)$, and therefore, g is strictly increasing in the interval $(0, +\infty)$. From the equality $g(0) = 0$ it follows that $g(\varepsilon) > 0$ for every $\varepsilon \in (0, +\infty)$.

Let us recall that $L_2[-1, 1]$ is the space of all classes of measurable functions f such that $\int_{-1}^{1} f^2(s)\,\mathrm{d}s < +\infty$. If $L_2[-1, 1]$ is endowed with the norm $\|f\|_2 = (\int_{-1}^{1} f^2(s)\,\mathrm{d}s)^{1/2}$, it is a uniformly convex Banach space [7].

Example 2. Let us consider the space $L_2[-1, 1]$. We denote the functions:

$$x_1(s) = \begin{cases} 0, & s \in [-1, 0], \\ s, & s \in [0, 1]; \end{cases} \qquad x_2(s) = x_1(-s);$$

$$x_3(s) = -x_1(s); \qquad\qquad x_4(s) = -x_1(-s).$$

Let us consider the sets $A_i \subset L_2[-1, 1]$ defined by $A_i = \{f \in L_2[-1, 1]: f(s) \geqslant x_i(s)\}$, $i = 1, 2$, and $A_j = \{f \in L_2[-1, 1]: f(s) \leqslant x_j(s)\}$, $j = 3, 4$. For any functions $f \in A_1 \cup A_2$ and $g \in A_3 \cup A_4$, we will use the notation $\pi(f) = 1$ and $\pi(g) = 2$.

We define the map $F: A_1 \to A_1$ by

$$F\big(f(s)\big)(t) = \mathrm{sign}\big(f(t)\big)\left(\frac{5}{6}t + \int_0^1 \frac{ts}{2} f(s)\,\mathrm{d}s\right) = \begin{cases} \frac{5}{6}t + \int_0^1 \frac{ts}{2} f(s)\,\mathrm{d}s, & t \geqslant 0, \\ 0, & t < 0. \end{cases}$$

We define a cyclic map $T(A_i) \subseteq A_{i+1}$ by

$$T\big(f_i(s)\big)(t) = (-1)^{i + \pi(f_i)} F\big((-1)^{\mathrm{sign}(f_i)} f_i((-1)^{i+1} s)\big)\big((-1)^i t\big) \qquad (20)$$

for $f \in A_i$, $i = 1, 2, 3, 4$. It is easy to observe from (20) that for $f_i \in A_i$, there hold $T(f_1(s))(t) = F(f_1(s))(-t) \in A_2$, $T(f_2(s))(t) = -F(f_2(-s))(t)v \in A_3$, $T(f_3(s))(t) = -F(-f_3(s))(-t) \in A_4$ and $T(f_4(s))(t) = F(-f_4(-s))(t) \in A_1$.

First, we calculate that $\mathrm{dist}(A_i, A_{i+1}) = \|x_i - x_{i+1}\|_2 = \sqrt{2}/\sqrt{3}$.

We will show that T is a 4-cyclic orbital Meir–Keeler contraction map. It is easy to see that $T(x_i) = x_{i+1}$, where we use the convention $x_{4+i} = x_i$. Let us choose $x = x_4$. Let $\varepsilon > 0$ be arbitrary. We put $\delta = (2\sqrt{2}/\sqrt{3} - 1)\varepsilon$. Let $y \in A_4$ be such that

$$\left\| T^{4n} x - Ty \right\|_2 < \frac{\sqrt{2}}{\sqrt{3}} + \varepsilon + \delta. \tag{21}$$

Let us put $Ty = f \in A_1$. Then inequality (21) is equivalent to the inequality

$$\int_0^1 f^2(s)\, \mathrm{d}s < \frac{2}{3}(1 + 2\varepsilon)^2 - \frac{1}{3}. \tag{22}$$

We will show that if inequality (22) holds, then there holds the inequality

$$\left\| T^{4n+1} x - Tf \right\|_2 = \| x_1 - Tf \|_2 < \frac{\sqrt{2}}{\sqrt{3}} + \varepsilon, \tag{23}$$

i.e.

$$\int_{-1}^0 \left(-\frac{5}{6}t - \int_0^1 \frac{ts}{2} f(s)\, \mathrm{d}s \right)^2 \mathrm{d}t < \left(\frac{\sqrt{2}}{\sqrt{3}} + \varepsilon \right)^2 - \frac{1}{3}.$$

Using Hölder's inequality and (19), we get the chain of inequalities

$$\int_0^1 \left(\frac{5}{6}t + \int_0^1 \frac{ts}{2} f(s)\, \mathrm{d}s \right)^2 \mathrm{d}t \leqslant \int_0^1 \left(\frac{5}{6}t + \frac{t}{2} \left(\int_0^1 s^2\, \mathrm{d}s \right)^{1/2} \left(\int_0^1 f^2(s)\, \mathrm{d}s \right)^{1/2} \right)^2 \mathrm{d}t$$

$$= \frac{1}{3} \left(\frac{5}{6} + \frac{1}{2\sqrt{3}} \left(\int_0^1 f^2(s)\, \mathrm{d}s \right)^{1/2} \right)^2$$

$$< \frac{1}{3} \left(\frac{5}{6} + \frac{1}{6} \left(2(1 + 2\varepsilon)^2 - 1 \right)^{1/2} \right)^2$$

$$< \left(\frac{\sqrt{2}}{\sqrt{3}} + \varepsilon \right)^2 - \frac{1}{3},$$

i.e. (23).

Consequently T is 4-cyclic orbital Meir–Keeler contraction map. It is easy to observe that x_i is a best proximity point of T in A_i.

The third example will be in infinite dimensional Banach space, which is not endowed with an Euclidian metric.

Proposition 2. *For any $d > 0$, $\varepsilon > 0$, $q > 1$, there exists $\delta = \delta(\varepsilon)$ such that there holds the inequality $(d + \varepsilon + \delta)^q - (d + \varepsilon)^q/2 + d^q/2 < (d + \varepsilon)^q$.*

Proof. Let us consider the function $f : [-\varepsilon, \varepsilon] \to \mathbb{R}$ defined as follows:

$$f(\delta) = (d + \varepsilon + \delta)^q - \frac{3}{2}(d + \varepsilon)^q + \frac{d^q}{2}.$$

The function f is continuous, and $f(0) = -(d+\varepsilon)^q/2 + d^q/2 < 0$. Thus, there exists δ_0 such that $f(\delta) < 0$ for every $\delta \in [-\delta_0, \delta_0]$. Consequently, there exists $\delta > 0$ such that the inequality $(d + \varepsilon + \delta)^q - (d + \varepsilon)^q/2 + d^q/2 < (d + \varepsilon)^q$ holds true. □

Example 3. Let us consider the space ℓ_q endowed with the norm $\|x\| = (\sum_{i=1}^{\infty} |x_i|^q)^{1/q}$. It is well known that ℓ_p is a uniformly convex Banach space. Let $\{e_i\}_{i=1}^{\infty}$ be the unit vector basis in ℓ_q. For any vector $x = \sum_{i=1}^{\infty} x_i e_i$, we will denote with $\text{supp}(x)$ the set of its nonzero coordinates. Let $z = \{z_i\}_{i=1}^{\infty} \in \ell_q$ be such that $z_i > 0$ for every $i \in \mathbb{N}$. Let $p \in \mathbb{N}$, $p \geqslant 3$, and let us define the sets

$$A_k = \left\{ x \in \ell_q \colon x = \sum_{i=0}^{\infty} x_{i+1} e_{pi+k},\ x_i \geqslant z_i \text{ for } i \in \mathbb{N} \right\}, \quad k = 1, 2, \dots p.$$

It is easy to see that A_k are convex and closed sets, and $A_k \cap A_j = \emptyset$ for any $k \neq j$. Thus, for any $x^{(k)} \in A_k$, $k = 1, 2, \dots p$, there holds $\text{supp}\, x^{(k)} \cap \text{supp}\, x^{(j)} = \emptyset$ provided that $k \neq j$. Consequently, $\|x^{(k)} \pm x^{(j)}\| = (\|x^{(k)}\|^q + \|x^{(j)}\|^q)^{1/q}$. It is easy to see that for any $x^{(k)} \in A_k$, there holds the inequality $\|x^{(k)}\| \geqslant \|z\|$. We will calculate the distance $\text{dist}(A_k, A_{k+1})$. For any $k = 1, 2, \dots, p$, where we have that $A_{p+1} = A_1$, there holds

$$\begin{aligned} \text{dist}(A_k, A_{k+1}) &= \inf\{\|x^{(k)} - x^{(k+1)}\| \colon x^{(k)} \in A_k,\ x^{(k+1)} \in A_{k+1}\} \\ &= \inf\{(\|x^{(k)}\|^q + \|x^{(k+1)}\|^q)^{1/q} \colon x^{(k)} \in A_k,\ x^{(k+1)} \in A_{k+1}\} \\ &\geqslant (2\|z\|^q)^{1/q} = \sqrt[q]{2}\|z\|. \end{aligned}$$

Thus, $\text{dist}(A_k, A_{k+1}) = \sqrt[q]{2}\|z\|$. Therefore, we get that

$$\text{dist}(A_1, A_2) = \text{dist}(A_2, A_3) = \cdots = \text{dist}(A_{p-1}, A_p) = \text{dist}(A_p, A_1).$$

Let us denote $d = \text{dist}(A_1, A_2) = \sqrt[q]{2}\|z\|$. For $x = \sum_{j=0}^{\infty} \sum_{i=1}^{p} x_{jp+i} e_{jp+i} \in \ell_q$, let us put $T_j(x) = (1/2)(x_{(j+1)p} + z_{j+1})e_{jp+1} + \sum_{i=2}^{p}(1/2)(x_{jp+i-1} + z_{j+1})e_{jp+i}$, $j = 0, 1, 2, \dots$. We define a map $T : \ell_q \to \ell_q$ by $T(\sum_{j=0}^{\infty}(\sum_{i=1}^{p} x_{jp+i} e_{jp+i})) = \sum_{j=0}^{\infty} T_j(x)$. For any $x^{(k)} \in A_k$, $k = 1, 2, \dots p-1$, we have $T(x^{(k)}) = T(\sum_{j=0}^{\infty} x_{j+1}^{(k)} \times e_{jp+k}) = \sum_{j=0}^{\infty}(1/2)(x_{j+1}^{(k)} + z_{j+1})e_{jp+k+1}$, and for $k = p$, we have $T(x^{(p)}) = T(\sum_{j=1}^{\infty} x_j^{(p)} e_{jp}) = \sum_{j=1}^{\infty}(1/2)(x_j^{(p)} + z_j)e_{(j-1)p+1}$. By the condition $x_i^{(k)} \geqslant z_i$ it follows that $(1/2)(x_i^{(k)} + z_i) \geqslant z_i$, and therefore, $T(A_k) \subseteq A_{k+1}$ for any $k = 1, 2, \dots, p$. Let us put for $k = 1, 2, \dots, p$, $z^{(k)} = \sum_{j=1}^{\infty} z_j e_{(j-1)p+k}$. Then there hold $\|z\| = \|z^{(k)}\|$ for all $k = 1, 2, \dots p$.

Then, for any $k = 1, 2, \dots, p-1$, there holds $T(z^{(k)}) = z^{(k+1)}$ and $T(z^{(p)}) = z^{(1)}$. For any $k, s \in \mathbb{N}$, there exist $m, r \in \mathbb{N}$ such that $r < p$ and $s + k = mp + r$. Then $T^s(z^{(k)}) = z^{(r)}$.

We will show that the map $T : \cup_{k=1}^{p} A_k \rightarrow \cup_{k=1}^{p} A_k$ satisfies the conditions of Theorem 2. The map T is a cyclic map, and $\operatorname{dist}(A_1, A_2) = \operatorname{dist}(A_2, A_3) = \cdots = \operatorname{dist}(A_{p-1}, A_p) = \operatorname{dist}(A_p, A-1)$.

It remains to show that T is a p-cyclic orbital Meir–Keeler contraction maps, i.e. it satisfies condition (3). We choose $x = z^{(1)} \in A_1$, and let $y \in A_1$ be arbitrary chosen. Then $\operatorname{supp}(T^{pn+k-1} z^{(1)}) \cap \operatorname{supp}(T^k y) = \emptyset$. It is easy to see that $T^{np} x = T^{np} z^{(1)} = z^{(1)}$ and $T^{np+k} x = T^{np+k} z^{(1)} = z^{(k+1)}$ for $k = 1, 2, \ldots, p-1$.

Let $\varepsilon > 0$ be arbitrary.

We will consider two cases:

Case 1. $\|T^{pn+k-1} x - T^k y\| < \operatorname{dist}(A_{i+k-1}, A_{i+k}) + \varepsilon$;
Case 2. $\|T^{pn+k-1} x, T^k y\| \geqslant \operatorname{dist}(A_{i+k-1}, A_{i+k}) + \varepsilon$.

Let us put $u = T^k y = \sum_{j=0}^{\infty} u_{j+1} e_{jp+k+1}$. Then $\operatorname{supp} z^{(k)} \cap \operatorname{supp} u = \emptyset$ and $\operatorname{supp} z^{(k+1)} \cap \operatorname{supp} Tu = \emptyset$. Then

$$\|Tu\| = \left\| \sum_{j=0}^{\infty} \frac{u_{j+1} + z_{j+1}}{2} e_{jp+k+2} \right\| = \left(\sum_{j=1}^{\infty} \left| \frac{u_j + z_j}{2} \right|^q \right)^{1/q} = \left\| \frac{u + z^{(k+1)}}{2} \right\|$$

and

$$\left\| T^{pn+k-1} x - u \right\| = \left(\left\| z^{(k)} \right\|^q + \|u\|^q \right)^{1/q}.$$

Case 1. Let there holds

$$\left\| T^{pn+k-1} x - u \right\| = \left\| T^{pn+k-1} x - T^k y \right\| < \operatorname{dist}(A_{i+k-1}, A_{i+k}) + \varepsilon.$$

From the inequality

$$
\begin{aligned}
\left\| T^{pn+k} x - T^{k+1} y \right\| &= \left\| z^{(k+1)} - Tu \right\| = \left(\left\| z^{(k+1)} \right\|^q + \|Tu\|^q \right)^{1/q} \\
&= \left(\left\| z^{(k+1)} \right\|^q + \left\| \frac{u + z^{(k+1)}}{2} \right\|^q \right)^{1/q} \\
&\leqslant \left(\left\| z^{(k+1)} \right\|^q + \frac{\|u\|^q}{2} + \frac{\left\| z^{(k+1)} \right\|^q}{2} \right)^{1/q} \\
&\leqslant \left(\left\| z^{(k+1)} \right\|^q + \|u\|^q \right)^{1/q} = \left(\|z\|^q + \|u\|^q \right)^{1/q} \\
&= \left\| T^{pn+k-1} x - u \right\| < \operatorname{dist}(A_{i+k-1}, A_{i+k}) + \varepsilon
\end{aligned}
$$

it follows that condition (3) holds true.

Case 2. Let there hold the inequality

$$
\begin{aligned}
(d + \varepsilon)^q &\leqslant \left\| T^{pn+k-1} x - T^k y \right\|^q = \left\| z^{(k)} - u \right\|^q \\
&= \left\| z^{(k)} \right\|^q + \|u\|^q = \|z\|^q + \|u\|^q.
\end{aligned}
$$

Consequently, we get the inequality

$$(d + \varepsilon)^q - 2\|z\|^q \leqslant \|u\|^q - \|z\|^q. \tag{24}$$

From Proposition 2 there exists $\delta > 0$ such that

$$(d + \varepsilon + \delta)^q - \frac{1}{2}\big((d + \varepsilon)^q - d^q\big) < (d + \varepsilon)^q.$$

Let $y \in A_1$ be such that there hold the inequalities

$$(d + \varepsilon)^q \leqslant \big\|T^{pn+k-1}x - T^k y\big\|^q = \big\|z^{(k)}\big\|^q + \|u\|^q \leqslant (d + \varepsilon + \delta)^q.$$

Then using (24), we get that

$$
\begin{aligned}
\big\|T^{pn+k}x - T^{k+1}y\big\|^q &= \big\|z^{(k+1)} - Tu\big\|^q = \big\|z^{(k+1)}\big\|^q + \|Tu\|^q \\
&= \|z\|^q + \left\|\frac{z^{(k+1)} + u}{2}\right\|^q \leqslant \|z\|^q + \frac{\|z^{(k+1)}\|^q}{2} + \frac{\|u\|^q}{2} \\
&= \|z\|^q + \|u\|^q - \frac{1}{2}\big(\|u\|^q - \|z\|^q\big) \\
&\leqslant (d + \varepsilon + \delta)^q - \frac{1}{2}\big((d + \varepsilon)^q - d^q\big) < (d + \varepsilon)^q.
\end{aligned}
$$

Consequently, T is p-cyclic orbital Meir–Keeler contraction map. It is easy to observe that $z^{(k)}$ is a best proximity point of T in A_k because $Tz^{(k)} = z^{(k+1)}$ and $\mathrm{dist}(A_k, A_{k+1}) = d(z^{(k)}, z^{(k+1)}) = d(z^{(k)}, Tz^{(k)})$.

We would like to pose an open question. A recent results in [9] gives a Variational principle, that can be applied for wide class of cyclic maps. Unfortunately, this result could not be applied for cyclic orbital Meir–Keeler contraction map. It will be interesting if the results from [9] can be generalized so that to be applied for cyclic orbital Meir–Keeler contraction map.

References

1. A. Amini-Harandi, Common best proximity points theorems in metric spaces, *Optim. Lett.*, **8**(2):581–589, 2014.

2. C.-M. Chen, Fixed point theorems for cyclic Meir–Keeler type mappings in complete metric spaces, *Fixed Point Theory Appl.*, **2012**:41, 2012.

3. R. Deville, G. Godefroy, V. Zizler, *Smoothness and Renormings in Banach Spaces*, Pitman Monogr. Surv. Pure Appl. Math., Vol. 64, Longman Scientific & Technical, Harlow, 1993.

4. C. Di Bari, T. Suzuki, C. Vetro, Best proximity points for cyclic Meir–Keeler contractions, *Nonlinear Anal., Theory Methods Appl.*, **69**(11):3790–3794, 2008.

5. A. Eldred, P. Veeramani, Existence and convergence of best proximity points, *J. Math. Anal. Appl.*, **323**(2):1001–1006, 2006.

6. M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant, V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis*, Springer, New York, 2001.

7. M. Fabian, P. Habala, P. Hájek, V. Montesinos, V. Zizler, *Functional Analysis and Infinite-Dimensional Geometry*, Springer, New York, 2011.

8. S. Gudder, D. Strawther, Strictly convex normed linear spaces, *Proc. Am. Math. Soc.*, **59**(2):263–267, 1976.

9. M. Ivanov, B. Zlatanov, N. Zlateva, A variational principle and best proximity points, *Acta Math. Sin., Engl. Ser.*, **31**(8):1315–1326, 2015.

10. E. Karapınar, I.M. Erhan, Best proximity point on different type contractions, *Applied Mathematics and Information Sciences*, **5**(3):558–569, 2011.

11. E. Karapinar, G. Petrusal, K. Tas, Best proximity point theorems for KT-types cyclic orbital contraction mappings, *Fixed Point Theory*, **13**(2):537–546, 2012.

12. E. Karapinar, S. Romaguera, K. Tas, Fixed points for cyclic orbital generalized contractions on complete metric spaces, *Cent. Eur. J. Math.*, **11**(3):552–560, 2013.

13. S. Karpagam, S. Agrawal, Best proximity point for p-cyclic Meir–Keeler contractions, *Fixed Point Theory Appl.*, **2009**:197308, 2009.

14. S. Karpagam, S. Agrawal, Existence of best proximity points of p-cyclic contractions, *Fixed Point Theory*, **13**(1):99–105, 2012.

15. S. Karpagam, Sushama Agrawal, Best proximity point theorems for cyclic orbital Meir–Keeler contraction maps, *Nonlinear Anal., Theory Methods Appl.*, **74**(4):1040–1046, 2011.

16. W. Kirk, P. Srinivasan, P. Veeramani, Fixed points for mapping satisfying cyclical contractive conditions, *Fixed Point Theory*, **4**(1):79–89, 2003.

17. T.-C. Lim, On characterizations of Meir–Keeler contractive maps, *Nonlinear Anal., Theory Methods Appl.*, **46**(1):113–120, 2001.

18. A. Meir, E. Keeler, A theorem on contractive mappings, *J. Math. Anal. Appl.*, **28**(2):326–329, 1969.

19. C. Mongkolkeha, P. Kumam, Some common best proximity points for proximity commuting mappings, *Optim. Lett.*, **7**(8):1825–1836, 2013.

20. M. Petric, B. Zlatanov, Best proximity point theorems for weak p-cyclic Kannan contractions, preprint, 2012.

21. M. Petric, B. Zlatanov, Best proximity points and fixed points for p-summing maps, *Fixed Point Theory Appl.*, **2012**:86, 2012.

22. V. Pragadeeswarar, M. Marudai, Best proximity points for generalized proximal weak contractions in partially ordered metric spaces, *Optim. Lett.*, **9**(1):105–118, 2015.

23. T. Suzuki, Some notes on Meir–Keeler contractions and L-functions, *Bull. Kyushu Inst. Technol., Pure Appl. Math.*, **53**:1–13, 2006.

24. B. Zlatanov, Best proximity points for p-summing cyclic orbital Meir–Keeler contractions, *Nonlinear Anal. Model. Control*, **20**(4):528–544, 2015.